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RESEARCH ARTICLE

On quintessential cosmological models and exponential potentials

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Abstract We first study dark energy models with a minimally-coupled scalar field and generalized exponential potentials, admitting exact solutions for the cosmological equations: actually, it turns out that for this class of potentials the Einstein field equations exhibit alternative Lagrangians, and are completely integrable and separable. We analyze their analytical solutions, especially discussing when they are compatible with a late time quintessential expansion of the universe. As a further issue, we discuss how quintessential scalar fields with exponential potentials can be connected to the inflationary phase, building up a quintessential inflationary scenario: actually, it turns out that the transition from inflation toward late-time exponential quintessential tail admits a kination period, which is an indispensable ingredient of this kind of theoretical models. All such considerations have been made by including also radiation into the model.

Keywords Theoretical cosmology · Dark energy · Observational cosmology

1 Introduction

Scalar fields have been used extensively in cosmology, both in minimal and non minimal coupling with geometry, since inflationary theories were first conceived in 1981 [1]. (An overview of scalar-tensor theories is given in Refs. [2,3].)

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When observational results on Type Ia supernovae (SNeIa) have strongly supported the possibility that the universe is now in an accelerated stage of expansion [4–7], it has become necessary to consider again a negative-pressure component. The late-time inflation so generated is however *soft* with respect to the earlier one and can also be based on the dominance of at least one scalar field, sometimes referred to as *quintessence* [8–12] or, earlier, as *x-field* [13,14]. There is now a wide agreement that a form of a so-called *dark energy* [15,16] has to be taken into account in any realistic cosmological model. Seen as a constant Λ-term [17–19], dark energy density has been usually taken as the vacuum energy density, which leads to discrepancies between theory and experiments [18–20].

On its side, the scalar field has gained new attention, and speculations about its nature and evolution have got new strength. Papers like Refs. [21,22] have been reconsidered again under a new light, since they show how a scalar field could give an accelerated cosmological expansion today. Many models have thereof been constructed, first of all trying to consider more appealing kinds of potentials driving the dynamics of the scalar field. Among others, the potential has in fact to be seen as an important ingredient of all the involved theoretical models, and many are the trials to give prescriptions in order to reconstruct its form according to observational data (see Refs. [23,24], for instance, but also Refs. [9–11,25–27]). Here we want to take into account a specific class of generalized exponential potentials useful for the present evolution of the universe, later investigating all the relative involved cases so derived in the cosmological equations.

As commented in Ref. [28], for example, exponential potentials can be found in theories with extra compact dimensions, such as Kaluza–Klein supergravity and superstring models [29,30]. Despite having been studied extensively in the literature, exponential quintessence has anyway been often discarded because of limits on big-bang nucleosynthesis or based on fine tuning arguments, for instance when there is a fixed point solution for the quintessence evolution in the universe. Nonetheless, it is possible that we have not yet reached this fixed point. As a matter of fact, for example, in Ref. [31] it is shown that there are regions of the parameter space of the simple exponential potential model for quintessence which are allowed by some observational constraints, and that the degree of fine tuning needed in these scenarios is not high [31–34]. As shown in Refs. [33–35], they have in fact proved useful in describing several features in the history of the universe, also reproducing the present acceleration and predicting eventual future deceleration, so avoiding the horizon problem that appears in the context of both perturbative quantum field and string theories. For example, the model for quintessence studied in Ref. [33] yields an eternally accelerating universe with an event horizon that seems to be incompatible with superstring theory [36], while, as shown in Refs. [28,37], one possible way to make this model compatible with current observations and with the absence of event horizons is to add a negative constant term to the scalar field potential, equivalent to having a negative cosmological constant [36]. As a matter of fact, we can consider an *effective* dark energy fluid as the source of the accelerated expansion, hence splitting its energy density into the sum of two components that might also work one against the other.

Interesting results can indeed be obtained by modelling dark energy by means of both a scalar field and a cosmological constant, which can indeed be incorporated into the quintessence potential as a constant shifting the minimum of the potential. On the other hand, the height of such a minimum can also be regarded as a part of the cosmological constant. (For the purpose of separating them, the possible non-vanishing height of that minimum can be included into the cosmological constant and then set to zero.) Either provided by various kinds of quantum and/or classical matter or originated in the intrinsic spacetime geometry, still there is no sufficient reason to set the cosmological constant to zero [36]. For example, mechanisms to generate a negative cosmological constant have been conceived in the context of spontaneous symmetry breaking [38,39], although astrophysical data suggest a positive cosmological constant. Also note that it is possible to obtain from the field equations that the quintessence potential should be a double exponential plus a constant term [40]. Actually, such a potential was already analytically found by some of us in 1990, using the Noether Symmetry Approach in cosmology for the first time [41]. In that paper, however, the constant term (resulting from the procedure itself and not guessed in advance) was intimately related to the coefficients of the two exponentials, and putting it to zero would annihilate the whole potential. Indeed, the fact that the cosmological constant could not be postulated a priori but results from the form of the potential itself is known [42]. On the other hand, in Ref. [43], it is also analyzed how quintessence is directly related not only with geometry but also with the cosmological constant, so that the latter turns out to be asymptotically deducible through the dynamics of the scalar-tensor theory itself [43–45].

On their side, the exponential-type potentials have received much attention since the late 1980's [21,22,46,47] (but see also Refs. [25,26,48–57], just to cite only some other papers). Besides the previous work made in Ref. [41], in Refs. [33,35] some of us have found general exact solutions for two classes of exponential potentials for a scalar field minimally coupled to gravity, in presence of another dust component (ordinary non-relativistic pressureless baryonic matter), also showing [58] that such solutions seem to fit SNeIa data given in Ref. [5]. (See Ref. [59] for much more on confrontation of theoretical predictions with observational data.) Furthermore, in Ref. [60] other approximate and exact solutions for cosmology with exponential potentials can be found, and until recent times this kind of potentials is still widely considered (as, for example, in Ref. [61], where the Noether Symmetry Approach [41,49] is used to probe the nature of dark energy, well underlining the role of the exponential potential in such a task). This seems to us a good motivation to investigate also some other generalizations of this kind of potentials. (Note that, among others, the hyperbolic sine potential can be easily assimilated to some of the potentials studied in the following.)

Moreover, for the class of generalized exponential potentials we want to deal with it is possible to exhibit some *special* properties: it turns out, actually, that the Einstein field equations are integrable (in the Liouville sense) and separable, that is they can be analytically integrated, at least by quadrature. Last, it is worth noting that our class of exponential potentials can be selected by finding the most general variables transformation which diagonalizes the scalar field kinetic-energy form, leaving the transformed Lagrangian *simple*.

We do not consider the presence of a constant Λ-term here, even if we find it arising from the theory itself, in which we generalize the exponential potential to the extent allowed by the particular technique used in Refs. [33,35] in order to integrate cosmological equations, still bearing in mind that a priori acceptable solutions, able of describing the present universe, should always allow a late-time acceleration. Such a technique is generally known as the Noether Symmetry Approach to cosmology [41,49] and, as said above, it leads to prefer the exponential-type potentials. As in Refs. [33,35] we anyway do not completely adopt it here, limiting ourselves to import from it only the change of variables needed to solve equations and suggested by the procedure itself. As a matter of fact, we have to point out that this transformation (together with the choice of a specific kind of potential) has to be considered as the main drawback of the approach above, leading to a point symmetry which certainly applies to the background evolution, but nothing still assures about the actual symmetries of the full theory. We analyze the exact solutions that we find in this way, discussing when they are compatible with a late time quintessential expansion of the universe.

As a further step, we also consider and illustrate a possible scenario in the framework of the *quintessential inflation paradigm* with scalar fields, where an inflationary potential drives also the quintessential phase of the scalar field evolution, by means of a simple exponential form of its late time tail. In this connection, we discuss how such an evolution mechanism for the scalar field potential can be compatible with the so-called inflation-kination transition, when the field energy density is dominated by the kinetic energy of the scalar field φ. All this clearly leaves apart the greater complexity involved by the more general forms of the potential studied above, and will be the subject of a forthcoming paper trying to discuss more in general the quintessential inflation. Even if in this second paper we should be able to overcome the dichotomy present in this work, where the two different parts of the paper could well live apart, we believe that it is nonetheless already interesting to offer both the issues in a single paper, trying to unify the whole subject a little forcibly.

In Sect. 2 we introduce the class of potentials and the general cosmological setting for further considerations. In Sect. 3, we systematically derive, when possible, general exact solutions case by case. In Sect. 4 we discuss connections between exponential potentials, quintessence and inflation. Finally, Sect. 5 is devoted to a conclusive discussion.

2 Cosmological models with (generalized) exponential potentials

Let us assume a Friedmann–Lemaître–Robertson–Walker (FLRW) metric and fix the curvature scalar $k = 0$, according to the CMBR observational data (see Refs. [62,63] for recent information). In what follows we have to consider the period of life of the universe after the decoupling time, in order to be realistic when taking the two components like dust and scalar field φ into account.

With φ minimally coupled to gravity, the cosmological equations are written as

$$3H^2 = \frac{8\pi G}{c^2}(\rho_m + \rho_\varphi), \tag{2.1}$$

$$\dot{H} + H^2 = -\frac{4\pi G}{c^2}(\rho_m + \rho_\varphi + 3(p_m + p_\varphi)), \tag{2.2}$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0. \tag{2.3}$$

The fluids filling the universe have equations of state given by

$$p_m = 0, \quad p_\varphi = w_\varphi \rho_\varphi, \tag{2.4}$$

being $\rho_m = Da^{-3}$, where the parameter $D \equiv \rho_{m0} {a_0}^3$ is determined by the current values of ρ_m and a, and

$$\rho_\varphi \equiv \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \quad p_\varphi \equiv \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \tag{2.5}$$

with

$$w_\varphi \equiv \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)}. \tag{2.6}$$

The equations above can then be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{c^2}\left(Da^{-3} + \frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right), \tag{2.7}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\frac{8\pi G}{c^2}\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right), \tag{2.8}$$

$$\ddot{\varphi} + 3\left(\frac{\dot{a}}{a}\right)\dot{\varphi} + V'(\varphi) = 0. \tag{2.9}$$

It can be shown [41,49] that the last two equations are also deduced from an action principle based on the *point* Lagrangian

$$\mathcal{L} = 3a\dot{a}^2 - \frac{8\pi G}{c^2}\left[a^3\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right) - D\right], \tag{2.10}$$

so that cosmological dynamics can be considered on a space with two *coordinates* a and φ, $\dot{a}$ and $\dot{\varphi}$ being the *velocities*. The fact that $\mathcal{L}$ has a constant additive term is understood by considering Eq. (2.7), which can be seen as $E_{\mathcal{L}} = 0$, where $E_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{a}} + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \mathcal{L}$ is the so called *energy function* associated with $\mathcal{L}$ and is zero for physical reasons (i.e., induced by the homogeneous and isotropic limit of Einstein's general field equations). In this way, once Eqs. (2.8) and (2.9) are solved, Eq. (2.7) is nothing but a constraint on the integration constants involved.

On defining

$$T \equiv 3a\dot{a}^2 - \frac{4\pi G}{c^2}a^3\dot{\varphi}^2, \quad U \equiv -\frac{8\pi G}{c^2}(a^3 V(\varphi) + D), \tag{2.11}$$

Equation (2.10) formally becomes $\mathcal{L} = T - U$. This has been already noted in Ref. [42], where we found the most general transformation

$$a = f(z, w) \quad \varphi = g(z, w) \tag{2.12}$$

leading to a *diagonalized* and simpler form of the *kinetic energy*

$$T' = \alpha^2 \dot{z}^2 - \beta^2 \dot{w}^2, \tag{2.13}$$

with α and β nonnegative real numbers. This simplification in T implies, of course, a complication in the transformed U'. However, in Ref. [41] we also found that there is at least one class of potentials not giving such a complicated expression for U', rendering, on the contrary, the transformed $\mathcal{L}'$ simpler than $\mathcal{L}$. It turns out that such potentials are exponentials:

$$V(\varphi) = V_0 \left(A^2 \exp(2C\varphi) + B^2 \exp(-2C\varphi) - 2AB \right), \tag{2.14}$$

with $C \equiv (3\pi G)/c^2$ and A, B real parameters, while V_0 simply fixes the scale of the potential. The above expression of $V(\varphi)$ in fact allows an exact integration of the cosmological equations [41,49]. Moreover, it is worth noting that, for such a class of exponential potentials,

$$V(\varphi) = V_1 \exp(2C\varphi) + V_2 \exp(-2C\varphi) + \lambda_0, \tag{2.15}$$

being V_1, V_2, λ_0 all free parameters, the Einstein field equations admit alternative Lagrangians, which is a circumstance somehow exceptional for a dynamical system, with many meaningful consequences: actually, it turns out that it is integrable (in the Liouville sense) and it is separable, i.e. there is a suitable change of variables by which it is splitted into separated one-dimensional systems (which are integrable by quadratures [64]). Note also that the λ_0-term acts as a cosmological constant and, being a product of the theoretical procedure, has to be taken into account *together* with the exponential part of the potential for the scalar field φ. As already commented in the introduction, this is not forbidden by theory and observations.

We thus feel motivated to continue considering exponential-like potentials. Let us take in what follows the class of potentials

$$V(\varphi) = A^2 \exp(\sigma\varphi) + \epsilon B^2 \exp(-\sigma\varphi) + \lambda, \tag{2.16}$$

where $\epsilon = \pm 1$, $\sigma \equiv \sqrt{12\pi G/c^2}$ is a fixed constant, A^2 and B^2 are arbitrary nonzero parameters, and $\lambda \lesseqgtr 0$ is the constant playing the role of the cosmological constant. Equation (2.16) generalizes the exponential potentials already considered in Ref. [33], which are

$$V(\varphi) = B^2 \exp(-\sigma\varphi), \quad V(\varphi) = A^2 \exp(\sigma\varphi) + B^2 \exp(-\sigma\varphi), \tag{2.17}$$

as well as those derived in Ref. [41]. The two cases above are therefore omitted in this paper. As noted in the introduction, also the case with both $\epsilon = +1$ and $\lambda = -2AB$

has been already treated and discussed in Refs. [41,42,49] and will not be touched upon again here.

Since the theory is invariant under $\varphi \to -\varphi$, the class of potentials in Eq. (2.16) is the most general one of this type and, as mentioned, also includes the first one shown in Eq. (2.17). $V(\varphi) = B^2 \exp(-\sigma\varphi)$ is in fact derived from the expression in Eq. (2.16) just setting (besides $\epsilon = +1$ and $\lambda = 0$) also $A^2 = 0$, or $B^2 = 0$ and $\varphi \to -\varphi$. The second potential in Eq. (2.17) is instead easily obtained when $\lambda = 0$ and $\epsilon = +1$, with $A^2 \neq 0$, $B^2 \neq 0$. (Of course, in order to get exact general solutions, the treatment has to be different from the beginning for each one of the potentials in Eqs. (2.16) and (2.17).)

As in Ref. [33], then, let us use again the transformation

$$a^3 = \frac{u^2 - v^2}{4}, \quad \varphi = \frac{1}{\sigma} \log \left| \frac{B(u+v)}{A(u-v)} \right|, \tag{2.18}$$

leading to the new variables u and v. Such a change of variables is invertible, provided that $a \neq 0$. Note that, when the solution for the scale factor eventually has a zero in the future, we cannot follow it beyond that point. We must in fact consider such a solution as good just *between* this zero and the eventual other one it can have in the past. (Later on, we will comment on this again.) Since we also want $a > 0$, Eq. (2.18) imposes the obvious restriction $u^2 > v^2$, or $(u+v)(u-v) > 0$, giving $u > v$ when both u and v are nonnegative. It is obvious that A and B can be always chosen both positive. In what follows, therefore, we shall consider

$$A > 0, \quad B > 0, \quad u > v, \quad u > 0, \quad v > 0. \tag{2.19}$$

We can thus disregard the absolute value sign in the expression for φ. By virtue of Eq. (2.18), the potential in Eq. (2.16) becomes

$$V(u, v) = AB \frac{(u+v)^2 + \epsilon(u-v)^2}{u^2 - v^2} + \lambda, \tag{2.20}$$

and we define

$$V_+(u, v) \equiv 2AB \frac{u^2 + v^2}{u^2 - v^2} + \lambda \tag{2.21}$$

for $\epsilon = +1$, and

$$V_-(u, v) \equiv 4AB \frac{uv}{u^2 - v^2} + \lambda \tag{2.22}$$

for $\epsilon = -1$.

Now, transforming the other terms in Eq. (2.10) by means of Eq. (2.18), we can eventually arrive at two expressions for the point Lagrangian, according again to the two opposite values of ϵ,

$$\mathcal{L}_{+} \equiv \dot{u}^2 - \dot{v}^2 + \frac{\sigma^2}{2}\left[(2AB+\lambda)u^2 + (2AB-\lambda)v^2 + 4D\right], \tag{2.23}$$

$$\mathcal{L}_{-} \equiv \dot{u}^2 - \dot{v}^2 + \frac{\sigma^2}{2}\left[4ABuv + \lambda(u^2 - v^2) + 4D\right]. \tag{2.24}$$

In order to find equations and solutions for the cosmological model, at this point, it is better to consider the various different cases separately, and this is what we are going to do in the next section, pointing out that we are facing what we can call *impure* quintessence models, because of the presence of the λ-term introduced in the theory by the expression of the potentials.

In the following, we fix four conditions (see Ref. [35] for further details). First of all, we set the origin of time by choosing $a(0) = 0$. This condition has to be interpreted just as an arbitrary choice of the time origin. The real beginning (of physical meaning) for the model starts a little bit afterwards, at a certain time t_1, without forbidding the substantial invertibility of the change of variables performed in Eq. (2.18). This delay is otherwise arbitrary, so that the setting we chose does not seem to exclude important cases, as said before, and leads to a great simplification in the formulae. Now, $a(0) = 0$ implies that we have thus to set $u(0) = 0$ *or* $v(0) = 0$ in evaluating initially Eq. (2.18).

The second condition here assumed is that the present time is the unit of time, $t_0 = 1$. Since t_1 is unknown, this is not exactly the age of the universe, but the difference can be considered irrelevant for our purposes; anyway, even if it were possible to avoid this condition, we thus get rid of a badly known quantity. Our third condition is to set $a_0 \equiv a(t_0) = a(1) = 1$, which fixes the normalization of the scale factor a as standard, while the fourth and last condition is to set $H(t_0 = 1) \equiv \mathcal{H}_0$. After the choice of t_0, this latter parameter turns out to be of order 1, even if it is not the same as the usual h. As a matter of fact, since we are using an arbitrary unit of time, such a parameter does not give any information on the observed value for H_0 [35].

As far as $\lambda = 0$ is concerned, as already said, some considerations are made in Ref. [33]. When $\lambda = A^2 = 0$ and $\epsilon = +1$, for instance, we should use the new change of variables

$$a^3 = uv, \tag{2.25}$$

$$\varphi = -\frac{1}{\sigma}\log\frac{u}{v}, \tag{2.26}$$

which easily leads to general exact solutions for $a(t)$ and $\varphi(t)$ [33]. For $\lambda = B^2 = 0$, everything remains the same, since there is symmetry in the potential in Eq. (2.16) with respect to a change of sign in φ. The situation with both $A^2 \neq 0$, $B^2 \neq 0$, and $\epsilon = +1$ has also been partially examined in Ref. [33].

In what follows, even if we will also consider situations with $\lambda \neq 0$, we will focus our attention first on the $\lambda = 0$-case with $A^2 \neq 0$, $B^2 \neq 0$ and $\epsilon = -1$ (when Eq. (2.18) is in order), choosing not to consider at all the situation with $\lambda = A^2 = 0$ and $\epsilon = -1$, since it involves an always negative potential and may also give a negative energy density, by virtue of Eq. (2.5). Most of all, let us note that it never allows the pressure to be negative, which forbids the possibility of describing the accelerated expansion we observe today.

We will also rule out the trivial case given by $V \equiv \lambda$ (i.e. what is known as the ΛCDM model), while the remaining possible situations will be dealt with systematically.

3 Solutions

Once we have fixed our Lagrangian expressions in Eqs. (2.23) and (2.24), we can soon derive the related equations for cosmology as Euler-Lagrange equations. If our transformation in Eq. (2.18) then works, such equations should turn out to be solvable, even if, of course, exact integration is not always easy. The analysis strongly depends on the relative values assumed by the constants involved. Thus, in what follows we separately discuss each situation generated, as a first step, by a different choice of λ values. Then, we pass to further investigate the features of the equations and their solutions, starting from consideration of the two allowed values of ϵ.

3.1 The $\lambda = 0$ case

As mentioned, this situation is what we can call a *pure* quintessence case and, with respect to the peculiar potential used, has already been partially treated. In Ref. [33], three cases have in fact been studied: (i) $A^2 = 0$ and $\epsilon = +1$, (ii) $B^2 = 0$, and (iii) $A^2 \neq 0$, $B^2 \neq 0$, and $\epsilon = +1$. The case with $A^2 = 0$ and $\epsilon = -1$ has already been touched upon at the end of Sect. 2, noting that it is not so interesting here in our considerations on current cosmology. On the other hand, when both A^2 and B^2 vanish, the potential vanishes and φ is free, giving $p_\varphi \equiv \rho_\varphi \equiv \dot{\varphi}^2/2 > 0$, and $w_\varphi = 1$. This means that the scalar field behaves like stiff matter and introduces a term $\propto a^{-6}$ in the cosmological equations, thus becoming soon negligible in the expanding evolution of the universe. In this way, φ would never produce a late-time inflation, for instance, as instead recent observations seem to require.

There only remains one case to discuss, i.e.,

$$A^2 \neq 0, \quad B^2 \neq 0, \quad \epsilon = -1, \tag{3.1}$$

which is in agreement with what we set before in Eq. (2.19), being $AB > 0$. With such assumptions, setting $\lambda = 0$ in Eq. (2.24) yields

$$\mathcal{L}_- \equiv \dot{u}^2 - \dot{v}^2 + \frac{24\pi G}{c^2}(ABuv + D), \tag{3.2}$$

so that the related Euler-Lagrange equations are

$$\ddot{u} = \omega^2 v, \quad \ddot{v} = -\omega^2 u, \tag{3.3}$$

where

$$\omega^2 \equiv \frac{12\pi GAB}{c^2} = \sigma^2 AB. \tag{3.4}$$

It can be shown that the solution reads as

$$u(t) = \left[\frac{c_3 + c_4}{\sqrt{2}\omega}\cosh\left(\frac{\omega t}{\sqrt{2}}\right) + c_2 \sinh\left(\frac{\omega t}{\sqrt{2}}\right)\right]\sin\left(\frac{\omega t}{\sqrt{2}}\right) - \left[\frac{c_3 - c_4}{\sqrt{2}\omega}\sinh\left(\frac{\omega t}{\sqrt{2}}\right) + c_1 \cosh\left(\frac{\omega t}{\sqrt{2}}\right)\right]\cos\left(\frac{\omega t}{\sqrt{2}}\right), \tag{3.5}$$

$$v(t) = \left[-\frac{c_3 - c_4}{\sqrt{2}\omega}\cosh\left(\frac{\omega t}{\sqrt{2}}\right) - c_1 \sinh\left(\frac{\omega t}{\sqrt{2}}\right)\right]\sin\left(\frac{\omega t}{\sqrt{2}}\right) + \left[\frac{c_3 + c_4}{\sqrt{2}\omega}\sinh\left(\frac{\omega t}{\sqrt{2}}\right) + c_2 \cosh\left(\frac{\omega t}{\sqrt{2}}\right)\right]\cos\left(\frac{\omega t}{\sqrt{2}}\right), \tag{3.6}$$

depending on four arbitrary parameters (u_1, u_2, t_1, and t_2). By virtue of the energy constraint $E_{\mathcal{L}_-} = 0$, which is written

$$\dot{u}^2 - \dot{v}^2 - 2\omega^2 uv - \frac{24\pi G}{c^2} D = 0, \tag{3.7}$$

Equation (3.4) yields

$$D = AB\left(\frac{{c_3}^2 - {c_4}^2}{2\omega^2} - c_1 c_2\right). \tag{3.8}$$

If we define $k_3 \equiv (c_3 + c_4)/(\sqrt{2}\omega)$ and $k_4 \equiv (c_3 - c_4)/(\sqrt{2}\omega)$ we then get $D = AB(k_3 k_4 - c_1 c_2)$. The initial scale factor at $t = 0$ is such that the volume $a^3(t)$ is at the *beginning* $a^3(0) = ({c_1}^2 - {c_2}^2)/4$, in light of Eq. (2.18). For sake of simplicity, we have fixed the origin of time in such a way as to get a vanishing scale factor. This, let us recall it again, is completely arbitrary, and we must always remember that the real beginning of time for the model under examination is actually afterwards, i.e. at an after-decoupling moment which can be arbitrarily delayed for the kind of considerations we are making here. This also means, then, that we are setting $c_1 = c_2$ or, possibly, $c_1 = c_2 = 0$ (since we have put $a(0) = 0$). Anyway, as required, both choices still allow D to be non vanishing and positive. (Of course, this would be gained even by taking $c_2 = 0$ simply.) We here put $c_1 = c_2 = 0$, so that we can write

$$D = ABk_3 k_4, \tag{3.9}$$

from which we find that the two constants k_3 and k_4 have the same sign, i.e., $k_3 k_4 > 0$, implying that ${c_3}^2 > {c_4}^2$.

Let us also note that, from Eqs. (3.5) and (3.6), we get $u(0) = -c_1$ and $v(0) = c_2$. Thus, independently of all considerations above, Eq. $(2.18)_2$ implies

$$\varphi(0) = \frac{1}{\sigma} \log \frac{B(u(0) + v(0))}{A(u(0) - v(0))} = \frac{1}{\sigma} \log \frac{B(-c_1 + c_2)}{A(-c_1 - c_2)}, \tag{3.10}$$

and the choice $c_1 = c_2$ or, equivalently, $a(0) = 0$, soon gives $\varphi(0) = 0$. (This is also true for $c_1 = c_2 = 0$, but one has then to be more cautious.)

By substituting Eqs. (3.5) and (3.6) into the expressions in Eq. (2.18) for $a^3(t)$ and $\varphi(t)$, after some algebra we find the cosmological solutions

$$a(t) = \frac{\sqrt[3]{-\frac{\beta^2\left[\cos(\sqrt{2}\omega_1 t)\cosh(\sqrt{2}\omega_1 t)-1\right]}{\omega_1{}^2}}}{2^{2/3}}, \tag{3.11}$$

$$\varphi(t) = -\log\left(\frac{B\left(\frac{\beta e^{-\frac{t\omega_1}{\sqrt{2}}}\left(e^{\sqrt{2}t\omega_1}+1\right)\sin\left(\frac{t\omega_1}{\sqrt{2}}\right)}{\sqrt{2}\omega_1} + \frac{\beta e^{-\frac{t\omega_1}{\sqrt{2}}}\left(e^{\sqrt{2}t\omega_1}-1\right)\cos\left(\frac{t\omega_1}{\sqrt{2}}\right)}{\sqrt{2}\omega_1}\right)}{A\left(\frac{\beta e^{-\frac{t\omega_1}{\sqrt{2}}}\left(e^{\sqrt{2}t\omega_1}+1\right)\sin\left(\frac{t\omega_1}{\sqrt{2}}\right)}{\sqrt{2}\omega_1} - \frac{\beta e^{-\frac{t\omega_1}{\sqrt{2}}}\left(e^{\sqrt{2}t\omega_1}-1\right)\cos\left(\frac{t\omega_1}{\sqrt{2}}\right)}{\sqrt{2}\omega_1}\right)}\right)\frac{1}{\sigma}. \tag{3.12}$$

The scale factor in Eq. (3.11) appears to describe a sort of cyclic universe with spatially flat hyper-surfaces. However the solution describing expansion should be considered in one cycle only, since we *squeezed* the whole evolution of the universe in the range [0, 1]. As we see in Figs. 1 and 2, such a solution makes it possible for us to describe the current accelerated expansion of the universe.

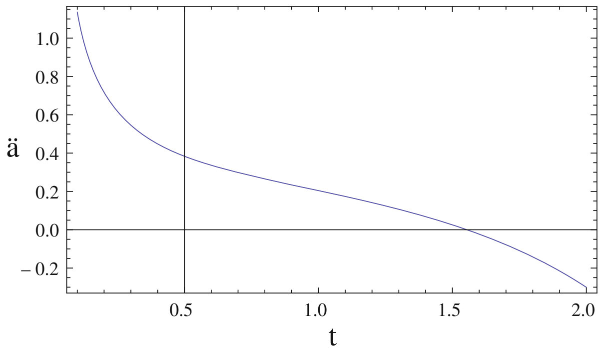

Fig. 1 The time evolution of the second derivative of the scale factor represented in Eq. (3.11) for $\omega = 1$, and $\beta = 1$: it turns out that such a model is compatible with a late time accelerated expansion

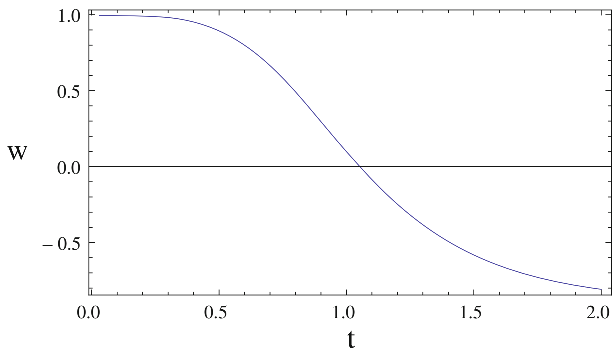

Fig. 2 The dark energy equation of state w parameter as a function of t, for the potential in Eq. (3.12) for $A = 10$, $B = 1$, $\omega = 1$ and $\sigma = 0.1$: it turns out that such a scalar field transits from $w = 1$ in the far past to $w = -1$ in the future

3.2 The $\lambda > 0$ case

Here, first of all, let us remind the reader that we are considering the *impure* quintessence situation. We still have to distinguish between the two cases with $\epsilon = +1$ or $\epsilon = -1$, i.e., between Lagrangian functions in Eq. (2.23) or (2.24), respectively. Then, as we can see, our considerations must also take the relative values of AB and λ into account. In this respect, as already noted, the situation given by $A^2 = B^2 = 0$ describes a universe filled in with dust and free scalar field in presence of a positive cosmological constant, since the potential $V(\varphi)$ reduces to λ only. Since we have discarded considering the contribution of only one exponential in the potential expression (and by virtue of Eq. (2.19)), this means that, as already required, we need to limit our considerations here to $AB > 0$.

3.2.1 The $\epsilon = +1$ value

First of all, let us note again that the case with $\epsilon = +1$ and $\lambda = -2AB$ has been already studied elsewhere [41,42,49] and would, however, lead to $\lambda < 0$, the product AB being positive assumed. Thus, we do not comment on it here. Now, on the other hand, bearing in mind that it must also be $u^2 > v^2$, the choice $\epsilon = +1$ always gives a positive potential for the scalar field and leads to two different cases, since we can write (with $2AB \neq \lambda$)

$$\frac{\sigma^2}{2}(2AB + \lambda) \equiv {\omega_1}^2, \tag{3.13}$$

$$\frac{\sigma^2}{2}(2AB - \lambda) \equiv \pm{\omega_2}^2, \tag{3.14}$$

so that Eq. (2.23) becomes

$$\mathcal{L}_+ \equiv \dot{u}^2 - \dot{v}^2 + {\omega_1}^2 u^2 \pm {\omega_2}^2 v^2 + 2\sigma^2 D. \tag{3.15}$$

We thus deduce the equations

$$\ddot{u} = {\omega_1}^2 u, \quad \ddot{v} = \pm {\omega_2}^2 v, \tag{3.16}$$

where the plus (minus) sign corresponds to the minus (plus) sign in the Lagrangian $\mathcal{L}_+$. In both cases we find

$$u(t) = \alpha \exp(\omega_1 t) + \beta \exp(-\omega_1 t), \tag{3.17}$$

α and β being integration constants. Furthermore, we get

$$v(t) = v_1 \sin(\omega_2 t + v_2) \tag{3.18}$$

when the plus sign ($2AB > \lambda$) is taken in the Lagrangian, or

$$v(t) = v_1 \exp(\omega_2 t) + v_2 \exp(-\omega_2 t) \tag{3.19}$$

when the minus sign ($2AB < \lambda$) is chosen therein (v_1 and v_2 are integration constants).

The integration constants are constrained by means of the equation $E_{\mathcal{L}_+} = 0$, which is

$$\dot{u}^2 - \dot{v}^2 - {\omega_1}^2 u^2 \pm {\omega_2}^2 v^2 - 2\sigma^2 D = 0. \tag{3.20}$$

This gives a relationship that, evaluated at $t = 0$, makes it possible to determine the more physically meaningful constant D in terms of the other constants, i.e.,

$$D = -\frac{\left({v_1}^2 {\omega_2}^2 + 4\alpha\beta {\omega_1}^2\right)}{2\sigma^2} > 0 \tag{3.21}$$

when the minus sign is taken in Eq. (3.20), and

$$D = \frac{2(v_1 v_2 {\omega_2}^2 - \alpha\beta {\omega_1}^2)}{\sigma^2} > 0 \tag{3.22}$$

with the plus sign. First focusing on the situation described by Eq. (3.18), i.e. choosing $2AB > \lambda$, we see that the inequality (3.21) implies that $-4\alpha\beta{\omega_1}^2 > {v_1}^2{\omega_2}^2$, i.e., $\alpha\beta < -{v_1}^2(2AB - \lambda)/[4(2AB + \lambda)]$, giving $\alpha\beta < 0$. Thus, when $\beta = -\alpha$, one finds $u(t) = 2\alpha \sinh(\omega_1 t)$, and we may express the volume $a^3(t)$ as the difference between the squares of a hyperbolic sine and a sine, where now $4\alpha^2{\omega_1}^2 > {v_1}^2{\omega_2}^2$. On the other hand, choosing $v_2 = 0$ does not change the sign of the value of D and implies $a(0) = 0$, i.e., we can fix the origin of time as before. Of course, putting $v_1 = 0$ also leads to $a(0) = 0$, but then we always have $v(t) = 0$, so that $a^3(t) = 4\alpha^2 \sinh^2(\omega_1 t)$, being $u^2(0) = 0$. Anyway, let us stress that, in the case under examination, $a(0)$ has to vanish in order to avoid an initial negative value for the scale factor.

At the same initial time $t = 0$, the scalar field generally has the constant value

$$\varphi(0) = \frac{1}{\sigma} \log \frac{B(\alpha + \beta + v_1 \sin v_2)}{A(\alpha + \beta - v_1 \sin v_2)}, \tag{3.23}$$

which gives rise to an undetermined form for $\beta = -\alpha$ and $v_2 = 0$, unlike what was found in the case studied above, when $\lambda = 0$.

The cosmological solutions can be then written as

$$a(t) = \left[\frac{4\alpha^2 \sinh^2(\omega_1 t) - {v_1}^2 \sin^2(\omega_2 t)}{4}\right]^{1/3}, \tag{3.24}$$

$$\varphi(t) = \frac{1}{\sigma} \log \frac{B[2\alpha \sinh(\omega_1 t) + v_1 \sin(\omega_2 t)]}{A[2\alpha \sinh(\omega_1 t) - v_1 \sin(\omega_2 t)]}, \tag{3.25}$$

so that the asymptotic behaviour of the scale factor is exponential (no hair behaviour). As before, it thus expresses a late-time *nonsoft* accelerated expansion of the universe, but a better agreement with observations obviously demands a more refined comparison with astrophysical data, and is postponed to future work.

On the other hand, discussing the other allowed situation, with $2AB < \lambda$, the condition (3.22) implies that $v_1 v_2 {\omega_2}^2 > \alpha\beta {\omega_1}^2$, i.e., $\alpha\beta < v_1 v_2 (\lambda - 2AB)/(2AB + \lambda)$, and now we can only say that $v_1 v_2 < 0$ gives $\alpha\beta < 0$, while $v_1 v_2 > 0$ leaves the possibility of having also $\alpha\beta > 0$. But let us note that $a(t) > 0$ involves $u - v > 0$ as needed, and therefore $u \neq 0$, which means that $\alpha\beta \neq 0$. If we then put $\beta = -\alpha$ (i.e., $\alpha\beta < 0$), we get $u(t) = 2\alpha \sinh \omega_1 t$, as before; this, in turn, requires that $\alpha^2 > -v_1 v_2 (\lambda - 2AB)/(2AB + \lambda)$, giving no restriction on the sign of $v_1 v_2$.

Thus, we can also choose $v_2 = -v_1$ (implying $\alpha^2 > {v_1}^2 (\lambda - 2AB)/(2AB + \lambda)$) and get $v(t) = 2v_1 \sinh(\omega_2 t)$, so that we may express the volume $a^3(t)$ as the difference between the squares of two hyperbolic sines. As before, this choice implies $a(0) = 0$ for every value of α and v_1. Moreover, at $t = 0$, the scalar field has in general the constant value

$$\varphi(0) = \frac{1}{\sigma} \log \frac{B(\alpha + \beta + v_1 + v_2)}{A(\alpha + \beta - v_1 - v_2)}, \tag{3.26}$$

which is again undetermined when we choose $\beta = -\alpha$ and $v_2 = -v_1$.

Now, the cosmological solutions are written as

$$a(t) = \left[\alpha^2 \sinh^2(\omega_1 t) - {v_1}^2 \sinh^2(\omega_2 t)\right]^{1/3}, \tag{3.27}$$

$$\varphi(t) = \frac{1}{\sigma} \log \frac{B[\alpha \sinh(\omega_1 t) + v_1 \sinh(\omega_2 t)]}{A[\alpha \sinh(\omega_1 t) - v_1 \sinh(\omega_2 t)]}. \tag{3.28}$$

The expression of $a(t)$ is non-negative only when the constants α, ω_1, v_1, and ω_2 are such that the difference occurring in $a(t)$ is positive. In such a case, when $\omega_2 < \omega_1$ we have an asymptotic exponential regime for the scale factor, as the one we already found before, with the other choice for the signs of the constants. The case with $\omega_2 > \omega_1$, on the other hand, has to be ruled out because it asymptotically leads to a non-physical situation, with a negative infinite value of the scale factor.

3.2.2 The $\epsilon = -1$ value

Let us consider, now, the Lagrangian in Eq. (2.24), resulting from the potential V_- expressed in Eq. (2.22). Since V_- has to be positive, as we have always chosen for the potential in this paper, one finds

$$4ABuv > -\lambda(u^2 - v^2), \tag{3.29}$$

having taken $\lambda > 0$, $u^2 > v^2$, $u > v$, and $AB > 0$.

Putting

$${\omega_1}^2 \equiv \sigma^2 AB, \quad {\omega_2}^2 \equiv \frac{1}{2}\lambda\sigma^2 \tag{3.30}$$

into Eq. (2.24) gives

$$\mathcal{L}_- \equiv \dot{u}^2 - \dot{v}^2 + 2{\omega_1}^2 uv + {\omega_2}^2(u^2 - v^2) + 2\sigma^2 D, \tag{3.31}$$

so that the resulting Euler-Lagrange equations are

$$\ddot{u} = {\omega_1}^2 v + {\omega_2}^2 u, \quad \ddot{v} = -{\omega_1}^2 u + {\omega_2}^2 v. \tag{3.32}$$

These equations are a linear system of homogeneous coupled ordinary differential equations of second order, which can be rewritten as

$$\begin{pmatrix} \ddot{u}(t) \\ \ddot{v}(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \tag{3.33}$$

where $a_{11} = a_{22} = \omega_2^2$, and $a_{12} = \omega_1^2 = -a_{21}$. The method of solving such a system reduces to diagonalizing the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{3.34}$$

Actually, if we introduce the rotation matrix $\mathbb{R}$ (whose columns are the eigenvectors $\vec{\psi}_1$, $\vec{\psi}_2$ of A), the transformation

$$\vec{y} = \mathbb{R}\vec{x} \tag{3.35}$$

(where we have set $\vec{y} = (y_1(t), y_2(t))$ and $\vec{x} = (u(t), v(t))$) decouples our starting system. One has

$$\ddot{\vec{y}} = \mathbb{R}A\mathbb{R}^{-1}\vec{y}, \tag{3.36}$$

where $\mathbb{R}A\mathbb{R}^{-1}$ is the eigenvalues' diagonal matrix. Such eigenvalues are

$$E_1 = \frac{1}{2}\left(-\sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}+a_{11}+a_{22}\right), \tag{3.37}$$

$$E_2 = \frac{1}{2}\left(\sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}+a_{11}+a_{22}\right), \tag{3.38}$$

and the eigenvectors of A associated with E_1 and E_2 are written as

$$\vec{\psi}_1 = -\frac{\left(\sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}-a_{11}+a_{22}\right)}{2a_{21}}\hat{e_1}+\hat{e_2}, \tag{3.39}$$

$$\vec{\psi}_2 = \frac{\left(\sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}+a_{11}-a_{22}\right)}{2a_{21}}\hat{e_1}+\hat{e_2}. \tag{3.40}$$

In terms of the transformed vector $\vec{y}$, the system in Eq. (3.32) reduces to a linear system of homogeneous decoupled ordinary differential equations of second order, which can now be integrated exactly:

$$\begin{pmatrix}\ddot{y}_1(t)\\ \ddot{y}_2(t)\end{pmatrix} = \begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}. \tag{3.41}$$

The solution, in its general form, is

$$y_1 = c_1 e^{\sqrt{E_1}t} + c_2 e^{-\sqrt{E_1}t}, \tag{3.42}$$

$$y_2 = c_3 e^{\sqrt{E_2}t} + c_4 e^{-\sqrt{E_2}t}. \tag{3.43}$$

The transformation $\mathbb{R}^{-1}\vec{y}$ provides the analytical expressions of $u(t)$ and $v(t)$:

$$u(t) = -\frac{1}{2}i\left(c_1 e^{t\sqrt{\omega_1^2-i\omega_2^2}} + c_2 e^{-t\sqrt{\omega_1^2-i\omega_2^2}} + ie^{-t\sqrt{\omega_1^2+i\omega_2^2}}\left(c_4 + c_3 e^{2t\sqrt{\omega_1^2+i\omega_2^2}}\right)\right), \tag{3.44}$$

$$v(t) = \frac{1}{2}\left(ie^{-t\sqrt{\omega_1^2-i\omega_2^2}}\left(c_2 + c_1 e^{2t\sqrt{\omega_1^2-i\omega_2^2}}\right) + c_3 e^{t\sqrt{\omega_1^2+i\omega_2^2}} + c_4 e^{-t\sqrt{\omega_1^2+i\omega_2^2}}\right), \tag{3.45}$$

the c_i (with $i = 1\ldots4$) being integration constants. It turns out that it is possible to set the constants to find a real-valued function representing the scale factor $a(t)$, according to Eq. (2.18), so as to obtain the following form of the scale factor:

$$\begin{aligned} a(t)^3 = &-\frac{\zeta^2}{16\omega_2^2} + \frac{\zeta^2 \sin\left(2\sqrt[4]{2}t\omega_2 \sin\left(\frac{\pi}{8}\right)\right)\sinh\left(2\sqrt[4]{2}t\omega_2\cos\left(\frac{\pi}{8}\right)\right)}{16\omega_2^2} \\ &+\left(\zeta^2\cos\left(2\sqrt[4]{2}t\omega_2\sin\left(\frac{\pi}{8}\right)\right)\cosh\left(2\sqrt[4]{2}t\omega_2\cos\left(\frac{\pi}{8}\right)\right)\right)\frac{1}{16\omega_2^2}, \end{aligned} \tag{3.46}$$

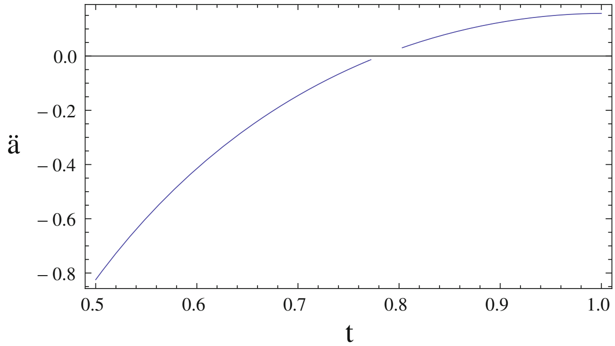

Fig. 3 The time evolution of the second derivative of the scale factor for $\omega_2 = 1$, $\zeta = 10$: it turns out that such a model is compatible with a late time accelerated expansion

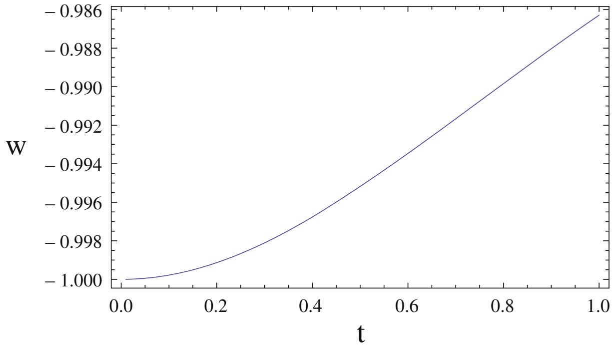

Fig. 4 The dark energy equation of state w parameter as a function of t, for $\omega_2 = 1$, $\zeta = 10$

ω_2 and ζ being positive constants. With such a choice of the integration constants we obtain a complex scalar field

$$\phi(t) = -\frac{\log\left(\frac{\lambda\left(\frac{(2-2i)\sinh\left(\sqrt{1+it}\right)}{\sqrt{2}\sinh\left(\sqrt{1-it}\right)-(1+i)\sinh\left(\sqrt{1+it}\right)}-i\right)}{2A^2}\right)}{\sigma}. \tag{3.47}$$

Through an appropriate choice of the integration constants it is possible to have a real-valued function. It turns out that such a cosmological solution of the Einstein field equations is able to describe a dark energy dominated evolution of the universe, as shown in Figs. 3 and 4.

3.3 The $\lambda < 0$ case

As above, we have again to discuss separately the two cases with $\epsilon = +1$ and $\epsilon = -1$, also considering that $AB > 0$ always. (Note that, should it be $A^2 = B^2 = 0$, we would be once again in a universe with a free scalar field plus dust, but, now, in presence of a negative cosmological constant.)

3.3.1 The $\epsilon = +1$ value

Starting from Eq. (2.23), let us remember that, by virtue of $AB > 0$, one has $\sigma^2(2AB+\lambda)/2 \equiv \pm{\omega_1}^2$ and $\sigma^2(2AB-\lambda)/2 \equiv {\omega_2}^2$. This means that we have to consider only one possible situation, because of the relative values of AB and λ.

As a matter of fact, the Lagrangian in Eq. (2.23) is written as

$$\mathcal{L}_+ \equiv \dot{u}^2 - \dot{v}^2 \pm {\omega_1}^2 u^2 + {\omega_2}^2 v^2 + 2\sigma^2 D, \tag{3.48}$$

giving rise to the equations

$$\ddot{u} = \pm{\omega_1}^2 u, \quad \ddot{v} = -{\omega_2}^2 v, \tag{3.49}$$

where the plus (minus) sign corresponds to the minus (plus) in Eq. (3.48). Now, the solution for $u(t)$ is

$$u(t) = u_1 \sin(\omega_1 t + u_2) \tag{3.50}$$

for the minus sign in Eq. (3.48), i.e., with $2AB < \lambda < 0$, and has therefore to be ruled out. One finds instead

$$u(t) = u_1 \exp(\omega_1 t) + u_2 \exp(-\omega_1 t) \tag{3.51}$$

when $2AB > \lambda$ (u_1 and u_2 being integration constants). The solution $v(t)$ is (in both cases)

$$v(t) = \alpha \sin(\omega_2 t + \beta), \tag{3.52}$$

α and β being integration constants. (In our considerations, of course, it has to be taken coupled to Eq. (3.51).)

The constraint equation $E_{\mathcal{L}_+} = 0$ now becomes (taking only the plus sign in Eq. (3.48))

$$\dot{u}^2 - \dot{v}^2 - {\omega_1}^2 u^2 - {\omega_2}^2 v^2 - 2\sigma^2 D = 0, \tag{3.53}$$

leading to

$$D = -\frac{\left(\alpha^2{\omega_2}^2 + 4u_1u_2{\omega_1}^2\right)}{2\sigma^2} > 0. \tag{3.54}$$

From Eq. (3.54) we have $u_1u_2 < -\alpha^2(2AB-\lambda)/[4(2AB+\lambda)] < 0$. Now, choosing $u_2 = -u_1$, one finds $u(t) = 2u_1 \sinh(\omega_1 t)$, so that the volume $a^3(t)$ is written as the difference $(4{u_1}^2 \sinh^2(\omega_1 t) - \alpha^2 \sin^2(\omega_2 t + \beta))/4$, with ${u_1}^2 > \alpha^2(2AB -$

$\lambda)/[4(2AB+\lambda)]$. If we then set $\beta = 0$, this is consistent with the physically required positive sign of D and, at the same time, ensures the vanishing of $a(0)$. Note, on the other hand, that the value $\alpha = 0$ gives $v(t) = 0$ at any t, and $a^3(t) = {u_1}^2 \sinh^2(\omega_1 t)$, so that this choice still yields $a(0) = 0$. If we do want to have both $a(0) = 0$ and assume $\alpha = 0$ (leading to a finite constant $\varphi(0)$ as well as an initially vanishing scale factor, as we already said), we instead (and unfortunately) find a constant scalar field at any time. Thus, even if it is true that, when both $u_2 = -u_1$ and $\beta = 0$, the argument of the logarithm in the φ-field becomes an undetermined form, the condition $a(0) = 0$ does require $u_2 = -u_1$ and $\beta = 0$.

With such choices the cosmological scale factor and the scalar field are

$$a(t) = \left(\frac{4{u_1}^2 \sinh^2(\omega_1 t) - \alpha^2 \sin^2(\omega_2 t)}{4}\right)^{1/3}, \tag{3.55}$$

$$\varphi(t) = \frac{1}{\sigma} \log \frac{B(2u_1 \sinh(\omega_1 t) + \alpha \sin(\omega_2 t))}{A(2u_1 \sinh(\omega_1 t) - \alpha \sin(\omega_2 t))}. \tag{3.56}$$

But, even if this solution were asymptotically good for describing an inflationary stage at present in the universe, we must rule it out because of the above considerations.

3.3.2 *The $\epsilon = -1$ value*

This situation has to be considered, now, starting from the Lagrangian in Eq. (2.24). Putting $\omega^2 \equiv \sigma^2 AB$ then yields

$$\mathcal{L}_- = \dot{u}^2 - \dot{v}^2 + 2\omega^2 uv + \frac{1}{2}\lambda\sigma^2(u^2 - v^2) + 2\sigma^2 D, \tag{3.57}$$

from which we deduce the equations

$$\ddot{u} = \frac{1}{2}\lambda\sigma^2 u + \omega^2 v, \quad \ddot{v} = \frac{1}{2}\lambda\sigma^2 v - \omega^2 u. \tag{3.58}$$

It turns out that this system is of the same kind as the one in Eqs. (3.32), with $a_{11} = a_{22} = \frac{1}{2}\lambda\sigma^2$, and $a_{12} = -a_{21} = \omega^2$. It turns out that the solutions can be represented as

$$\begin{aligned} a(t) = {} & \frac{\beta^{2/3}\sqrt[3]{e^{\sqrt{2}\sqrt{\lambda}\sigma t} - 1}\, e^{-\frac{t\left(\sqrt{\lambda\sigma^2+2\omega^2}+2\sqrt{\lambda}\sigma\right)}{3\sqrt{2}}}}{2\sqrt[3]{\lambda}\sigma^{2/3}\sqrt[6]{\lambda\sigma^2+2\omega^2}} \\ & \times \left\{ 2\sqrt{\lambda}\sigma e^{\frac{t\left(2\sqrt{\lambda\sigma^2+2\omega^2}+\sqrt{\lambda}\sigma\right)}{\sqrt{2}}} + \sqrt{\lambda\sigma^2+2\omega^2}\, e^{t\sqrt{\frac{\lambda\sigma^2}{2}+\omega^2}} \right. \\ & \left. -\sqrt{\lambda\sigma^2+2\omega^2}\, e^{\frac{t\left(\sqrt{\lambda\sigma^2+2\omega^2}+2\sqrt{\lambda}\sigma\right)}{\sqrt{2}}} - 2\sqrt{\lambda}\sigma e^{\frac{\sqrt{\lambda}\sigma t}{\sqrt{2}}} \right\}^{\frac{1}{3}}, \end{aligned} \tag{3.59}$$

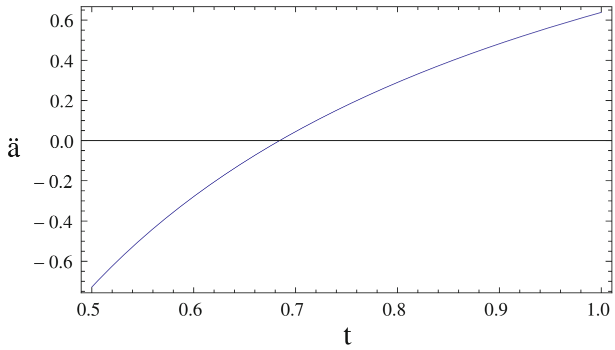

Fig. 5 The time evolution of the second derivative of the scale factor represented in Eq. (3.59) for $A = 1$, $\lambda = 1$, and $\sigma = 1$: it turns out that such a model is compatible with a late time accelerated expansion

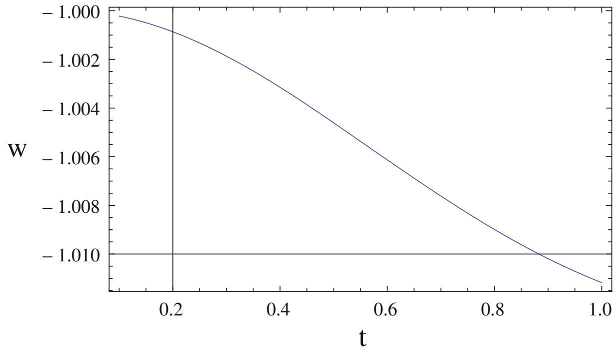

Fig. 6 The dark energy equation of state w parameter as a function of t, for the potential in Eq. (3.60) for $A = 1$, $\lambda = 1$, and $\sigma = 1$: it turns out that such a scalar field can be interpreted as a *superquintessential field*

$$\varphi(t) = -\frac{1}{\sigma}\left\{\log\left[\lambda\sqrt{\lambda\sigma^2+2}e^{t\sqrt{2\lambda\sigma^2+4}}\left(e^{\sqrt{2}\sqrt{\lambda}\sigma t}-1\right)\right]\right.$$
$$-\log\left[2A^2\left(4A^2\sqrt{\lambda\sigma^2+2}e^{\sqrt{2}t\left(\sqrt{\lambda\sigma^2+2}+\sqrt{\lambda}\sigma\right)}\right.\right.$$
$$\left.\left.\left.+2\sqrt{\lambda}\sigma e^{\frac{t\left(\sqrt{\lambda\sigma^2+2}+\sqrt{\lambda}\sigma\right)}{\sqrt{2}}}\left(e^{t\sqrt{2\lambda\sigma^2+4}}-1\right)+\sqrt{\lambda\sigma^2+2}e^{t\sqrt{2\lambda\sigma^2+4}}\right)\right]\right\}. \tag{3.60}$$

Once again, it is possible to describe the current accelerated expansion of the universe, as shown in Figs. 5 and 6.

Finally, it is worth noting that for consistency we have explicitly checked that the solutions corresponding to $\lambda = 0$ can be obtained as limit for $\lambda \to 0$ of the solutions obtained for $\lambda \neq 0$.

4 Connection between quintessence and inflation for exponential potential models

During inflation, the scalar field φ is supposed to be highly excited and slowly evolving ("rolling down") to the minimum of the potential. In this phase the potential energy of the inflaton field dominates the energy density of the universe. After the period of inflation, φ oscillates rapidly around the minimum of the potential and, by virtue of coupling with other scalar and spinor fields, massive and massless particles are created and the universe reheats, starting the standard post-Big-Bang evolution. Simple models of inflation use a scalar field with a potential of the form $V(\varphi) = \frac{1}{2}m^2\varphi^2$, where m is the mass of the inflaton field. In the standard scenario the energy of the inflaton field is then transformed into mass-energy of created particles. Another possibility can be offered, for example, by a quartic potential

$$W(\varphi) = \alpha \left(\varphi^2 - \delta^2\right)^2, \tag{4.1}$$

where α and δ are parameters.

Many efforts have been made for constructing unified frameworks for inflation and quintessence which employ a unique scalar field to drive both stages (see, for instance, Refs. [65–68]). Actually, in such scenarios the scalar field responsible of late time acceleration is nothing else but the remnant of the one which caused inflation at early time. This implies that a successful model of quintessential inflation is subject to the constraints of both inflation and quintessence simultaneously. For example, the minimum of the potential must not have been yet reached by the scalar field (generally, this requirement is satisfied by assuming the presence of a quintessential tail, i.e. by assuming potentials with the minimum displaced at infinity). Moreover, φ should not decay completely into a thermal bath of particles in order to survive until today, just to drive the late phase of accelerating expansion. As a consequence of this, the universe undergoes a period of *kination expansion*, when its energy density is dominated by the kinetic energy of φ. In this context, the standard reheating mechanism usually assumed to generate the primordial plasma does not work; however, the mechanism of gravitational particle production can still reheat the universe in the framework of quintessential inflation. Actually, even if the amount of radiation produced by such a mechanism is largely sub-dominant when compared with the energy field contribution during kination, the energy density of kination is redshifted by the cosmological expansion much faster than the radiation density, and starts dominating at some temperature.

Generally, the constraints and requirements which should be satisfied by quintessential inflation are fulfilled by using a multi-branch scalar potential, where the change of the potential, when the field moves from the inflationary to the quintessential frame of its evolution, is fixed by hand [65] or is the outcome of a phase transition arranged by the interaction with other scalar fields (see, for instance, Ref. [69]). Recently, the possibility has been investigated to connect the inflationary and quintessential expansion of the universe within the theoretical framework of particle production, usually developed in the very early universe. (Even if the quantum aspect of the creation mechanism is not yet very well known, some classical aspects, due to kinetic collisions in the hot dense regions of the early universe, have been discussed in the literature.) This suggests that the same mechanism may occur in the late universe, also leading to late time cosmic acceleration (see, for instance, Refs. [70,71]). In other approaches, a large variety of quintessential inflationary potentials are derived from theories of non-minimally coupled gravity (see, for instance, Ref. [72]).

In our model the quintessential inflation is formulated in terms of a multi-branch scalar field, driving both the inflationary and the quintessential phases of the evolution of the universe. The quintessential tail is realized through a single exponential potential (hence choosing the latter as our working potential for quintessence); on the contrary, to describe the inflationary plateau we do not fix any inflationary potential, but we propose a parametrization of the inflationary scalar field equation of state, and implement the transition from an inflationary stage to a kination evolution, which is characterized by the value $w_\varphi = 1$ of the equation of state, and corresponds just to the asymptotic in the past value for the equation of state of the exponential potential scalar field, as calculated from Eq. (2.6) [33,35].

4.1 The fiducial cosmological model

In order to illustrate our paradigm, we use, as fiducial cosmological model, the one considered in Refs. [35,59]. This model is based on the simplest form of exponential potential of the quintessence field

$$V(\varphi) = V_0 e^{-\sqrt{\frac{3}{2}}\varphi}, \tag{4.2}$$

and the assumption that the universe is spatially flat and filled in with dark matter and scalar field. (We postpone to a forthcoming paper the detailed investigation of the impact of a kination-dominated phase generated by the class of potentials described in the first part of this paper.) The equations that determine the dynamics of this model are (i) the Friedmann equation

$$3H^2 = \varrho_m + \varrho_\varphi, \tag{4.3}$$

where we use units in which $8\pi G = c = 1$, $H = \frac{\dot{a}}{a}$ is the Hubble constant, $\varrho_m \sim a^{-3}$ is the density of matter, and $\varrho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$ is the energy density of the scalar field, (ii) the Raychaudhuri equation

$$2\dot{H} + 3H^2 = -\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right), \tag{4.4}$$

and (iii) the generalized Klein–Gordon equation describing the evolution of the scalar field

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0. \tag{4.5}$$

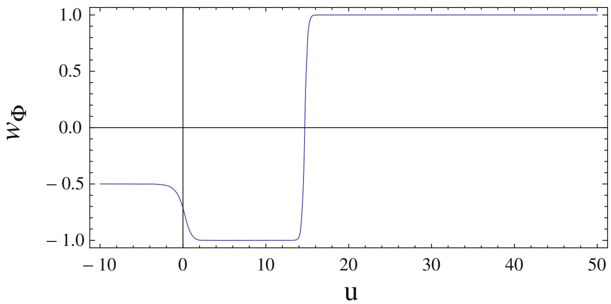

Fig. 7 The dark energy equation of state w_φ parameter as a function of u

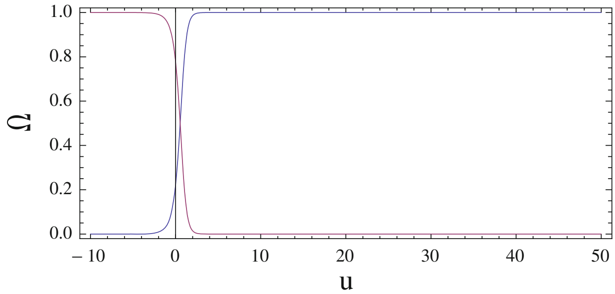

Fig. 8 Omega parameters as functions of u, where Ω_φ is marked in *green* and Ω_m in *blue* (colour figure online)

These equations take a simple form when instead of t as an independent variable one uses $a(t)$, the scale factor. Introducing a new independent variable by $u = \log(1+z) = -\log\left(\frac{a(t)}{a_0}\right)$, where a_0 is the present value of the scale factor and z is the redshift, and rescaling other variables as $\bar{\varphi} = \frac{1}{\sqrt{3}}\varphi$, $\bar{\varrho}_i = \frac{\varrho_i}{3H_0^2}$ (where $i = m, \varphi$), $\bar{V}_0 = \frac{V_0}{3H_0^2}$ and $\bar{H} = \frac{H}{H_0}$, where V_0 is a parameter and H_0 the present value of the Hubble constant, we thus obtain the following set of equations that contain only dimensionless variables:

$$\bar{H}^2 = \frac{\bar{\varrho}_m + \bar{V}}{1 - \frac{1}{2}\bar{\varphi}'^2}, \tag{4.6}$$

$$\bar{H}^2\bar{\varphi}'' - 3\left(\frac{1}{2}\bar{\varrho}_m + \bar{V}\right)\bar{\varphi}' + \frac{d\bar{V}}{d\bar{\varphi}} = 0, \tag{4.7}$$

with prime denoting derivative with respect to u. Such equations can be solved analytically but we here limit ourselves to display the plots of some important quantities of the model, such as $w_\varphi(u)$, $\Omega_m(u)$ and $\Omega_\varphi(u)$. (See Figs. 7 and 8 for their behaviours.)

This model of the universe is described by an exact solution of the dynamical equations [33,35]. The arbitrary parameters that appear in the solution are determined by specifying the initial conditions; we in fact require that at the present time, e.g. at $u = 0$, one should have $\Omega_m(u = 0) = 0.3$ and $\Omega_\varphi(u = 0) = 0.7$. The variable u is such that it decreases as time increases, and, at early times, the scalar field is almost constant and only recently it starts increasing. On the other hand, the potential of the scalar field at early times is constant and only recently is rapidly decreasing, while in the future it assumes a constant value again. The dark energy equation of state w_φ parameter in the far past is equal to -1, so that at the early stage of the evolution of the universe dark energy behaves as a cosmological constant, but (as is shown in Fig. 8, through the behaviours of the Ω parameters) it only recently has started dominating the expansion rate of the universe.

The quintessential exponential potential admits, as said, exact solutions of the Einstein field equations. Actually, from them we find [33]

$$\varphi = -\sqrt{\frac{2}{3}}\log\left(\frac{2}{1+t^2}\right), \tag{4.8}$$

which was obtained by using as suitable unit of time the age of the universe, i.e. $t_0 = 1$. Thus, at the present time $\varphi_0 = 0$. At the time of reheating t is virtually zero, so that we may set $\varphi_{\rm in} \equiv -\sqrt{\frac{2}{3}\log(2)}$. We now shift φ to $x \equiv \varphi - \varphi_{\rm in}$, so that the potential may be written as

$$V = 4\exp\left(-\sqrt{\frac{3}{2}}x\right), \tag{4.9}$$

where the prefactor 4 is due to the particular choice of units (and, of course, is also the initial value for V) and represents the value of the effective cosmological constant at that time, so that it has the dimension of $[\text{length}]^{-1}$. (In our units, the unit length is of the order of Hubble length, just as the current estimate for Λ. An effective cosmological constant close to 4 is thus slightly greater than this and slowly evolves towards a smaller value of 2 nowadays.)

If we consider more realistically the inclusion of radiation, too, into such a model, the dynamical equations do not have analytical solutions, as far as we know, and therefore we have to rely on numerical computations. Following the procedure used above, we again use the variable u instead of time t together with the rescaled variables, so that the equations contain only dimensionless variables and can now be written in the form

$$\bar{H}^2 = \frac{\bar{\varrho}_m + \bar{\varrho}_r + \bar{V}}{1 - \frac{1}{2}\bar{\varphi}'^2}, \tag{4.10}$$

$$\bar{H}^2\bar{\varphi}'' - \left(\bar{\varrho}_r + \frac{3}{2}\bar{\varrho}_m + 3\bar{V}\right)\bar{\varphi}' + \frac{d\bar{V}}{d\bar{\varphi}} = 0, \tag{4.11}$$

where $\bar{\varrho}_r \sim a^{-4}$ is the rescaled energy density of radiation.

We can numerically solve this system of coupled equations, specifying the initial condition at $u = 60$, for example, and assuming that $\varphi(30)$, $\varphi'(30)$, and $H(30)$ have the same values as in the case without radiation, while $\Omega_r(60)$ is just the rescaled present value of Ω_r. Some results of numerical integration are shown in Figs. 9 and 10.

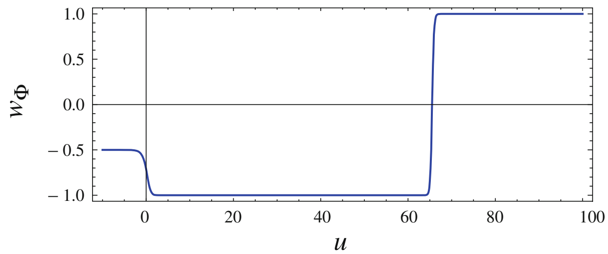

Fig. 9 The dark energy equation of state w_φ parameter as a function of u with radiation included

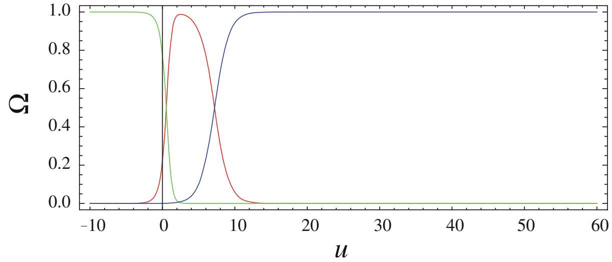

Fig. 10 Omega parameters as functions of u in the universe filled in with matter, radiation and scalar field. Ω_φ is marked in *green*, Ω_r in *red* and Ω_m in *blue* (colour figure online)

It can be seen that the presence of radiation is slightly changing the behaviour of the scalar field, its potential, the Hubble constant, and the w parameter of the dark energy equation of state. As expected, now, only the evolution of the Ω parameters is substantially different. At the initial time, again when $u = 30$, radiation dominates the expansion rate of the universe, with dark energy and matter being subdominant, at a redshift z of about 5,000; the energy densities of matter and radiation become equal and, for a relatively short period, the universe becomes matter dominated until, at a redshift of about 1, dark energy starts dominating the expansion rate of the universe. With these results we then confirm the independent investigations of effects of radiation on the evolution of the quintessence field by França and Rosenfeld [73]. From our results it indeed follows that, during the epoch of nucleosynthesis ($z \sim 10^9$), the energy density of the scalar field is much smaller than the energy density of radiation. In particular, during such an epoch the kinetic term in the scalar field energy density vanishes, and the potential term is constant, so that the dark-energy term acts as an effective cosmological constant Λ and does not affect the process of primordial nucleosynthesis.

The presence of such an effective cosmological constant Λ in that early period of the universe is a sign of the often *artificial* separation operated a priori between *pure* and *impure* quintessence models. Actually, we here note that, even if we did not introduce any Λ-term in the theory, it *effectively* can come out as a byproduct of the form of the potential. What is more important, in our opinion, is how the universe behaves as a whole, while the a priori presence or not of the Λ-term is not really meaningful in a scalar-tensor theory of cosmology.

4.2 Considerations about the inflation-kination transition

On the other hand, as remarked above, an indispensable ingredient of quintessential inflationary scenarios is the existence of an early kination-dominated (KD) era, where the universe is dominated by the kinetic energy of the quintessence field. During this era, the expansion rate of the universe is larger compared to its value during the usual radiation-domination (RD) epoch. A generic reasonable inflation-kination period can be described by two main parametrizations of the transition, without specifying the details of the model. One is a polynomial parametrization for $a^2(\eta)$ (η being the conformal time) [74], and the other is a parametrization of the equation of state w, smoothly connecting the inflation ($w = -1$) and kination ($w = 1$) regimes. For example, in Ref. [75] a hyperbolic tangent parametrization is used to study gravitational reheating in quintessential inflation. Moreover, the impact of a kination-dominated phase generated by a quintessential exponential model or by quintessential power law, or also by a running kinetic inflation model, has been already investigated in literature (see for instance Refs. [75–77]).

Here our approach is different, since we do not fix a single-branch scalar field potential which drives both the early-time inflationary and the late-time quintessence evolution of the universe. We instead want to show that a quintessential exponential potential tail can be (at least *by hand*) connected to the inflation through a kination dominated era, because the *asymptotic* value (in the past) of the equation of state, w_φ, is just[1] $w_\varphi = 1$, which characterizes the kination phase. We illustrate such a mechanism by using a new parametrization of the scalar field equation of state, which could also be used to study statistical properties of massive non-relativistic bosons arising at the first stage of reheating as a result of a quantum decay of a classical quantum field in inflationary cosmological model without slow rolling [78].

The FLRW evolution of the universe in the inflationary epoch is described by the equations

$$3H^2 = \rho_\varphi, \tag{4.12}$$

$$\frac{\ddot{a}}{a} = -\frac{\rho_\varphi + 3P_\varphi}{2}, \tag{4.13}$$

where we are using dimensionless units and $a_0 = \sqrt{\frac{3}{8\pi}} l_{Pl} \frac{E_{Pl}}{E_{GUT}}^2 \simeq 10^{-25}$ cm. Since the quantities ρ_ϕ and P_ϕ are related by the equation of state $P_\varphi = w_\varphi \rho_\varphi = \left(\gamma_\varphi - 1\right) \rho_\varphi$, their evolution is described by the continuity equation

$$\dot{\rho_\varphi} + 3H(\rho_\varphi + P_\varphi) = 0. \tag{4.14}$$

From Eqs. (4.12), (4.13), and (4.14) it turns out that

$$\gamma_\varphi = -\frac{2}{3}\frac{\dot{H}}{H^2}, \tag{4.15}$$

[1] It is worth noting that this is the case of several quintessential solutions with exponential potential, as for instance the ones described in Eqs. (3.11) and (3.12).

and for $\gamma_\varphi = 0$ we have an exponential behaviour for $a(t)$; if $\gamma_\varphi = \gamma_0 \neq 0$ is constant, there is a power-law expansion, and we have inflation iff $\gamma_0 < \frac{2}{3}$, the more de Sitter-like, the closer γ_0 to zero.

All this suggests the possibility of parametrizing the dynamics of the universe during such a phase in terms of $\gamma_\varphi(t)$. Indeed, we have that

$$H(t) = \frac{2}{3}\left(\int \gamma_\varphi dt\right)^{-1}, \tag{4.16}$$

$$\rho_\varphi(t) = \frac{4}{9}\left(\int \gamma_\varphi dt\right)^{-2}, \tag{4.17}$$

$$P_\varphi = \frac{4}{9}\left(\gamma_\varphi - 1\right)\left(\int \gamma_\varphi dt\right)^{-2}, \tag{4.18}$$

$$V(\varphi(t)) = H^2(t)\left(1 - \frac{1}{2}\gamma_\varphi\right), \tag{4.19}$$

$$\varphi(t) = \int \sqrt{-\frac{2}{3}\dot{H}}dt. \tag{4.20}$$

The above Eqs. (4.16)–(4.20) are parametric equations for the potential V, and can allow to reconstruct the scalar field potential in terms of γ_φ. Thus, let us consider a scalar field which, describing an inflationary stage at the beginning, undergoes a phase transition into a kination phase. This can be achieved by choosing a special form for γ_φ (and, therefore, w_φ), and reconstructing the potential.

Our choice for γ_φ is

$$\gamma_\varphi = \alpha \frac{1}{1 + \frac{1}{2}\exp(-\beta(t - t_f))}, \tag{4.21}$$

where β and t_f are parameters indicating the rate of the phase transition and the time at which inflation ends, respectively. On the other hand, the parameter α is related to the *asymptotic* value of w_φ, which in our model is $w_\varphi = 1$. For different values of α, it is possible to obtain a transition from vacuum into dust or radiation. Moreover, by using a slightly different parametrization of the equation of state, as

$$\gamma(\varphi) = \frac{A}{e^{\alpha(t-t_0)} + 1} - \frac{B}{e^{\beta(-(t-t_f))} + 1} - 1, \tag{4.22}$$

it is also possible to obtain a double transition from vacuum into kination and into radiation, as illustrated in Fig. 11. Such a form, even if arbitrarily assigned, can easily implement the transition from an inflationary stage to a kination one, as we see in Figs. 12 and 13, where we plot the behaviour of the equation of state and the reconstructed scalar field potential, respectively. We have to note that the potential changes rather roughly from its value during inflation to its final values, and that the *asymptotic in the future* value for the equation of state $w_\varphi = 1$ corresponds to the *asymptotic in the past* value for the quintessential exponential potential. Thus, these behaviours can be

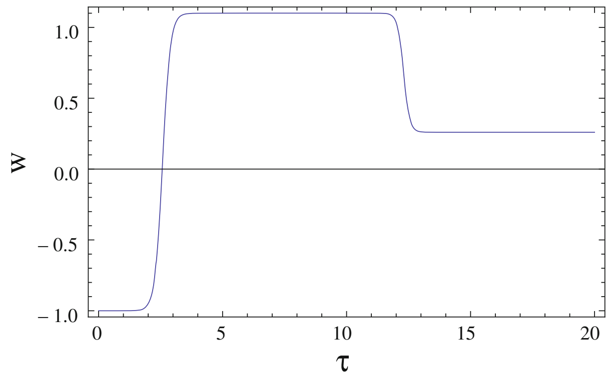

Fig. 11 We show the double transition from vacuum into kination and into radiation obtained by using the parametrization of Eq. (4.22), with $A = 2.1$, $B = 0.84$, $\beta = 8.14$, $\alpha = -6.43$, $t_f = 12.3$, $t_0 = 2.58$

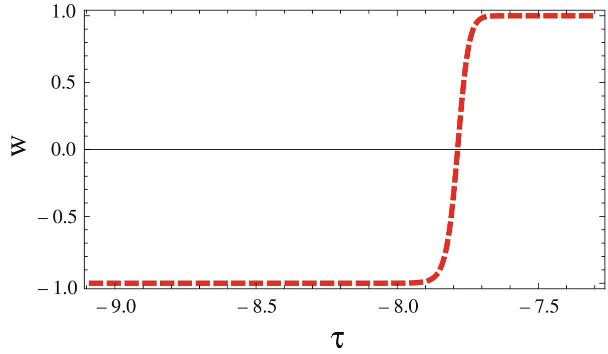

Fig. 12 Equation of state w_φ as a function of time, when the universe is dominated by kination at the end of inflation. The parametrization of Eq. (4.21) is used, with $\alpha = 2$, $\beta = 1.32$, $t_f = 16.9$

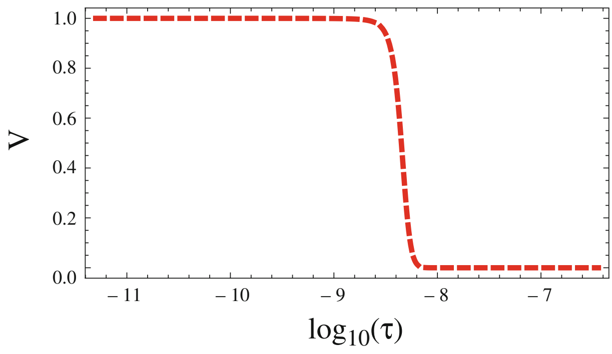

Fig. 13 Behaviour of the scalar field potential as a function of time, when the universe is dominated by kination at the end of inflation. The parametric equations for the potential V in Eq. (4.19) are used, with γ_φ given in Eq. (4.21) and $\alpha = 2$, $\beta = 1.32$, $t_f = 16.9$

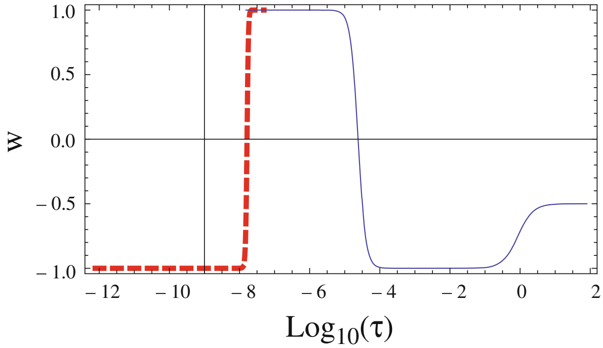

Fig. 14 Evolution of the equation of state from the inflationary to the quintessential stage. *Dashed in red*, the transition is plotted from the inflation into the kination when $w_\varphi = 1$; we see that such an *asymptotic in the future* value for the equation of state corresponds to the *asymptotic in the past* value for the quintessential exponential potential (*solid line*) (colour figure online)

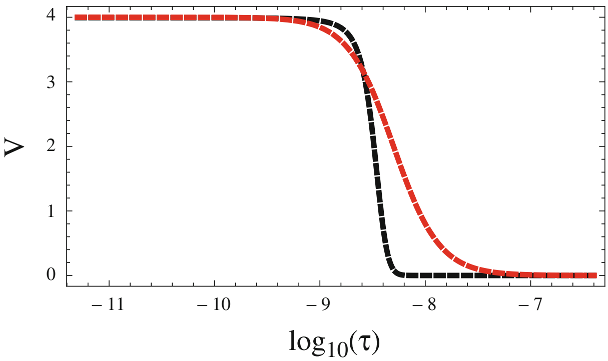

Fig. 15 Behaviour of the scalar field potential during the two stages: it turns out that the quintessential exponential potential (*red dashed line*) asymptotically behaves like the inflationary one (*black dashed line*) (colour figure online)

connected, as shown in Fig. 14, where the time-scale has been arbitrarily taken in order to show all the evolution from inflation into kination, toward the late-time exponential quintessential stage, and the values of the parameters α, β and t_f are fixed so as to obtain the optimal link between the early and late time evolutions. Finally, in Fig. 15 we compare the behaviours of the scalar field potentials (with radiation included) during the two stages: it turns out that the quintessential exponential potential asymptotically behaves like the inflationary one.

5 Discussion and conclusions

After the introduction of scalar fields in inflationary cosmology, the quite recent reconsideration of their crucial importance relies upon the fact that they can improve a dynamical mechanism for giving rise today to a repulsive component in cosmic energy, the dark energy. In this paper dealing with exponential-like potentials for the scalar field, our attention has been generally focused on how they could characterize current cosmology. At first, we have discussed some models generalizing the simple exponential form of the potential, in order to derive general exact solutions. The technique used has been very simple, being based on a given *ad hoc* change of variables. Such a procedure is generally possible either *by chance* or because there exists a sort of a method to deduce that useful transformation. In our work we have adopted the second procedure, i.e. the Noether Symmetry Approach to cosmology, borrowing from it here not only the suggestion on the kind of useful transformations to apply to the variables a and φ involved, but also on the *natural* kind of potential $V(\varphi)$ to be studied. On the other hand, the results we have found in this first part of the paper are not always easily discernable. Basically, they have been mathematically derived but not yet appropriately discussed on physical ground. This deserves, of course, further investigations in a forthcoming paper, but we can however try here to sum up what has come out as more interesting for nowadays cosmology.

When $\lambda = 0$ in the potential, first of all, the only case we have discussed here is for $A^2 \neq 0$, $B^2 \neq 0$, $\epsilon = -1$. In such a case the solution, which at first glance appears to describe a sort of cyclic universe with spatially flat hyper-surfaces, allows us to describe the current accelerated expansion of the universe, with a quintessential scalar field, whose equation of state transits from $w = 1$ in the far past to $w = -1$ in the future. The situation with $\lambda > 0$, on the other hand, presents an accelerated evolution for $\epsilon = \pm 1$.

In a forthcoming paper we are going into details of the cosmological evolution, with regard to each specific solution discussed above, by performing the necessary confrontation of these theoretical outputs with the observational data sets. We are going to investigate also whether such models can be interpreted as the Einstein frame counterpart of alternative gravity models formulated into the Jordan frame by a conformal transformation (see [79,80]). However, it is worth noting that the Jordan and Einstein frames cannot be physically equivalent according to the choice of observable quantities. Indeed, the Jordan frame is mapped into the Einstein frame with a minimally-coupled scalar field but at the price of coupling matter to the scalar field (see [81–83]). The second part of the paper is still connected with the exponential potential. But, now, we instead focus on considering and illustrating a possible quintessential inflationary scenario, formulated in terms of a multi-branch scalar field, driving both the inflationary and the quintessential phases of the evolution of the universe. The quintessential tail is realized through an exponential potential (choosing, thus, the latter as our working potential for quintessence); on the contrary, for describing the inflationary plateau we do not fix any inflationary potential, but we propose a parametrization of the inflationary scalar field equation of state, and implement the transition from an inflationary stage to a kination evolution, which is characterized by the value $w_\varphi = 1$ of the equation of state, just corresponding to the asymptotic in the past value for the equation of state of the exponential potential scalar field. It turns out that the reconstructed potential changes rather roughly from its value during inflation to its final values, and the *asymptotic in the future* value for the equation of state $w_\varphi = 1$ corresponds to the *asymptotic in the past* value for the quintessential exponential potential. Let us stress again that these behaviours have been connected by hand, and the exponential form of the scalar field potential driving the late stage of the universe could indeed be the asymptotic late time behaviour of the inflationary scalar field, which transits from inflation into kination, toward the late quintessential stage. It is worth noting that such a conclusion is somehow independent of the mechanism proposed for the evolution of the scalar field potential, being only based on a parametric description of the very early inflationary dynamics of the universe, and on the property of the equation of state $w_\varphi = 1$, which characterizes our exponential potential. In our cited forthcoming paper we are also going to select, among all solutions, the cases which *preserve* such asymptotic behaviour with $w_\varphi = 1$.

This mechanism for driving the transition from the inflationary evolution toward the late time accelerated expansion has indeed to be considered as mainly exploratory, and some topics need much more investigation. For example, particles' production, that is the gravitational production during the reheating or the preheating, in which particles are produced by virtue of the variation of the classical inflaton field, needs to be investigated in a forthcoming paper, considering the scalar field potential given by Eq. (4.19).

To conclude, let us note that considering radiation in the model only changes the evolution of the Ω parameters. Radiation initially dominates on matter, while later on the energy densities of matter and radiation become equal; after that, for some time matter dominates in the universe, while dark energy starts dominating the expansion rate of the universe only afterwards. These results seem to confirm other investigations of effects of radiation on the evolution of the quintessence field, according to which the energy density of the scalar field during the epoch of nucleosynthesis ($z \sim 10^9$) is much smaller than the energy density of radiation. While the kinetic term in the scalar field energy density vanishes and the potential term becomes constant, the dark-energy term in fact behaves like an effective cosmological constant, not affecting the process of primordial nucleosynthesis.

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References

1. Guth, A.H.: Phys. Rev. D **23**, 347 (1981)
2. Damour, T., Esposito-Farese, G.: Class. Quantum Gravit. **9**, 2093 (1992)
3. Fujii, Y., Maeda, K.: The Scalar-Tensor Theory of Gravitation. Cambridge University Press, Cambridge (2004)
4. Perlmutter, S. et al.: Nature **391**, 51 (1998)
5. Perlmutter, S. et al.: Astrophys. J. **517**, 565 (1999)
6. Riess, A.G. et al.: Astron. J. **116**, 1009 (1998)
7. Garnavich, P.M. et al.: Astrophys. J. **509**, 74 (1998)
8. Ostriker, J.P., Steinhardt, P.J.: Nature **377**, 600 (1995)
9. Caldwell, R.R., Dave, R., Steinhardt, P.J.: Phys. Rev. Lett. **80**, 1582 (1998)
10. Zlatev, I., Wang, L., Steinhardt, P.J.: Phys. Rev. Lett. **82**, 896 (1999)
11. Steinhardt, P.J., Wang, L., Zlatev, I.: Phys. Rev. D **59**, 123504 (1999)
12. Wang, L. et al.: Astrophys. J. **530**, 17 (2000)
13. Turner, M.S., White, M.: Phys. Rev. D **56**, 4439 (1997)
14. Chiba, T., Sugiyama, N., Nakamura, T.: Mon. Not. R. Astron. Soc. **289**, L5 (1997)

15. van den Bergh, S.: In: Martinez, V.J., Trimble, V. (eds.) Development of Modern Cosmology. ASP Conference Series (2000)
16. Turner, M.S.: In: Campbell, M., Wells, T.M. (eds.) The Proceedings of Physics in Collision (Ann Arbor, MI, 24–26 June 1999), World Scientific, NJ (2000)
17. Carroll, S.M., Press, W.H., Turner, E.L.: Ann. Rev. Astron. Astrophys. **30**, 499 (1992)
18. Carroll, S.M.: Living Rev. Relativ. **4**, 1–56 (2001). http://www.livingreviews.org/Articles/Volume4/2001-1carroll
19. Sahni, V., Starobinsky, A.: Int. J. Mod. Phys. D **9**, 373 (2000)
20. Weinberg, S.: Rev. Mod. Phys. **61**, 1 (1989)
21. Ratra, B., Peebles, P.J.E.: Phys. Rev. D **37**, 3406 (1988)
22. Peebles, P.J.E., Ratra, B.: Astrophys. J. Lett. **325**, L17 (1988)
23. Saini, T.D., Raychaudhury, S., Sahni, V., Starobinsky, A.A.: Phys. Rev. Lett. **85**, 1162 (2000)
24. Rubano, C., Barrow, J.D.: Phys. Rev. D **64**, 127301 (2001)
25. Liddle, A.R., Scherrer, R.J.: Phys. Rev. D **59**, 023509 (1999)
26. Brax, Ph., Martin, J.: Phys. Lett. B **468**, 40 (1999)
27. Binetruy, P.: Int. J. Theor. Phys. **39**, 1859 (2000)
28. Cardenas, R., Gonzales, T., Martin, O., Quiros, I.: arXiv:astro-ph/0210108v1
29. Halliwell, J.: Phys. Lett. B **185**, 341 (1987)
30. Sahfi, Q., Wetterich, C.: Nucl. Phys. B **289**, 787 (1987)
31. Lopes, U., Rosenfeld, R.: JHEP **0210**, 015 (2002)
32. Kolda, C., Lahneman, W.: arXiv:hep-ph/0105300
33. Rubano, C., Scudellaro, P.: Gen. Relativ. Gravit. **34**, 307 (2002)
34. Rubano, C., Sereno, M.: Mon. Not. R. Astron. Soc. **335**, 30 (2002)
35. Rubano, C. et al.: Phys. Rev. D **69**, 103510 (2004)
36. Gu, J.-A., Hwang, W.-Y.P.: Phys. Rev. D **73**, 023519 (2006)
37. Cardenas, R., Gonzales, T., Martin, O., Quiros, I.: arXiv:astro-ph/0209524v1
38. Shapiro, I.L., Sola, J.: Phys. Lett. B **475**, 236 (2000)
39. Gu, J.-A., Hwang, W.-Y.P.: Mod. Phys. Lett. A **17**, 1979 (2002)
40. Sen, A.A., Sethi, S.: Phys. Lett. B **532**, 159 (2002)
41. de Ritis, R., Marmo, G., Platania, G., Rubano, C., Scudellaro, P., Stornaiolo, C.: Phys. Rev. D **42**, 1091 (1990)
42. de Ritis, R., Marmo, G., Platania, G., Rubano, C., Scudellaro, P., Stornaiolo, C.: Phys. Lett. A **149**, 79 (1990)
43. de Ritis, R., Marino, A.A.: Phys. Rev. D **64**, 083509 (2001)
44. Capozziello, S., de Ritis, R., Marino, A.A.: Helv. Phys. Acta **69**, 241 (1996)
45. Capozziello, S., de Ritis, R.: Gen. Relativ. Gravit. **29**, 1425 (1997)
46. Barrow, J.D.: Phys. Lett. B **187**, 12 (1987)
47. Burd, A.B., Barrow, J.D.: Nucl. Phys. B **308**, 929 (1988)
48. Wetterich, C.: Astron. Astrophys. **301**, 321 (1995)
49. Capozziello, S., de Ritis, R., Rubano, C., Scudellaro, P.: Riv. Nuovo Cimento **19**(4), 1–114 (1996)
50. Wetterich, C.: Nucl. Phys. B **302**, 668 (1998)
51. Copeland, E.J., Liddle, A.R., Wands, D.: Phys. Rev. D **57**, 4686 (1998)
52. Ferreira, P.G., Joyce, M.: Phys. Rev. D **58**, 023503 (1998)
53. Fabris, J.C., Goncalves, S.V.B., Tomimura, N.A.: Class. Quantum Gravit. **17**, 2983 (2000)
54. Barreiro, T., Copeland, E.J., Nunes, N.J.: Phys. Rev. D **61**, 127301 (2000)
55. Sahni, V., Wang, L.: Phys. Rev. D **62**, 103517 (2000)
56. Batista, A.B., Fabris, J.C., Goncalves, S.V.B., Tossa, J.: Int. J. Mod. Phys. A **16**, 4527 (2001)
57. Andrianov, A.A., Cannata, F., Kamenshchik, A.Y.: JCAP **10**, 004 (2011)
58. Pavlov, M., Rubano, C., Sazhin, M., Scudellaro, P.: Astrophys. J. **566**, 619 (2002)
59. Demianski, M., Piedipalumbo, E., Rubano, C., Tortora, C.: Astron. Astrophys. **431**, 27 (2005)
60. Kehagias, A., Kofinas, G.: Class. Quantum Gravit. **21**, 3871 (2004)
61. Basilakos, S., Tsamparlis, M., Paliathanasis, A.: Phys. Rev D **83**, 103512 (2011)
62. Scott, D., Smoot, G.F.: arXiv:1005.0555v1 [astro-ph.CO] (2010)
63. Komatsu, E. et al.: Astrophys. J. Suppl. **192**, 18 (2011)
64. de Ritis, R., Rubano, C., Scudellaro, P.: Europhys. Lett. **32**, 185 (1995)
65. Peebles, P.J., Vilenkin, A.: Phys. Rev. D **59**, 063535 (1999)
66. Dimopoulos, K., Valle, J.W.F.: Astropart. Phys. **18**, 287 (2002)

67. Neupane, I.P., Scherrer, C.: JCAP **05**, 009 (2008)
68. Bastero-Gil, M.: Phys. Lett. B **678**, 157 (2009)
69. Kinney, W.H., Riotto, A.: Astropart. Phys. **10**, 387 (1999)
70. Lima, J.A.S., Silva, F.E., Santos, R.C.: Class. Quantum Gravit. **25**, 205006 (2008)
71. Debnath, S., Sanyal, A.K.: Class. Quantum Gravit. **28**, 145015 (2011)
72. Kaganovich, A.B.: Phys. Rev. D **63**, 025022 (2001)
73. França, U., Rosenfeld, R.: JHEP **10**, 015 (2002)
74. Ford, L.H.: Phys. Rev. D **35**, 2955 (1987)
75. Chun, E.J., Scopel, S., Zaballa, I.: JCAP **07**, 022 (2009)
76. Gómez, M.E., Lola, S., Pallis, C., Rodríguez-Quintero, J.: JCAP **01**, 027 (2009)
77. Nakayama, K., Takahashi, F.: JCAP **11**, 09 (2010)
78. Capozziello, S., de Ritis, R., Dymnikova, I., Rubano, C., Scudellaro, P.: Phys. Lett. A **203**, 18 (1995)
79. Bamba, K., Capozziello, S., Nojiri, S., Odintsov, S.D.: Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests. arXiv:1205.3421v2 [gr-qc]
80. Capozziello, S., De Laurentis, M.: Phys. Rep. **509**, 167–321 (2011)
81. Capozziello, S., Martin-Moruno, P., Rubano, C.: Phys. Lett. B **689**, 117–121 (2010)
82. Capozziello, S., de Ritis, R., Marino, A.A.: CQ Gravit. **14**, 3259 (1997)
83. Demianski, M., Piedipalumbo, E., Rubano, C., Scudellaro, P.: Astron. Astrophys. **481**, 279 (2008)