The soft consensus model: dynamics, complexity, and modularity

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Abstract. We consider the network dynamics of the soft consensus model and we investigate the dynamical behavior of two important measures of network structure, complexity and modularity. The dynamical mechanism of the soft consensus model is driven by the minimization of a cost function combining a collective measure of dissensus with an individual mechanism of opinion changing aversion. The dissensus measure plays a key role in the model and induces a network of pairwise interactions between the individual opinions. The pairwise interaction coefficients take values in the unit interval and define the adjacency matrix of an undirected weighted graph. The measures of complexity and modularity express in two different ways how far the network structure is from some appropriate, transitive partitioning. We illustrate their dynamical behavior in the context of the soft consensus model by means of a number of computer simulations.

1 The soft consensus model

Let \( \{i = 1, \ldots, n\} \) be an index set for a group of agents. The individual opinions of these agents are denoted by \( r_i \in [0, 1] \) with \( i = 1, \ldots, n \). The soft consensus network model of group decision making [11,12,13,6,5,3,4] is based on a graph representation in which each node \( i \) encodes the individual opinion value \( r_i \), and each edge \((i, j)\) encodes the interaction coefficient (strength of interaction) between agents \( i \) and \( j \), expressed in the \((0, 1)\) interval: larger interaction values correspond to stronger interactions. Consider the decreasing sigmoid function

\[
\sigma(x) = \frac{1}{1 + e^{\beta(x-\alpha)}} \in (0,1)
\]

with two free parameters \( 0 < \alpha < 1 \) and \( \beta > 0 \). The former is a threshold parameter, \( \sigma(\alpha) = \frac{1}{2} \), and the latter controls the polarization of the sigmoidal response around the
threshold value, $\sigma'(\alpha) = -\frac{\beta}{\alpha}$. The interaction coefficients $v_{ij}(\alpha, \beta) \in (0, 1)$ defining the adjacency matrix of an undirected weighted graph are computed by applying the non-linear $\sigma$ filter to the square distance between the two opinion values $r_i$ and $r_j$,

$$v_{ij} = v_{ji} = \sigma\left((r_i - r_j)^2\right), \quad v_i = \sum_{j \neq i} \frac{v_{ij}}{n-1}$$

where $v_i(\alpha, \beta) \in (0, 1)$ corresponds to the average interaction coefficient between decision maker $i$ and the rest of the group. In this framework the parameter $\alpha$ sets the threshold value separating small from large opinion distances and the parameter $\beta$ controls how sharply the interaction coefficients respond to this differentiation: small opinion distances imply strong interactions and large opinion distances imply weak interactions. The interaction coefficients can also be used to define, with respect to decision maker $i$, the mean opinion of the rest of the group,

$$\bar{r}_i = \frac{\sum_{j \neq i} v_{ij} r_j}{\sum_{j \neq i} v_{ij}}$$

In the soft consensus model as in [3] each agent $i = 1, \ldots, n$ is represented by a pair of connected nodes, a primary node (dynamic) and a secondary node (static). The $n$ primary nodes form a fully connected subnetwork and each of them encodes the individual opinion of a single agent. The $n$ secondary nodes, on the other hand, encode the individual opinions originally declared by the agents, denoted $s_i \in [0, 1]$, and each of them is connected only with the associated primary node. The dynamical process of opinion change corresponds to the gradient descent optimization of a cost function $W$, depending on both the present and the original network configurations. The value of $W$ combines a measure $V$ of the overall dissensus in the present network configuration with a measure $U$ of the overall change from the original network configuration. The various interactions involving node $i$ are modulated by interaction coefficients whose role is to quantify the strength of the interaction. The consensual interaction between primary nodes $i$ and $j$ is modulated by the interaction coefficient $v_{ij} \in (0, 1)$, whereas the inertial interaction between the primary node $i$ and the associated secondary node is modulated by the interaction coefficient $u_i \in (0, 1)$. In the soft consensus model the values of these interaction coefficients are given by the derivative $f' = \sigma$ of a scaling function $f$ according to

$$v_{ij} = f'\left((r_i - r_j)^2\right), \quad v_i = \sum_{j \neq i} \frac{v_{ij}}{n-1}, \quad u_i = f'\left((r_i - s_i)^2\right)$$

where the scaling function is given by $f(x) = -\frac{1}{\beta} \ln \left(1 + e^{-\beta(x-\alpha)}\right)$. The average preference $\bar{r}_i$ is again given by (3), and represents the average preference of the remaining decision makers as seen by decision maker $i = 1, \ldots, n$. The construction of the cost function $W$ that drives the dynamics of the soft consensus model is as follows. The individual dissensus cost $V(i)$ is given by

$$V(i) = \sum_{j \neq i} \frac{V(i,j)}{n-1}, \quad V(i,j) = f\left((r_i - r_j)^2\right)$$
and the individual opinion changing cost $U(i)$ is

$$U(i) = f\left((r_i - s_i)^2\right).$$  \hfill (6)$$

Summing over the various decision makers we obtain the collective dissensus cost $V$ and inertial cost $U$,

$$V = \frac{1}{4} \sum_i V(i), \quad U = \frac{1}{2} \sum_i U(i)$$

(7)

with conventional multiplicative factors of $\frac{1}{4}$ and $\frac{1}{2}$, which ensure that no explicit numerical coefficient will be present in (13). The full cost function is then $W = (1 - \lambda)V + \lambda U$ with $\lambda \in [0, 1]$. The consensual network dynamics, which can be regarded as an unsupervised learning algorithm, acts on the individual opinion variables $r_i$ through the iterative process

$$r_i \leadsto r'_i = r_i - \gamma \frac{\partial W}{\partial r_i}. \hfill (8)$$

Considering the effect of the two dynamical components $V$ and $U$ separately, we obtain

$$\frac{\partial V}{\partial r_i} = v_i (r_i - \bar{r}_i), \hfill (9)$$

where the coefficients $v_i$ were defined in (4) and the average preference $\bar{r}_i$ was defined in (3). Therefore

$$r'_i = (1 - \gamma v_i) r_i + \gamma v_i \bar{r}_i. \hfill (10)$$

On the other hand, we obtain

$$\frac{\partial U}{\partial r_i} = u_i (r_i - s_i), \hfill (11)$$

where the coefficients $u_i$ were defined in (4), and therefore

$$r'_i = (1 - \gamma u_i) r_i + \gamma u_i s_i. \hfill (12)$$

The full dynamics associated with the cost function $W = (1 - \lambda)V + \lambda U$ acts iteratively according to

$$r'_i = (1 - \gamma ((1 - \lambda) v_i + \lambda u_i)) r_i + \gamma ((1 - \lambda) v_i \bar{r}_i + \gamma u_i s_i). \hfill (13)$$

In the iterative soft consensus dynamics each individual opinion $r_i$ is combined with the context opinion $\bar{r}_i$ and the original opinion $s_i$ to obtain the new opinion $r'_i$. The coefficients of the convex combination are based on $v_i$ and $u_i$ and have to be recomputed at each iteration. The iterative dynamics of the so-called social influence networks [8,10,9] is based on a similar convex combination paradigm but in that case the convex combination coefficients are constant.
2 Measures

Complexity and modularity are measures of network structure that express in two different ways how far a network is from some appropriate, transitive partitioning.

Given a graph, a partitioning constitutes an equivalence relation over the set of vertices, and can be thought of as another undirected, unweighted graph of the same order, expressed in terms of the elements \( x_{ij} \in \{0, 1\} \) of the corresponding adjacency matrix. To ensure consistency, we enforce the following constraints for all triples of vertices, with \( i, j, k = 1, \ldots, n \),

\[
\begin{align*}
  x_{ii} &= 1 \quad \text{(reflexivity)} \quad (14) \\
  x_{ij} &= x_{ji} \quad \text{(symmetry)} \quad (15) \\
  x_{ij} + x_{jk} - 2x_{ik} &\leq 1 \quad \text{(transitivity)} \quad (16)
\end{align*}
\]

The last three inequalities ensure that, for every triple \( i, j, k = 1, \ldots, n \), when two edges exist the third edge exists as well.

2.1 Complexity

Complexity is a function of a particular partitioning of a network, informally defined as the minimal number of edge operations, either additions or removals, that are necessary to turn a graph into its transitive counterpart.

Note that this approach constitutes a relaxation of the original proposal of [1], which considered edge removals only.

Given a graph and a possible partitioning, the associated complexity is defined as

\[
C = \sum_{i,j} (1 - a_{ij}) x_{ij} + a_{ij} (1 - x_{ij}). \quad (17)
\]

The problem of finding the partitioning that minimizes \( C \) while satisfying all of (14) can be solved exactly for small \( n \) by means of Integer Linear Programming.

2.2 Modularity

Modularity is a benefit function associated with a particular partitioning of a network, roughly defined as the number of edges within groups minus the expected number of such edges [18].

Given a graph and a possible partitioning, the associated modularity is defined more precisely [16] as

\[
Q = \frac{1}{a} \sum_{i,j} \left( a_{ij} - \frac{a_i a_j}{a} \right) x_{ij}, \quad (18)
\]

where \( a_i = \sum_j a_{ij} \) and \( a = \sum a_i \).

Once again, the problem of finding the partitioning that maximizes \( Q \) under the conditions (14) can be solved exactly for small \( n \) using Integer Linear Programming [2].
3 Simulation results

We have examined the evolution of the soft consensus model in five distinct cases, each corresponding to a different initial configuration of the preferences of eight agents. These configurations have been build by assigning initial values close to four “centers” at 0.2, 0.4, 0.6 and 0.8; the number of nodes initially placed at each position is reflected in the naming.

The parameter $\alpha$ was fixed to be 0.04, yielding a critical interaction distance of 0.2, while $\beta$ was allowed to take either the value $\beta_1 \approx 114.88$, determined such that $f'(0) = 0.99$, or the arbitrary value $\beta_2 = 40$. Finally, the parameter $\lambda$ was assigned the value $\frac{1}{3}$, which ensures a balanced effect of the two dynamical components $V$ and $U$.

For each case analyzed, we present the graphs of the model dynamics, showing the evolution in the individual preferences for the two values of $\beta$ considered, as well as the graphs of the associated measures.

To allow for a better comparison, the two measures have been linearly scaled to the interval [0, 1] by considering:

- for the modularity, the minimal and maximal values attainable in ungraded graphs, namely $-\frac{1}{2}$ and 1, as proven in [2];
- for the complexity, a minimal value of 0 and, as maximal value, the highest complexity obtained by analyzing all bipartite graphs of order 8 (the rationale behind this choice being the fact that bipartite graphs achieve a minimal modularity value).

3.1 General remarks

Firstly, we note that the complexity value peaks at least once as the model evolves over time, typically whenever transitions in the network configuration take place. However, asymptotically it takes a value which is systematically lower than the initial one, indicating that, following one or more negotiating moments during which interactions among agents make it difficult to sharply define groups, the model reaches stability having achieved a complexity reduction, as sought.

Moreover, these peaks in complexity often correspond to inflection points for the modularity measure, i.e. points in which trend inversions occur.

3.2 Case-by-case analysis

2+2+2+2 (Figure 1) Since the nodes have been strategically placed at distances slightly lower than the critical one, the model yields, for both values of $\beta$, a final configuration where all agents converge around the central value 0.5. From the point of view of the complexity, this is clearly identified as a great simplification in the network layout, even more so when using $\beta_1$, which leads to an asymptotic value very close to zero. As for the modularity, it can be easily proven [7] that the network itself, considered as a partition with a single group, has modularity zero, and it is in fact approximately to this value (i.e., the corresponding normalized value $\frac{1}{3}$) that this measure tends asymptotically.
For the $3+1+2+2$ case, we note again that the model evolves similarly regardless of the choice of $\beta$, with $\beta_2$ corresponding to a much faster interaction and a slightly more blurred picture of the measures’ dynamics. It is evident, however, that in both cases the joining of the single node and the bigger group, as well as the joining of the two identically sized groups, are positively rewarded both by the complexity, which reaches a value very close to 0, and the modularity, which is in fact known to unfavorably weigh groups consisting of a single node [2]. Similar considerations apply to the configuration $4+1+1+2$.

On the contrary, the $4+2+1+1$ case allows us to highlight a number of interesting differences. In the first place, we note that, depending on the value of $\beta$ considered, the model evolves in significantly different ways. In the case of $\beta_2$, all nodes but one tightly converge around the same value, leaving a group with a single node that is negatively considered by the modularity, which decreases. On the other hand, the complexity is positively affected by the creation of a large, mostly transitive group which is weakly linked to the single node, and thus decreases over the course of the simulation. A similar consideration for the complexity applies when considering $\beta_1$, whereas the modularity ultimately rises back to a value only close to the initial one, which can be explained by the relatively high distance between the merged single nodes. Similar remarks can be made about the $5+1+1+1$ case.

References


Fig. 1: The 2+2+2+2 case
Fig. 2: The 3+1+2+2 case

(a) Model dynamics using $\beta_1$  
(b) Model dynamics using $\beta_2$  
(c) Measures using $\beta_1$  
(d) Measures using $\beta_2$

Fig. 3: The 4+1+1+2 case

(a) Model dynamics using $\beta_1$  
(b) Model dynamics using $\beta_2$  
(c) Measures using $\beta_1$  
(d) Measures using $\beta_2$
(a) Model dynamics using $\beta_1$

(b) Model dynamics using $\beta_2$

(c) Measures using $\beta_1$

(d) Measures using $\beta_2$

Fig. 4: The 4+2+1+1 case

(a) Model dynamics using $\beta_1$

(b) Model dynamics using $\beta_2$

(c) Measures using $\beta_1$

(d) Measures using $\beta_2$

Fig. 5: The 5+1+1+1 case