THE ROLE OF POINT SOURCES AND THEIR POWER FLUXES IN THE LINEAR SAMPLING METHOD

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Abstract. In this paper we investigate the linear sampling method by focusing on energy conservation inside a lossless background, in the case of a three-dimensional, impenetrable and acoustic scattering set-up. We analyze, from both a numerical and a theoretical viewpoint, how average power fluxes are carried throughout the host medium by the flow tubes of radiating fields. As a result, the far-field equation, which is the core of the linear sampling method, can be regarded as a physical constraint linking the power flux of the scattered wave with that of the field radiated by a point source. Then, we show that this constraint, together with appropriate assumptions on the flow tubes of the scattered field, gives rise to a physical framework whereby some theoretical flaws of the linear sampling method can be overcome.

Key words. Qualitative inverse scattering, linear sampling method.

AMS subject classifications. 45A05, 65R20, 65R30, 65R32, 78A46.

1. Introduction. Among qualitative methods in inverse scattering [8], the linear sampling method (LSM) [6, 9, 10, 13] is the earliest and probably the most popular. However, its theoretical foundation is not so sound as one might desire, since there is no clear link between its actual implementation and the general theorem inspiring the algorithm [7, 8]: in other terms, the theory available suggests, but does not justify the LSM. This gap is explained in Section 2 of our paper, soon after a short review of the LSM.

In order to address this issue, some efforts have been made by the scientific community since the appearance of the first papers concerning the LSM. In this context, a prominent role is certainly played by the factorization method (FM) [17, 18]: it can be considered, in some sense, a modified version of the LSM, whereby the foundational problems affecting the latter are completely removed. Moreover, as a notable by-product, the FM can also inspire a revisititation and a consequent mathematical justification of the LSM [3, 4, 16]. However, the price to be paid for applying the FM is a non-negligible restriction of the class of scattering conditions and of the measurement set-up: hence, so far, the FM is somewhat less general [7, 8] than the LSM, and the same limitations clearly affect the FM-based justifications of the LSM. These points are briefly recalled and discussed in Section 3 of our paper.

To overcome this drawback, a physics-based approach has been recently proposed in [2], whereby the far-field equation at the basis of the LSM (which does not follow from physical laws) is regarded as a constraint on the power fluxes carried by the scattered field considered in the implementation of the LSM. According to this approach, the LSM is explained from a physical viewpoint by numerically observing the behavior of the flow lines of the scattered field and then by proving that such behavior forces the LSM to work as expected. In principle, this physical interpretation is feasible without restrictions on the scattering conditions: however, in [2], it is discussed in detail only for a two-dimensional, penetrable and electromagnetic scattering set-up.

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Then, we want to address here also the three-dimensional, impenetrable and acoustic case, in order to show the flexibility of this approach: the generalization is not trivial, as shown e.g. by the technical result in Appendix A. Moreover, differently from [2], also the FM is taken into account and briefly discussed within this framework. Finally, in the present paper we numerically investigate also the case where the emitting and receiving antennas are placed according to an aspect-limited configuration.

To work out our physical interpretation of the LSM, in Section 4 we analyze the mechanism of power flux transport throughout the background medium for acoustic waves, both from a global and local viewpoint: in the latter case, the importance of the flow tubes of the scattered field is highlighted. Then, the far-field equation is regarded as a constraint on the power fluxes in the far-field region, whereby the flux at infinity of the scattered field can be made arbitrarily close to the flux of the field radiated by a point source placed in the background. In Section 5, a technical result concerning the power fluxes across the boundary of the scatterer is proved, by taking inspiration from an analogous property of the FM: this result ensures the expected performance of the LSM close to the edge of the obstacle and involves neither the far-field equation nor the flow lines of the scattered field. In Section 6 the behavior of the flow lines is numerically investigated, while in Section 7 we prove that such behavior, together with the constraint expressed by the far-field equation, suffices to satisfy the hypotheses implying the technical result presented in Section 5, thus explaining how the LSM can correctly visualize the boundary of the scatterer. Section 8 extends the previous investigation, both from a numerical and theoretical viewpoint, to the visualization provided by the LSM outside the scatterer. Our conclusions and hints for future work are presented in Section 9, while in Appendix A we prove a minimum property for cones inscribed in a half-sphere: this result is an important tool for investigating the power fluxes across the boundary of the scatterer.

2. The scattering problem and the LSM. In this section we recall some basic notations and properties of direct and qualitative inverse scattering problems in the time-harmonic regime, by focusing, in particular, on the 3D acoustic case for an impenetrable, sound-soft obstacle put inside a homogeneous and non-absorbing host-medium; we refer to [8, 12] for details. If \( \omega \) denotes the angular frequency of the wave and \( c_0 \) the speed of sound in the background, the corresponding wavenumber is \( k = \omega/c_0 > 0 \). The obstacle takes up a bounded and open \( C^2 \)-domain \( D \subset \mathbb{R}^3 \), such that \( \mathbb{R}^3 \setminus \bar{D} \) is connected (but \( D \) does not need to be connected). Factoring out the time dependence \( e^{-i\omega t} \), we denote by \( u^i = u^i(x) \) the incident wave (assumed to be an entire solution of the Helmholtz equation with wavenumber \( k \)), by \( u^s \) the corresponding scattered field, and by \( u = u^i + u^s \) the resulting total field. Then, the direct scattering problem can be formulated as the following exterior Dirichlet problem.

**Problem 2.1.** Given \( u^i \) as above, find \( u^s \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C^{1,\alpha}(\mathbb{R}^3 \setminus D) \) such that

\[
\begin{cases}
\Delta u^s(x) + k^2 u^s(x) = 0 & \text{for } x \in \mathbb{R}^3 \setminus \bar{D}, \\
u^s(x) = -u^i(x) & \text{for } x \in \partial D, \\
lim_{r \to \infty} \left[ r \left( \frac{\partial u^s}{\partial r} - iku^s \right) \right] = 0, & \text{(2.1)}
\end{cases}
\]

where \( r = |x| \) and the limit (2.1)(c), which represents the Sommerfeld radiation condition, holds uniformly in all directions \( \hat{x} = \frac{x}{|x|} \in \Omega := \{ x \in \mathbb{R}^3 : |x| = 1 \} \).
Problem 2.1 is well-posed: in particular, the operator mapping the boundary data $-u^t$ into the solution $u^s$ is continuous from $C^{1,\alpha}(\partial D)$ into $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$. Moreover, the scattered field $u^s$ admits the following asymptotic representation, uniformly in $\hat{x}$:

$$u^s(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}) + O(r^{-2}) \quad \text{as } r \to \infty,$$

(2.2)

where the function $u_\infty \in L^2(\Omega)$ is called the far-field pattern of the scattered field $u^s$.

A particular case of importance in our framework occurs when the incident field is a plane wave propagating along the direction identified by the unit vector $\hat{d} \in \Omega$, i.e., $u^t(x, \hat{d}) = e^{ikx \cdot \hat{d}}$; the far-field pattern of the corresponding scattered field $u^s(x, \hat{d})$ will be denoted with $u_\infty(\hat{x}, \hat{d})$. Then, the inverse problem we are interested in can be formulated as follows.

**Problem 2.2.** Given the far-field pattern $u_\infty(\hat{x}, \hat{d})$ for all incidence and observation directions $\hat{x}, \hat{d} \in \Omega$, determine the boundary $\partial D$ of the obstacle.

A qualitative approach to solving Problem 2.2 is provided by the LSM. The formulation of this method relies on the following mathematical tools:

a) the far-field operator $F : L^2(\Omega) \to L^2(\Omega)$, with $Fg(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, \hat{d})g(\hat{d})d\sigma(\hat{d})$.

b) the far-field pattern $\Phi(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x} \cdot z}$ of the fundamental solution $\Phi(x, z) = \frac{1}{4\pi} e^{-ik|x-z|}$ of the Helmholtz equation;

c) the far-field equation, written (for each sampling point $z \in \mathbb{R}^3$) in the unknown $g_z \in L^2(\Omega)$ as

$$Fg_z(\hat{x}) = \Phi(\hat{x}, z), \quad \hat{x} \in \Omega.$$  

(2.4)

It is worth noting that, in general, this equation is not solvable for almost all (f.a.a.) sampling point $z \in \mathbb{R}^3$, i.e., $\Phi_z(\cdot, z)$ does not belong to the range of $F$ f.a.a. $z \in \mathbb{R}^3$.

However, this range is dense in $L^2(\Omega)$, as explained in the following point d);

d) the general theorem, as we call it, which can be stated as follows.

**Theorem 2.1.** (General theorem) Let $D \subset \mathbb{R}^3$ be a nonempty, open, bounded and $C^2$-domain, such that $\mathbb{R}^3 \setminus D$ is connected; let us assume that $k^2$ is not a Dirichlet eigenvalue for the negative Laplacian in $D$. Then:

(i) if $z \in D$, for every $\varepsilon > 0$ there exists a solution $g_z^\varepsilon \in L^2(\Omega)$ of the inequality $\|Fg_z^\varepsilon - \Phi(z, \cdot)\|_{L^2(\Omega)} \leq \varepsilon$ such that, for every $z^* \in \partial D$,

$$\lim_{\varepsilon \to 0^+} \|g_z^\varepsilon\|_{L^2(\Omega)} = \infty \quad \text{(a)} \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \|v_{g_z^\varepsilon}\|_{H^1(\Omega)} = \infty, \quad \text{(b)}$$

(2.5)

where $v_{g_z^\varepsilon}$ is the Herglotz wave function with kernel $g_z^\varepsilon$;

(ii) if $z \notin D$, for every $\varepsilon > 0$ and $\delta > 0$ there exists a solution $g_z^{\varepsilon, \delta} \in L^2(\Omega)$ of the inequality $\|Fg_z^{\varepsilon, \delta} - \Phi(z, \cdot)\|_{L^2(\Omega)} \leq \varepsilon + \delta$ such that

$$\lim_{\delta \to 0} \|g_z^{\varepsilon, \delta}\|_{L^2(\Omega)} = \infty \quad \text{(a)} \quad \text{and} \quad \lim_{\delta \to 0} \|v_{g_z^{\varepsilon, \delta}}\|_{H^1(\partial D)} = \infty, \quad \text{(b)}$$

(2.6)
where \( v_{g;\,\cdot;\,\cdot} \) is the Herglotz wave function with kernel \( g_{z;\,\cdot;\,\cdot}^{\cdot;\,\cdot} \).

Theorem 2.1 establishes, in particular, the denseness of the range of \( F \), as well as the existence of \( \epsilon \)-approximate solutions of the far-field equation such that their \( L^2 \)-norm can be used as an indicator function for the support of the scatterer, i.e., a function that is bounded inside the obstacle, blows up on its boundary and remains arbitrarily large outside, as indicated by limits (2.5)(a) and (2.6)(a).

Then, inspired by Theorem 2.1, the algorithm of the LSM can be summarized by the following steps: 1) choose a grid of sampling points \( z \) in a bounded region \( Z \) containing \( D \); 2) for each \( z \in Z \), compute the Tikhonov regularized solution \([14, 21]\) \( g_z^{\alpha} \) of the far-field equation \( (2.4) \), written in an angle-discretized and noisy version; 3) fix the value \( \alpha^*(z) \) of the regularization parameter \( \alpha \) by means of the generalized discrepancy principle \([21]\); 4) visualize the boundary \( \partial D \) of the scatterer as the set of grid points where the (discretized) \( L^2 \)-norm of \( g_{\alpha^*(z)} := g_z^{\alpha^*(z)} \) grows up.

Evidently, there is a gap between Theorem 2.1 and the previous algorithm, which is not only due to the discretization and the noise affecting the far-field pattern, but is much deeper: indeed, a priori there is no reason why the regularized solution of the far-field equation for each \( z \) should behave (when regarded as a function of \( z \)) as an indicator function, i.e., in agreement with limits (2.5)(a) and (2.6)(a). In particular, no useful convergence result can be obtained as \( \alpha \to 0^+ \): indeed, since the range \( \mathcal{R}(F) \) of the far-field operator \( F \) is dense in \( L^2(\Omega) \) and \( \Phi_{\infty}(\cdot, z) \) does not belong, in general, to \( \mathcal{R}(F) \), we have that \( \lim_{\alpha \to 0^+} \|g_z^{\alpha}\|_{L^2(\Omega)} = \infty \) f.a.a. \( z \in \mathbb{R}^3 \) \([14, 21]\).

This is also the reason why an alternative version of part (b) of Theorem 2.1, as proposed e.g. at p. 166 of \([18]\), does not fill in this gap: indeed, according to \([18]\), for \( z \notin D \) one can in particular identify the regularization parameter \( \alpha \) with the tolerance \( \epsilon \) and then find

\[
\lim_{\alpha \to 0^+} \|Fg_z^{\alpha} - \Phi_{\infty}(\cdot, z)\|_{L^2(\Omega)} = 0, \quad \lim_{\alpha \to 0^+} \|g_z^{\alpha}\|_{L^2(\Omega)} = \infty. \tag{2.7}
\]

Nevertheless, these two limits hold f.a.a. \( z \in \mathbb{R}^3 \) and not only for \( z \notin D \), since they follow from the denseness of \( \mathcal{R}(F) \) and from the fact that \( \Phi_{\infty}(\cdot, z) \notin \mathcal{R}(F) \) f.a.a. \( z \in \mathbb{R}^3 \) (cf. \([14]\), p. 54 and \([21]\), p. 19). Hence, the typical behavior of \( z \mapsto \|g_z^{\alpha}(z)\|_{L^2(\Omega)} \) as an indicator function cannot be explained by simply considering that \( \alpha^*(z) \to 0^+ \): at least, one should prove a complementary property, whereby a vanishing regularization parameter is only enforced when \( z \in D \) approaches the boundary \( \partial D \), or when \( z \notin D \). (In fact, our approach will provide such enforcement, as we shall explain in Remarks 7.2 and 8.1.)

For future purpose, we briefly recall that analogous results and problems hold for an aspect-limited configuration of antennas \([8, 18]\). Let \( \Omega' \) and \( \Omega'' \) be the two (measurable) subsets of \( \Omega \) identifying the incidence and observation angles, respectively. Then, we can define the modified far-field operator \( F_{al} : L^2(\Omega') \to L^2(\Omega'') \) and introduce the modified far-field equation \( F_{al} g_z(x) = \Phi_{\infty}(x, z) \) in the unknown \( g_z \in L^2(\Omega'') \), for \( x \in \Omega'' \). By definition, an \( \epsilon \)-approximate solution \( g_z^{\epsilon} \) of this equation verifies the inequality \( \|F_{al} g_z^{\epsilon} - \Phi_{\infty}(\cdot, z)\|_{L^2(\Omega'')} \leq \epsilon \). We shall come back to the aspect-limited case later on.

Among the efforts to overcome this theoretical drawback, the FM certainly plays a major role: then, the next section is devoted to briefly describing this qualitative method and the insight it can give into the LSM.

Throughout the paper, we shall adopt the following notations for spheres and
balls: if $x_0 \in \mathbb{R}^3$ and $R > 0$, then

\[
B(x_0, R) := \{ x \in \mathbb{R}^3 : |x - x_0| < R \}, \quad B_R := \{ x \in \mathbb{R}^3 : |x| < R \}, \quad B := B_1; \quad (2.8)
\]

\[
\Omega(x_0, R) := \{ x \in \mathbb{R}^3 : |x - x_0| = R \}, \quad \Omega_R := \{ x \in \mathbb{R}^3 : |x| = R \}, \quad \Omega := \Omega_1. \quad (2.9)
\]

However, the symbols denoting centers and radii may vary, depending on the context.

### 3. The LSM justified by the FM.

The classical Problem 2.1 can be regarded as a particular case of the following weak boundary-value problem, which involves the space $H_{loc}^1(\mathbb{R}^3 \setminus D) := \{ u : u \in H^1( (\mathbb{R}^3 \setminus D) \cap B_R) \forall R > 0 : (\mathbb{R}^3 \setminus D) \cap B_R \neq \emptyset \}$.

**Problem 3.1.** Given $\varphi \in H^{1/2}(\partial D)$, determine $u^* \in H_{loc}^1(\mathbb{R}^3 \setminus D)$ such that

\[
\begin{aligned}
\Delta u^* + k^2 u^* &= 0 & \text{in } \mathbb{R}^3 \setminus D, & & (a) \\
u^* &= \varphi & \text{on } \partial D, & & (b) \\
\lim_{r \to \infty} |r \left( \frac{\partial u^*}{\partial r} - iku^* \right)| &= 0. & & (c)
\end{aligned}
\]

Problem 3.1 is well-posed [8]: as a trivial consequence, the next problem is too.

**Problem 3.2.** Given $\varphi \in H^{1/2}(\partial D)$, determine the far-field pattern $u_\infty \in L^2(\Omega)$ of the field $u^* \in H_{loc}^1(\mathbb{R}^3 \setminus D)$ solving Problem 3.1.

As proved in [17], the far-field operator $F$ having as integral kernel the far-field pattern determined by Problem 3.2 is normal (but only for a full-view configuration of antennas [8, 18]); moreover, if $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $D$ (as we shall always assume throughout the paper), $F$ is injective with dense range. The solution operator of Problem 3.2 is denoted by $G : H^{1/2}(\partial D) \to L^2(\Omega)$, with $G(\varphi) := u_\infty$. In particular, remembering that $Fg$ is the far-field pattern of the field $u^*_\infty(x)$ given by (2.3), we have $G(-v_\infty|_{\partial D}) = Fg$. This amounts to establishing the factorization $F = GH$, where $H : L^2(\Omega) \to H^{1/2}(\partial D)$ is the Herglotz operator defined as $Hg(x) := \int_{\Omega} e^{ikx \cdot \hat{d}} g(\hat{d}) d\sigma(\hat{d})$, for $x \in \partial D$. Moreover, two important properties can be established:

1. $G$ provides an exact characterization of $D$, i.e., $\Phi_\infty(\cdot, z) \in \mathcal{R}(G) \iff z \in D$;
2. $\mathcal{R}(G) = \mathcal{R}(F^* F)^{1/4}$, where $F^*$ denotes the adjoint of $F$.

The two previous points suggest to replace the far-field equation (2.4), involved in the LSM, with the following one:

\[
(F^* F)^{1/4} \tilde{g}_z(\hat{x}) = \Phi_\infty(\hat{x}, z),
\]

which forms the core of the FM and provides an exact characterization of the domain $D$, since it is (uniquely) solvable in $L^2(\Omega)$ if and only if $z \in D$. Moreover, since $\mathcal{R}(F^* F)^{1/4}$ is dense in $L^2(\Omega)$, using Tikhonov regularization to approxi-mately solve (3.2) is justified by two convergence/divergence results: a) for $z \in D$, $
\lim_{\alpha \to 0^+} \| \tilde{g}_z \|_{L^2(\Omega)} = \| \tilde{g}_z \|_{L^2(\Omega)} \in \mathbb{R}^+$, where $\tilde{g}_z$ is the unique solution of equation (3.2); b) for $z \notin D$, $
\lim_{\alpha \to 0^+} \| \tilde{g}_z \|_{L^2(\Omega)} = \infty$.

For future purpose, we also observe that, even in absence of noise, the FM (as well as the LSM) has only a limited resolution power, i.e., the indicator function $\| \tilde{g}_z \|_{L^2(\Omega)}$ cannot exhibit a step-like behavior in the transition from the interior to the exterior of the scatterer, but rather becomes larger and larger as the sampling point $z$ approaches $\partial D$ from inside (and remains arbitrarily large outside). This property,
which is analogous to the limit (2.5)(a) for the LSM, is formalized by the following theorem (for a proof, see e.g. [4]).

**Theorem 3.1.** For each \( z \in D \), let \( \tilde{g}_z \in L^2(\Omega) \) be the unique solution of equation (3.2). Then, for any \( z^* \in \partial D \), the limit \( \lim_{z \to z^*} \| \tilde{g}_z \|_{L^2(\Omega)} = \infty \) holds.

The FM can also be extended to some situations where the far-field operator is not normal, but it remains, so far, considerably less general than the LSM [8]. However, the theoretical foundation of the latter is notably improved by the FM, when applicable. Then, let us briefly recall how this improvement can be achieved.

The first step, made in [3], is to identify an appropriate family of regularization methods \( \tilde{R}_\alpha \) for the far-field operator \( F \), i.e., \( \tilde{R}_\alpha h := \sum_{n=1}^{\infty} q_{\{\alpha_n, g_n, h_n\}}^n(h, h_n) g_n \), where \( \{\mu_n, g_n, h_n\}_{n=1}^\infty \) is a singular system of \( F \) such that \( h_n = s_n g_n \) with \( s_n \in \mathbb{C} \), \( |s_n| = 1 \), and \( q : (0, +\infty) \times (0, \|F\|) \to \mathbb{R} \) verifies three conditions detailed in [3]. For our purposes, it suffices to observe that Tikhonov regularization is obtained from the particular choice \( q(\alpha, \mu) = \frac{\mu^2}{\alpha + \mu^2} \). Then, the following theorem can be proved [3].

**Theorem 3.2.** If \( \tilde{R}_\alpha : L^2(\Omega) \to L^2(\Omega) \) is a regularization method for \( F \) as above, then \( R_\alpha := -H \tilde{R}_\alpha : L^2(\Omega) \to H^{1/2}(\partial D) \) is a regularization method for \( G \).

In our framework, this theorem is important and can be paraphrased as follows. It is well known [12] that having a control over the norm of far-field patterns does not allow, in general, a control over the corresponding fields outside the scatterer. In other terms, small (in the \( L^2(\Omega)-\)norm) far-field patterns do not need to correspond to small (in any reasonable norm) fields in a finite region surrounding the obstacle. In particular, when the Tikhonov regularized solution \( g^\alpha_z \) of the far-field equation (2.4) is considered (for \( z \in D \)), the fact that the far-field pattern \( Fg^\alpha_z - \Phi(\cdot, z) \) can be made arbitrarily small in \( L^2(\Omega) \) for \( \alpha \to 0^+ \) does not imply that the field characterized by this far-field pattern is small in \( H^1(\partial B_R \setminus D) \), with \( B_R \supset D \). However, Theorem 3.2, by exploiting the normality of \( F \), states just this implication: indeed, by this theorem, we have that, for \( z \in D \), \( -Hg^\alpha_z \to \Phi(\cdot, z) \) in \( H^{1/2}(\partial D) \) as \( \alpha \to 0^+ \) and then the well-posedness of Problem 3.1, together with definition (2.3), implies that \( u^\alpha_{gz} \to \Phi(\cdot, z) \) in \( H^1(B_R \setminus D) \) as \( \alpha \to 0^+ \). In other terms, establishing that the regularization of \( F \) is inherited by \( G \), in the sense of Theorem 3.2, amounts to stating that this regularization, together with the operator \( -H \), is responsible for the stable back-propagation (up to the scatterer boundary) of the information expressed in the far-field region by the far-field equation. We shall revisit this point in the next section.

Another important consequence of Theorem 3.2, developed in [4, 18] (for Tikhonov regularization only), consists in the possibility of replacing the usual indicator function of the LSM, i.e., \( z \mapsto \|g^\alpha_z\|_{L^2(\Omega)} \), with the absolute value of the Herglotz wave function with kernel \( g^\alpha_z \), i.e., with \( z \mapsto |v_{gz}(z)| \); indeed, if the FM can be applied, it is possible to show that the new indicator function satisfies good convergence/divergence properties, i.e.,

\[
\lim_{\alpha \to 0^+} |v_{gz}(z)| < \infty \quad \text{for} \quad z \in D \quad \text{(a)} \quad \text{and} \quad \lim_{n \to \infty} |v_{g_{zn}}(z)| = \infty \quad \text{for} \quad z \notin \hat{D} \quad \text{(b)} \quad (3.3)
\]

for an appropriate vanishing sequence \( \{\alpha_n\}_{n=0}^\infty \). The two limits in (3.3) give rise to a new and mathematically justified version of the LSM: it is understood that the gap with respect to the algorithm actually implemented in numerical applications can be filled by assuming that discretization effects are not important and that the presence of noise affecting the far-field operator entails, in any case, ‘small’ values of the regularization parameter \( \alpha \).
Summing up, the FM can give a satisfactory justification of the LSM, provided that 1) the class of scattering conditions is conveniently restricted; 2) the traditional indicator function \( z \mapsto \| g_\alpha^z \|_{L^2(\Omega)} \) is replaced by \( z \mapsto v g_\alpha^z(z) \); 3) the regularization parameter is assumed to tend to zero.

In the following, we shall face the problem of understanding the LSM in a completely different perspective, by relying on a physical interpretation of the far-field equation in terms of power fluxes and by investigating how these fluxes are propagated throughout the background. In this framework, 1) the class of scattering conditions does not need to be restricted (anyway, only the impenetrable case will be explicitly discussed here; for two-dimensional penetrable scatterers, see [2]); 2) the usual indicator function \( z \mapsto \| g_\alpha^z \|_{L^2(\Omega)} \) is maintained; 3) the regularization parameter is not assumed to tend to zero.

However, we point out that, unlike the previous \textit{a priori}, FM-based justification, our approach is an \textit{a posteriori} one, since we need to make some assumptions on the geometry of the power fluxes in the background: although not based on theoretical predictions, these assumptions are supported by numerical simulations (also when \( F \) is not normal) and generalize in a rather natural way the main features of the field radiated by a point-like source placed in the host medium. In spite of this theoretical limitation, we shall be able to provide sufficient conditions ensuring that the \( L^2 \)-norm of any \( \epsilon \)-approximate (and, in particular, Tikhonov regularized) solution of the far-field equation behaves as a good indicator function for the support of the scatterer as \( z \) varies in \( \mathbb{R}^3 \).

4. Flow tubes of the scattered field. Numerical simulations show that the LSM can work even when the FM is not applicable: this suggests that the back-propagation of the constraint expressed by the far-field equation from the far-field to the near-field region is not necessarily related to the normality of \( F \) and to the regularization of the far-field equation, as discussed in the previous section. Rather, it seems to be related to the physics of wave propagation or, more precisely, to how power is carried throughout the background.

To pursue this approach, we recall that time-harmonic sound waves of small intensity, propagating in a homogeneous, isotropic and inviscid fluid surrounding the obstacle \( D \), can be described [12] by a velocity potential \( U(x,t) = \text{Re} \{ u(x)e^{-i\omega t} \} \) such that

\[
v(x) = \frac{1}{\rho_0} \nabla u(x) \quad \text{and} \quad p(x) = i\omega u(x) \Rightarrow v(x) = \frac{1}{i\omega\rho_0} \nabla p(x),
\]

where \( p = p(x) \) and \( v = v(x) \) are the (complex-valued) velocity and excess pressure of the fluid, respectively, while \( \rho_0 \) is its constant equilibrium density. In particular, the second equality in (4.1) shows that \( p \) and \( u \) differ by a multiplicative constant only; then, Problems 2.1 and 2.2, although written in terms of \( u, u^s, u^t \), can also be regarded as a problem for \( p, p^s, p^t \), as we shall do in the following. Of course, the physical pressure and velocity fields are given by:

\[
v_{ph}(x,t) = \text{Re} \{ v(x)e^{-i\omega t} \} \quad \text{and} \quad p_{ph}(x,t) = \text{Re} \{ p(x)e^{-i\omega t} \}.
\]

Let us now consider a surface \( \Sigma \subset \mathbb{R}^3 \setminus \bar{D} \) admitting a piecewise smooth parametric representation \( \tilde{\Sigma} : A \to \mathbb{R}^3 \), with \( A \subset \mathbb{R}^2 \) and \( \tilde{\Sigma}(A) = \Sigma \) (throughout the paper, the piecewise smoothness of the parametric representations of surfaces or curves will be always understood); moreover, let \( \nu = \nu(x) \) be the unit normal to \( \Sigma \) at \( x \), chosen as outward when \( \Sigma \) is closed. The traction forces induced in the background fluid by the
scattered pressure field $p^s$ develop a power per unit surface $P(x, t) = p^s_{ph}(x, t) v^s_{ph}(x, t) \cdot \nu(x)$, outgoing from $\Sigma$; by using relations (4.1) and (4.2), this power can be computed as

$$P(x, t) = \frac{1}{2} \text{Re} \left\{ e^{-2i\omega t} p^s(x) \nabla p^s(x) + \frac{1}{i\omega \rho_0} p^s(x) \nabla \bar{p}^s(x) \right\} \cdot \nu(x), \quad (4.3)$$

where the dot and the bar denote the canonical scalar product in $\mathbb{C}^3$ and the complex conjugation respectively. Expression (4.3) suggests computing the average $P_{av}(x)$ of $P(x, t)$ over one period $T = 2\pi/\omega$, i.e.,

$$P_{av}(x) = \frac{1}{T} \int_0^T P(x, t) dt = \frac{1}{2} \text{Re} \left\{ \frac{1}{i\omega \rho_0} \bar{p}^s(x) \nabla p^s(x) \right\} \cdot \nu(x). \quad (4.4)$$

Then, the average power flux across the surface $\Sigma$ is given by

$$F_\Sigma(p^s) = \frac{1}{2} \int_{\Sigma} \text{Re} \left\{ \frac{1}{i\omega \rho_0} \bar{p}^s(x) \nabla p^s(x) \right\} \cdot \nu(x) d\sigma(x), \quad (4.5)$$

where $d\sigma(x)$ is the standard measure on $\Sigma$. The right-hand side of (4.5) is the flux across $\Sigma$ of the vector field $S$ associated with $p^s$ via the following definition:

$$S(x) := \frac{1}{2} \text{Re} \left\{ \frac{1}{i\omega \rho_0} \bar{p}^s(x) \nabla p^s(x) \right\} = \frac{1}{4\omega \rho_0} \left\{ \bar{p}^s(x) \nabla p^s(x) - p^s(x) \nabla \bar{p}^s(x) \right\}, \quad (4.6)$$

which is analogous to the Poynting vector in the electromagnetic case [2]. For brevity, we shall speak of the ‘flux of the scattered field’, or the ‘flux of $p^s$’.

Since $p^s$ satisfies the Helmholtz equation (2.1)(a) in the background fluid, it is easy to show that the vector field $S(x)$ is divergence free in $\mathbb{R}^3 \setminus D$. As a consequence, if $E \subset \mathbb{R}^3$ is a compact integration domain such that its boundary $\Sigma := \partial E$ is a simple and closed surface enclosing no subset of $D$, by Gauss’ divergence theorem we have $F_\Sigma(p^s) = 0$. For the same reason, if $E_1, E_2 \subset \mathbb{R}^3$ are two compact domains such that $E_1 \subset E_2$ and their boundaries $\Sigma_1$ and $\Sigma_2$ are non-intersecting, simple and closed surfaces surrounding the whole scatterer $D$, by applying Gauss’ theorem in $E_2 \setminus E_1$ we have

$$F_{\Sigma_1}(p^s) = F_{\Sigma_2}(p^s). \quad (4.7)$$

This is consistent with the fact that the scatterer can be regarded [5] as an unknown equivalent source radiating the scattered field $p^s$ in the background: if there are no sources inside $\Sigma$, the flux of $p^s$ across $\Sigma$ is zero; if $\Sigma$ encloses the whole source, the flux across $\Sigma$ is completely determined by the source itself. In the latter case, one can choose $\Sigma = \Omega_R$ in (4.5), with $R$ large enough. Then, remembering (4.6), any of the fluxes in (4.7) can be computed as the flux at infinity of $p^s$, i.e.,

$$F_\infty(p^s) := \lim_{R \to \infty} \int_{\Omega_R} S(x) \cdot \nu(x) d\sigma(x) =$$

$$= \frac{1}{4\omega \rho_0} \lim_{R \to \infty} \int_{\Omega} \left[ \bar{p}^s \frac{\partial p^s}{\partial r} - p^s \frac{\partial \bar{p}^s}{\partial r} \right] (R, \hat{x}) R^2 d\sigma(\hat{x}), \quad (4.8)$$

where $(R, \hat{x})$ are modulus and direction of $x \in \Omega_R$, while $d\sigma(\hat{x}) = \sin \theta d\theta d\varphi$ is the surface element on the unit sphere $\Omega$. In order to compute the limit (4.8), we observe
that the Sommerfeld radiation condition (2.1)(c) and the asymptotic behavior (2.2) together imply
\[
\frac{\partial p^*}{\partial r} = ik\frac{e^{ikr}}{r}p_\infty(\hat{x}) + o(r^{-1}) \quad \text{as } r \to \infty.
\] (4.9)

Hence, by using (2.2) and (4.9), we obtain
\[
\hat{p}^*\frac{\partial p^*}{\partial r}(R, \hat{x}) = \frac{ik}{k^2} |p_\infty(\hat{x})|^2 + o(R^{-2}) \quad \text{as } R \to \infty,
\] (4.10)
uniformly in \(\hat{x}\). Then, by substituting (4.10) into (4.8), we find
\[
\mathcal{F}_\infty(p^*) = \frac{k}{2\omega \rho_0} \|p_\infty\|_{L^2(\Omega)}^2.
\] (4.11)

We now point out that equality (4.7) is a way to express, at a global level, the conservation of energy inside a lossless background. However, for our purposes, it is also interesting to study how power is locally transported (and conserved) from the near-field to the far-field region. In this framework, a key-role is played by the flow lines of the vector field \(S(x)\) associated with \(p^*\) (also called, for brevity, ‘flow lines of \(p^*\)). For any point \(x_0 \in \mathbb{R}^3 \setminus \bar{D}\), the flow line \(\zeta_{x_0}(\tau)\) of \(S(x)\) starting from \(x_0\) is the unique solution (considered only for \(\tau \geq 0\)) of the Cauchy problem
\[
\frac{dx}{d\tau}(\tau) = S(x(\tau)), \quad \text{with } x(0) = x_0.
\] (4.12)
The local existence and uniqueness of the solution to problem (4.12) are ensured by the fact that \(p^* \in C^2(\mathbb{R}^3 \setminus \bar{D})\), as stated in Problem 2.1, and then \(S \in C^1(\mathbb{R}^3 \setminus \bar{D})\) by definition (4.6). The case \(x_0 \in \partial D\) is more delicate: while existence of solutions still holds [20], uniqueness does not, since \(p^* \in C^{1,\alpha}(\mathbb{R}^3 \setminus \bar{D})\), then \(S \in C^{0,\alpha}(\mathbb{R}^3 \setminus \bar{D})\) and, in particular, \(\bar{S}\) may not be Lipschitz on domains containing subsets of \(\partial D\). Thus \(x_0 \in \partial D\) may be a ramification point for the solutions of (4.12): we shall discuss this issue in Section 8. Finally, if \(x_0\) is such that \(S(x_0) = 0\), the flow line starting from \(x_0\) collapses into \(x_0\) itself, which is then called a critical point of \(S\) [1]. In our framework, we shall focus on non-critical initial points \(x_0\) and assume that the flow lines are defined for all \(\tau \geq 0\).

We now focus on flow tubes as bundles of flow lines. Let \([a, b]\) be a non-empty and closed interval in \(\mathbb{R}\) and \(\tilde{\gamma} : [a, b] \to \mathbb{R}^3\) the parametric representation of a simple and closed curve, i.e., \(\tilde{\gamma}\) is an injective map on \([a, b]\), but \(\tilde{\gamma}(a) = \tilde{\gamma}(b)\). We assume that the image \(\gamma := \tilde{\gamma}(a, b)\) is transverse with respect to the flow lines, i.e., we require that, for any point \(x_0 \in \gamma\), the image \(\zeta_{x_0} := \tilde{\zeta}_{x_0}([0, +\infty))\) of the flow line \(\tilde{\zeta}_{x_0}(\tau)\) starting from \(x_0\) intersects \(\gamma\) only in \(x_0\) itself. Then, for each \(x_0 \in \gamma\), consider \(\zeta_{x_0}(\tau)\) and define the bundle of (images of) flow lines \(T_\gamma := \cup_{x_0 \in \gamma} \zeta_{x_0}\); if \(T_\gamma\) is homotopic to a cylinder, it will be called a flow tube of \(p^*\) (starting from \(\gamma\)). Any curve like \(\gamma\) (i.e., simple, closed and transverse) will be named a transverse section of the flow tube \(T_\gamma\).

We observe that \(T_\gamma \subset \mathbb{R}^3 \setminus \bar{D}\), since the field \(p^*\) is only defined on \(\mathbb{R}^3 \setminus \bar{D}\).

Flow tubes are responsible for the transport of power (i.e., of information) in the background. In particular, they are the key-tool for understanding how the constraint expressed in the far-field region by the far-field equation can contain information on the shape and location of the scatterer, which is placed in the near-field region. To this end, we need to assume that flow tubes verify some regularity conditions. First,
a flow tube starting from a curve $\gamma$ should reach the far-field region: in particular, it should not refoel on the scatterer. Second, in the far-field region this tube should identify a definite set of directions $W_{\infty}(\gamma) \subset \Omega$, so that a local counterpart of (4.11), with $L^2(\Omega)$ replaced by $L^2(\bar{W}_{\infty}(\gamma))$, could be considered (cf. (4.14) in the following).

The latter requirement is made rather natural by the fact that the vector field $\mathcal{S}$ tends to become radial in the far-field region. This can be shown by analyzing the behavior of the three spherical components of $\nabla p^s(r, \theta, \varphi) = \frac{\partial p^s}{\partial r}\hat{e}_r + \frac{1}{r} \frac{\partial p^s}{\partial \theta}\hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial p^s}{\partial \varphi}\hat{e}_\varphi$, where $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi)$ are the intrinsic coordinate vectors associated to a point of spherical coordinates $(r, \theta, \varphi)$. From (4.9), we have that the radial component of $\nabla p^s$ decays as $1/r$ when $r \to \infty$; instead, the transverse components of $\nabla p^s$ are $O(r^{-2})$ as $r \to \infty$, as shown by using the series representation (see [11], pp. 72-75).

$\Phi^s(x) = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{\Phi_n(\theta, \varphi)}{r^n}$, which holds for all $r > R$ such that $\Omega_R \subset \mathbb{R}^3 \setminus D$, and can be differentiated term by term with respect to $r, \theta, \varphi$ any number of times. Then, remembering relations (4.6) and (4.10), we can conclude that the radial component of $\mathcal{S}$ is increasingly dominant over the transverse ones in the far-field region.

Let us now formalize the previous discussion in the following definition.

**Definition 4.1.** Given a flow tube $T_\gamma \subset \mathbb{R}^3 \setminus D$ as above, it is called ‘regular’ if there exist $R_0 > 0$ and $a, b \in \mathbb{R}$, with $a < b$, such that the set $T_{R_0} := T_\gamma \setminus B_{R_0}$ admits a parametrization defined on $A := [R_0, +\infty) \times [a, b] \subset \mathbb{R}^2$, i.e.,

$$\bar{T}_{R_0} : A \to \mathbb{R}^3, \quad (R, s) \mapsto \bar{T}_{R_0}(R, s),$$

with the following properties:

(i) $\forall \bar{R} \geq R_0$, the map $\bar{T}_{R_0}(R, \cdot) : [a, b] \to \mathbb{R}^3$ is the parametric representation of a transverse section $\gamma_R$ of $T_{\gamma}$, with $\gamma_R = T_{\gamma} \cap \Omega_R$;

(ii) $\forall s_0 \in [a, b]$, the point $\bar{T}_{R_0}(R, s_0) \in \mathbb{R}^3$ tends to a definite direction as $R \to \infty$, i.e., $\exists \lim_{R \to \infty} \frac{\bar{T}_{R_0}(R, s_0)}{R} = : \bar{\gamma}_{\infty}(s_0) \in \Omega$, where $\bar{\gamma}_{\infty}(\cdot) : [a, b] \to \Omega$ is the parametric representation of a simple and closed curve $\gamma_{\infty} \subset \Omega$.

Finally, the open and connected domain enclosed by $\gamma_{\infty}$ on $\Omega$ is denoted with $W_{\infty}(\gamma)$, and its area $|W_{\infty}(\gamma)|$ is called the ‘asymptotic angular width’ of $T_\gamma$.

Apart from technicalities, Definition 4.1 simply describes a generalization of some relevant features of the field radiated by a point-like source in a homogeneous background, whereby the flow lines spread out from the source itself and can be continued up to infinity according to an asymptotically definite direction, with no narrowing of the corresponding flow tubes.

Under the previous regularity assumption, it is easy to describe how power is locally preserved and carried throughout the background. Given a regular flow tube $T_{\gamma}$, consider a finite portion $\Sigma_1$ and $\Sigma_2$ be two open and bounded surfaces with boundaries $\gamma_1$ and $\gamma_2$. Moreover, let $\Sigma_1$ and $\Sigma_2$ be two open and bounded surfaces with boundaries $\gamma_1$ and $\gamma_2$, respectively, such that $\Sigma := T_{\gamma} \cup \Sigma_1 \cup \Sigma_2$ is a simple and closed surface enclosing no subsets of $D$. Then, the power flux of $p^s$ across $\Sigma$ vanishes, as stated soon before equality (4.7). Since the lateral boundary $T_{\gamma}^{12}$ of $S$ is entirely formed by flow lines of $p^s$, i.e., lines that are tangent in each point to the vector field $\mathcal{S}$, the flux of $p^s$ across $T_{\gamma}^{12}$ is zero too, as shown by relations (4.5) and (4.6). Hence, by possibly interchanging the names of $\Sigma_1$ and $\Sigma_2$, we can choose the orientations of the normals to $\Sigma_1$ and $\Sigma_2$ in such a way that the flux of $p^s$ through $\Sigma_1$ entering into the field tube is equal to the flux through $\Sigma_2$ outgoing from it, and also equal to the flux at infinity outgoing from $W_{\infty}(\gamma)$, i.e.,

$$\mathcal{F}_{\Sigma_1}(p^s) = \mathcal{F}_{\Sigma_2}(p^s) = \mathcal{F}_{W_{\infty}(\gamma)}(p^s) = \frac{k}{2\omega p_0} \|p^s\|^2_{L^2(\bar{W}_{\infty}(\gamma))},$$

(4.14)
where the last inequality holds as a local version of (4.11). Property (4.14) expresses the transmission (and conservation) of power inside each regular flow tube.

Also for future purpose, it is interesting to focus on the fundamental solution $\Phi(x, \omega) = \frac{1}{4\pi} e^{-ik|x-z|}$ of the Helmholtz equation. In absence of scatterers, $\Phi_\omega := \Phi(\cdot, z)$ is the radiating field generated in the background by an isotropic point source placed at $z$. It is easy to realize that the flow lines of $\Phi_\omega$ are half-lines outgoing from $z$ (but not defined at the singularity point $z$). Since the far-field pattern of $\Phi_\omega$ is given by $\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi} e^{-ik \hat{x} \cdot z}$, we can use (4.11) to compute the flux at infinity of $\Phi_\omega$ as

$$\mathcal{F}_\infty(\Phi_\omega) = \frac{k}{8\pi \omega \rho_0}, \quad (4.15)$$

More generally, since $\frac{1}{4\pi} e^{-ik \hat{x} \cdot z} = \frac{1}{4\pi}$, the flux of $\Phi_\omega$ conveyed by any one of its flow tubes (which are portions of cones with vertex at $z$) is expressed according to (4.14) as

$$\mathcal{F}_{\Sigma_1}(\Phi_\omega) = \mathcal{F}_{\Sigma_2}(\Phi_\omega) = \mathcal{F}_{W_\infty(\gamma)}(\Phi_\omega) = \frac{k}{2\omega \rho_0} \|\Phi_\omega\|^2_{L^2(W_\infty(\gamma))} = \frac{k}{8\pi \omega \rho_0} \frac{\beta[W_\infty(\gamma)]}{4\pi},$$

where $\beta[W_\infty(\gamma)]$ is the amplitude (in steradians) of the solid angle identified by $W_\infty(\gamma) \subset \Omega$ and by the origin $O$ of $\Omega$ as its vertex.

The previous considerations naturally lead us to reformulate the constraint expressed by the far-field equation (2.4) as follows. Let $g^\epsilon_\omega \in L^2(\Omega)$ be an $\epsilon$-approximate solution of (2.4), i.e., such that $\|Fg^\epsilon_\omega - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} \leq \epsilon$. The last inequality immediately implies that $\|Fg^\epsilon_\omega - \Phi_\infty(\cdot, z)\|^2_{L^2(\Omega')} \leq \epsilon^2$ for any (measurable) subset $\Omega' \subset \Omega$. Then, a simple computation shows that

$$\|Fg^\epsilon_\omega\|^2_{L^2(\Omega')} - \|\Phi_\infty(\cdot, z)\|^2_{L^2(\Omega')} \leq \epsilon'(\Omega'), \quad (4.17)$$

having defined $\epsilon'(\Omega') := \epsilon \left(2 \|\Phi_\infty(\cdot, z)\|_{L^2(\Omega')} + \epsilon\right)$. Note that $\|\Phi_\infty(\cdot, z)\|_{L^2(\Omega')}$ and consequently $\epsilon'(\Omega')$ are independent of $z$. Moreover, we recall that $Fg^\epsilon_\omega$ is the far-field pattern of the scattered field given by (2.3), i.e., with a slight change of notations,

$$p^\epsilon_\omega(x) := \int_{\Omega} p^\epsilon(x, \hat{d}) g^\epsilon_\omega(\hat{d}) d\sigma(\hat{d}) \quad \text{for } x \in \mathbb{R}^3 \setminus D, \quad (4.18)$$

and that the flux at infinity of any radiating field is proportional to the squared $L^2$-norm of its far-field pattern, as shown by (4.11) or by its local version (4.14). Then, relation (4.17) means that, for any observation solid angle $\Omega'$ in the far-field region, the flux of the scattered field $p^\epsilon_\omega$ and the flux of the elementary source $\Phi(\cdot, z)$ can be made arbitrarily close to each other, by choosing $\epsilon$ small enough.

Analogous considerations can be made for an aspect-limited configuration of antennas, as briefly described at the end of Section 2: in relation (4.17), it suffices to replace $F$ with the modified far-field operator $F_{\text{d}} : L^2(\Omega') \rightarrow L^2(\Omega')$ and to require that $\Omega'$ is any (measurable) subset of the region $\Omega'$ where the receivers are placed; in definition (4.18), the integration domain $\Omega'$ should be replaced by the set $\Omega'$ of incidence directions, but the field $p^\epsilon_\omega$ is anyway defined in all $\mathbb{R}^3 \setminus D$.

The next sections are devoted to understanding how the far-field constraint (4.17), back-propagated to the near-field region by regular flow tubes, can induce a behavior of the indicator function that is in agreement with limits (2.5)(a) and (2.6)(a).
5. Boundary power fluxes. In Section 3 we saw that the FM can provide a good indicator function $||\tilde{g}_z||_{L^2(\Omega)}$ for the support of the scatterer: in particular, let us now focus on the statement of Theorem 3.1. Apart from technicalities, this result can be explained as follows. Since $\mathcal{R}(G) = \mathcal{R}(F^*F)^{1/4}$, for each $\tilde{g} \in L^2(\Omega)$ the function $(F^*F)^{1/4} \tilde{g} \in L^2(\Omega)$ is the far-field pattern of a radiating solution $p^r[\tilde{g}]$ of the Helmholtz equation in $\mathbb{R}^3 \setminus D$. (Note that $p^r[\tilde{g}]$ is different from $p^r_\beta$, as it would be given by (2.3), and that, in general, it is not even attainable as a scattered field.) In particular, consider the solution $\tilde{g}_z$ of equation (3.2) for $z \in D$: the equality of the two far-field patterns $(F^*F)^{1/4} \tilde{g}_z(\hat{x})$ and $\Phi_\infty(\hat{x}, z)$ implies, by Rellich’s lemma, that the corresponding fields $p^r[\tilde{g}_z]$ and $\Phi(\cdot, z)$ are equal in $\mathbb{R}^3 \setminus D$. But $\Phi(\cdot, z)$ is singular in $z$: then, as $z \rightarrow z^* \in \partial D$, the field $p^r[\tilde{g}_z]$ should become singular too, and this is possible only if $\tilde{g}_z$ fails to be in $L^2(\Omega)$.

Now, the fact that, for $z \in D$, $p^r[\tilde{g}_z]$ and $\Phi(\cdot, z)$ coincide in $\mathbb{R}^3 \setminus \hat{D}$ implies that the fluxes of $p^r[\tilde{g}_z]$ and $\Phi(\cdot, z)$ across any surface in the background are equal: this simple observation suggests an interesting result and generalization in the framework of the LSM, as we are going to show.

To this end, we first need to consider the following geometrical construction (see Figure 5.1). Given $z^* \in \partial D$, let $U_{z^*} \subset \mathbb{R}^3$ be a neighborhood of $z^*$; for each $z \in U_{z^*} \cap D$, define $r_z := |z - z^*|$, $\Omega(z^*, r_z) := \{x \in \mathbb{R}^3 : |x - z^*| = r_z\}$, and $\bar{\Omega}(z^*, r_z) := \Omega(z^*, r_z) \cap (\mathbb{R}^3 \setminus D)$. Since $D$ is a $C^2$-domain, for each $z^* \in \partial D$ the tangent plane $\tau(z^*)$ is well-defined: then, for $r_z$ small enough (say $r_z < r_0$), $\Omega(z^*, r_z)$ is a connected and simply connected subset of $\Omega(z^*, r_z)$ that well approximates a half-sphere. Now, for $r_z < r_0$, consider the solid angle $\hat{z}(z^*, r_z)$ having its vertex in a point $z \in D \cap \Omega(z^*, r_z)$ and identifying the subset $\hat{\Omega}(z^*, r_z)$ on $\Omega(z^*, r_z)$: as $r_z$ decreases, $\hat{z}(z^*, r_z)$ better and better approximates the vertex angle of a cone inscribed in a half-sphere. Then, if we denote by $\beta_z := |\hat{z}(z^*, r_z)|$ the amplitude (in steradians) of $\hat{z}(z^*, r_z)$, it follows from Theorem A.1 in the Appendix A that $\beta_z$ is bounded from below, i.e., $\liminf_{z \rightarrow z^*} \beta_z = 2\pi \left(1 - \frac{\sqrt{2}}{2}\right) > 0$.

On the other hand, the flux across $\hat{\Omega}(z^*, r_z)$ due to the elementary and isotropic source placed at $z$ is directly proportional to $\beta_z$, as highlighted by (4.16). Then,
remembering the previous limit, we find

$$\lim_{z \to z^*} \mathcal{F}_{\tilde{\Omega}(z^*, r_z)}(\Phi_z) = \frac{k}{8\pi \omega \rho_0} \frac{2\pi (1 - \sqrt{\frac{\pi}{2}})}{4\pi} = \frac{k (2 - \sqrt{\frac{\pi}{2}})}{32\pi \omega \rho_0} > 0, \tag{5.1}$$

which states that, although the area of $\tilde{\Omega}(z^*, r_z)$ vanishes as $z \to z^*$, the flux of $\Phi_z$ across it has a positive lower bound. This is not surprising, since $\Phi_z$ is singular in $z$; however, it is interesting to observe that the very same property, in the FM, holds for the radiating field $p^s[\hat{g}_z]$, since $p^s[\hat{g}_z] = \Phi_z$ in $\mathbb{R}^3 \setminus \bar{D}$ (for $z \in D$).

Notably, if we now assume an analogous property for the scattered field considered in the LSM (cf. (4.18)), we can prove that the indicator function blows up as $z \to z^*$, in analogy with Theorem 3.1. This is shown by the following theorem, which makes no use of the far-field equation.

**Theorem 5.1.** Let $z^*$, $U_z$, $\Omega(z^*, r_z)$ and $\tilde{\Omega}(z^*, r_z)$ be as above. Suppose that a function $g_z \in L^2(\Omega)$ is associated to each $z \in U_z \cap D$. Consider the scattered field $p^s_z$ defined in analogy with (4.18), and assume that the corresponding flux across $\tilde{\Omega}(z^*, r_z)$ is bounded from below as $z \to z^*$, i.e., there exists $c > 0$ such that

$$\lim_{z \to z^*} \mathcal{F}_{\tilde{\Omega}(z^*, r_z)}(p^s_z) = c. \tag{5.2}$$

Then, $\lim_{z \to z^*} \|g_z\|_{L^2(\Omega)} = \infty$.

**Proof.** Assume, by contradiction, that the limit does not hold. Then, there exist a sequence $\{z_n\}_{n=0}^\infty \subset U(z^*) \cap D$ and a positive constant $K > 0$ such that, for all $n \in \mathbb{N}$, we have $\|g_{z_n}\|_{L^2(\Omega)} \leq K$. Setting $r_n := r_{z_n}$, we are going to obtain a contradiction with hypothesis (5.2) by proving that $\mathcal{F}_{\tilde{\Omega}(z^*, r_n)}(p^s_{z_n})$ vanishes as $n \to \infty$.

Let $\hat{d}(\theta, \varphi)$ be the unit vector of the direction identified by the spherical angles $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi]$ (the ambiguities arising for $\theta = \pi$ or $\varphi = 2\pi$ are irrelevant in this context). Thus, if we regard the incident plane wave $p^i(x, \theta, \varphi) = \iota^{ikx} \cdot \hat{d}(\theta, \varphi)$ as a function of $x, \theta, \varphi$, it is easy to realize that $p^i$, as well as its first and second partial derivatives with respect to the Cartesian coordinates $x_j$ (for $j = 1, 2, 3$), are continuous functions on the compact set $A := \partial D \times [0, \pi] \times [0, 2\pi]$, and then uniformly continuous on $A$ itself. The latter property implies that

$$\lim_{(\theta, \varphi) \to (\theta_0, \varphi_0)} \|p^i(\cdot, \theta, \varphi) - p^i(\cdot, \theta_0, \varphi_0)\|_{1, \alpha, \partial D} = 0 \quad \forall (\theta_0, \varphi_0) \in [0, \pi] \times [0, 2\pi], \tag{5.3}$$

where, following the notations of [12], $\|\|_{1, \alpha, \partial D}$ is the norm in the space $C^{1, \alpha}(\partial D)$. Then, by the well-posedness of Problem 2.1, we have

$$\lim_{(\theta, \varphi) \to (\theta_0, \varphi_0)} \|p^s(\cdot, \theta, \varphi) - p^s(\cdot, \theta_0, \varphi_0)\|_{1, \alpha, \mathbb{R}^3 \setminus \bar{D}} = 0 \quad \forall (\theta_0, \varphi_0) \in [0, \pi] \times [0, 2\pi]. \tag{5.4}$$

In particular, the functions $p^s(x, \cdot, \cdot)$ and $\frac{\partial p^s}{\partial x_j}(x, \cdot, \cdot)$ are continuous in $[0, \pi] \times [0, 2\pi]$ uniformly with respect to $x \in \mathbb{R}^3 \setminus D$. On the other hand, the continuity in $\mathbb{R}^3 \setminus D$ of $p^s(\cdot, \theta, \varphi)$ and $\frac{\partial p^s}{\partial x_j}(\cdot, \theta, \varphi)$ for each $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi]$ is a trivial consequence of the well-posedness of Problem 2.1. As a result, both $p^s$ and $\frac{\partial p^s}{\partial x_j}$ are continuous in $\mathbb{R}^3 \setminus \bar{D} \times [0, \pi] \times [0, 2\pi]$.

Consider now the open ball $B_R$, with $R > 0$ so large that $\bar{D} \subset B_R$, and define $G := B_R \setminus \bar{D}$. The continuity of $p^s$ and $\frac{\partial p^s}{\partial x_j}$ on the compact set $A' := G \times [0, \pi] \times [0, 2\pi]$
implies that there exist constants $M_1, M_2 > 0$ such that

$$ |p^s(x, \theta, \varphi)| \leq M_1, \quad \left| \frac{\partial p^s}{\partial x}(x, \theta, \varphi) \right| \leq M_2 \quad \forall (x, \theta, \varphi) \in A'. \quad (5.5) $$

Then, by using the Cauchy-Schwarz inequality, from relations (4.18), (5.5) and the assumption $\|g_{n, x}\|_{L^2(\Omega)} \leq K$, we obtain

$$ |p^s_{z_n}(x)| \leq \int_{\Omega} |p^s(x, d)g_{n, x}(d)| \, ds(d) = \int_0^\pi \int_0^{2\pi} |p^s(x, \theta, \varphi)| \cdot |g_{n, x}(\theta, \varphi)| \, d\varphi \sin \theta \, d\theta \leq
\leq M_1 \sqrt{4\pi} \|g_{n, x}\|_{L^2(\Omega)} \leq 2 \sqrt{\pi} M_1 K := Q_1 \quad \forall x \in \tilde{G}, \, \forall n \in \mathbb{N}, \quad (5.6) $$

and analogously, by Lebesgue’s dominated convergence theorem,

$$ \left| \frac{\partial p^s_{z_n}}{\partial x^i}(x) \right| \leq M_2 \sqrt{4\pi} \|g_{n, x}\|_{L^2(\Omega)} \leq 2 \sqrt{\pi} M_2 K := Q_2 \quad \forall x \in \tilde{G}, \, \forall n \in \mathbb{N}. \quad (5.7) $$

Let us now recall the expression of $F_{\tilde{G}}(z^*, r_n) \left( p_{z_n}^s \right)$ as given by (4.5) and (4.6), i.e.,

$$ F_{\tilde{G}}(z^*, r_n) \left( p_{z_n}^s \right) = \frac{1}{4\lambda \rho_0} \int_{\tilde{G}(z^*, r_n)} \left[ p_{z_n}^s \frac{\partial p_{z_n}^s}{\partial \nu} - p_{z_n}^s \frac{\partial p_{z_n}^s}{\partial \nu} \right] (x) \, ds(x). \quad (5.8) $$

It is clear that $\tilde{G}(z^*, r_n) \subset \tilde{G}$ for $n$ large enough: then, we can use inequalities (5.6) and (5.7) to bound (5.8) as

$$ \left| F_{\tilde{G}}(z^*, r_n) \left( p_{z_n}^s \right) \right| \leq \frac{1}{2\lambda \rho_0} \int_{\tilde{G}(z^*, r_n)} \left| p_{z_n}^s(x) \frac{\partial p_{z_n}^s}{\partial \nu}(x) \right| \, ds(x) \leq \frac{\sqrt{3}|\tilde{G}(z^*, r_n)|}{2\lambda \rho_0} Q_1 Q_2, \quad (5.9) $$

where $|\tilde{G}(z^*, r_n)|$ denotes the area of the surface $\tilde{G}(z^*, r_n)$. As $n \to \infty$, this area vanishes: then, from the last inequality in (5.9), we have $\lim_{n \to \infty} \left| F_{\tilde{G}}(z^*, r_n) \left( p_{z_n}^s \right) \right| = 0$, in contradiction with (5.2). \qed

It is clear that Theorem 5.1 can be immediately adapted to the aspect-limited case: it suffices to properly restrict the set of incidence directions, i.e., to require that $d(\theta, \varphi) \in \Omega'$.

The next step is now to formulate sufficient conditions for the fulfilment of the hypotheses of Theorem 5.1. According to the framework outlined in Section 4, these conditions should be related to the far-field equation and to the regular behavior of the flow tubes starting from the boundary of the scatterer and reaching the far-field region. Then, in the following section we numerically investigate this behavior, while in Section 7 the numerical results will be used to identify and formalize the sufficient conditions we are looking for.

6. Numerical simulations: the case of a sampling point inside the scatterer. We are going to consider two scattering experiments. For both of them, the scatterer is a sound-soft sphere placed in air and centred at the origin of an orthogonal Cartesian coordinate system $(x_1, x_2, x_3)$; moreover, the wavelength of the incident pressure field is $\lambda = 1$ m and the radius of the sphere is $a = 0.3 \, \lambda$. In both cases, the LSM is implemented as described e.g. in [9, 13]. However, the former experiment deals with a full-view configuration of the emitting and receiving antennas, which entails the normality of the far-field operator $F$ [3]; instead, the latter considers an aspect-limited configuration, whereby all antennas are placed in the far-field
half-sphere $\Omega_{x_2 \geq 0}$ contained in the half-space $x_2 \geq 0$. This means that the set of incidence directions is $\Omega^i = -\Omega_{x_2 \geq 0} = \Omega_{x_2 \leq 0}$, while the set of observation directions is $\Omega^o = \Omega_{x_2 \geq 0}$; accordingly, the modified far-field operator $F_{al} : L^2(\Omega^i) \to L^2(\Omega^o)$ is not normal [8, 18].

For each experiment, we use $11 \times 22$ incident plane waves and $10 \times 20$ measurement points. Both the incidence and observation angles are chosen according to a uniform discretization step for the spherical angles: $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ for the full-view case; $\theta \in [0, \pi]$ and $\varphi \in [0, \pi]$ for the aspect-limited case. For each incident wave, the far-field pattern of the corresponding scattered field is computed by exploiting the analytical results in [15] and then blurred by 3% Gaussian noise. The visualizations provided by the LSM for the two experiments are shown in Figure 6.1. In general, for each figure, the panels on the left and on the right will refer to the full-view and to the aspect-limited case, respectively. Then, in panels (a) and (c) of Figure 6.1 we show the reconstructed sphere together with a schematic representation of the positions of all antennas in the far-field region, while in panels (b) and (d) we plot a zoomed visualization of the scatterer. In particular, panel (d) highlights that, in the aspect-limited case, the quality of the reconstruction is lower, due to a lengthening along the $x_2$-axis.

We now investigate the behavior of the flow lines of the scattered field: more precisely, given a sampling point $z \in D = B_a$, we compute a discretized version of the field (4.18), where $g^e_z$ is given by the LSM. Then, for each point of a uniform and cubic grid with discretization step equal to $\lambda/10$ in a cubic domain with side at least $10\lambda$ long, we plot a discretized and normalized version of the vector $S$ (see (4.6)) associated with the field (4.18). For the clarity of visualization, only zoomed plane sections, parallel to the coordinate axes, will be shown.

We first consider a sampling point placed at the center of the sphere: the resulting behavior of the vector field $S$ is shown in Figure 6.2. The three panels (a), (b) and (c) on the left are obtained, for the full-view configuration, by orthogonally projecting onto the planes $x_1 = 0, x_2 = 0, x_3 = 0$ respectively the vectors $S$ applied into the grid points of such planes. The same holds true for the panels (d), (e) and (f) on the right, except that they refer to the aspect-limited case: accordingly, we added a half-circle or a circle to indicate the position of the antennas in each plane section. In spite of normalization, the arrows representing $S$ may be, in general, of different lengths: shorter arrows indicate that the corresponding $S$ is not parallel to the plane section under exam. From Figure 6.2, the radial behavior of $S$ with respect to $z$ is evident in the full-view case (panels (a)-(c)), while it undergoes a slight deterioration for the aspect-limited case, as highlighted by panels (d) and (f): notably, such deterioration is negligible for the plane section $x_2 = 0$ (panel (e)), which corresponds to a planar full-view configuration. The observed behaviors turn out to be fully preserved also when larger investigation domains are visualized.

This (approximately) radial behavior of $S$ with respect to the sampling point is not related to the symmetric position chosen for the previous sampling point: on the contrary, it is verified for any $z$ inside the scatterer, or even on its boundary. For example, in the three pairs of panels (a)-(d), (b)-(e) and (c)-(f) of Figure 6.3 we can respectively see the three plane sections $x_1 = 0$, $x_2 = 0.2$ and $x_3 = 0$ for the sampling point $z = (0, 0.2, 0)$. Figure 6.4 is organized in an analogous way, but is concerned with the case of a sampling point on the boundary, i.e., $z = (0, 0.3, 0)$, and the corresponding plane sections $x_1 = 0$, $x_2 = 0.3$ and $x_3 = 0$. In any case, an approximate radially of the flow lines of $S$, and a consequent regularity of the flow
tubes starting from the boundary of the scatterer, is suggested by these (and others, not shown here) numerical simulations.

7. **Back-propagation of information from the far-field to the near-field region.** We can now understand how the constraint expressed by the far-field equation is back-propagated to the near-field region. More precisely, taking inspiration from the numerical results of Section 6, we can identify sufficient conditions ensuring that the lower bound (5.2), concerning the flux outgoing from a small sphere centered at a point of $\partial D$, follows from the energy conservation inside a regular flow tube linking that portion of $\partial D$ to the far-field region, where the far-field equation forces the
Fig. 6.2. Qualitative behavior of the vector field $S$ for a sampling point $z$ (red bullet) at the center of the sphere (black contour). Full-view configuration of antennas: (a) plane section $x_1 = 0$; (b) plane section $x_2 = 0$; (c) plane section $x_3 = 0$. Aspect-limited configuration of antennas: (d) plane section $x_1 = 0$; (e) plane section $x_2 = 0$; (f) plane section $x_3 = 0$. 

Fig. 6.3. Qualitative behavior of the vector field $S$ for a sampling point $z = (0, 0, 2, 0)$ (red bullet) inside the sphere (black contour). Full-view configuration of antennas: (a) plane section $x_1 = 0$; (b) plane section $x_2 = 0.2$; (c) plane section $x_3 = 0$. Aspect-limited configuration of antennas: (d) plane section $x_1 = 0$; (e) plane section $x_2 = 0.2$; (f) plane section $x_3 = 0$. 
Fig. 6.4. Qualitative behavior of the vector field $\mathbf{S}$ for a sampling point $z = (0, 0.3, 0)$ (red bullet) on the boundary of the sphere (black contour). Full-view configuration of antennas: (a) plane section $x_1 = 0$; (b) plane section $x_2 = 0.3$; (c) plane section $x_3 = 0$. Aspect-limited configuration of antennas: (d) plane section $x_1 = 0$; (e) plane section $x_2 = 0.3$; (f) plane section $x_3 = 0$. 
tube itself to convey a positive flux. The next theorem formalizes these considerations (for the case of a full-view configuration of antennas).

Theorem 7.1. Let \( z^* \), \( U_{z^*}, \Omega(z^*, r_z) \) and \( \tilde{\Omega}(z^*, r_z) \) be as in Theorem 5.1. Given \( \epsilon > 0 \), for each \( z \in U_{z^*} \cap D \) consider an \( \epsilon \)-approximate solution of the far-field equation, i.e., a function \( p_z^\epsilon \in L^2(\Omega) \) such that \( \|Fg_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} \leq \epsilon \). Moreover, assume that, for each \( z \in U_{z^*} \cap D \), the curve \( \gamma_z \) := \( \partial \Omega(z^*, r_z) - \Omega(z^*, r_z) \cap \partial D \) is a transverse section of a regular flow tube \( T_{z^*} \) of the scattered field \( p_z^\epsilon \), with asymptotic angular width \( W_{\infty}(\gamma_z) \). Finally, assume that there exists an open and connected domain \( W_{\infty}^* \subset \Omega \) such that its area \( |W_{\infty}^*| \) is positive and \( \lim_{z \to z^*} |W_{\infty}^* \Delta W_{\infty}(\gamma_z)| = 0 \), where \( \Delta \) denotes the symmetric difference between two sets. Then, for any such \( g_z^\epsilon \in L^2(\Omega) \), the limit \( \lim_{z \to z^*} \|g_z^\epsilon\|_{L^2(\Omega)} = \infty \) holds, provided that \( \epsilon \) is small enough.

Proof. The power flux of the scattered field \( p_z^\epsilon \) across \( \tilde{\Omega}(z^*, r_z) \) is preserved, by energy conservation, along the regular flow tube \( T_{z^*} \), as highlighted in (4.14). In particular, with reference to property (i) of Definition 4.1, the flux outgoing from \( \tilde{\Omega}(z^*, r_z) \) equal to the flux across the domain \( W_{\infty}(\gamma_z) \) enclosed on \( \Omega_{\infty} \) by the section \( (\gamma_z)_R = T_{z^*} \cap \Omega_R \), for all \( R \geq R_0 \). Then, remembering also (4.17), we have for each \( z \in U(\cdot, r_z) \cap D \):

\[
F_{\tilde{\Omega}(z^*, r_z)}(p_z^\epsilon) = \frac{1}{4i\omega \rho_0} \int_{\tilde{\Omega}(z^*, r_z)} \left[ \frac{\partial p_z^\epsilon}{\partial \nu} - \frac{\partial p_z^\epsilon}{\partial \nu} \right] (x) \, d\sigma(x) = \frac{1}{4i\omega \rho_0} \lim_{R \to \infty} \int_{W_{\infty}(\gamma_z)} \left[ \frac{\partial p_z^\epsilon}{\partial \nu} - \frac{\partial p_z^\epsilon}{\partial \nu} \right] (R, \hat{x}) R^2 \, d\sigma(\hat{x}) = \frac{k}{2\omega \rho_0} \|Fg_z^\epsilon\|_{L^2(W_{\infty}(\gamma_z))}^2 \geq \frac{k}{2\omega \rho_0} \left\{ \|\Phi_\infty(\cdot, z)\|_{L^2(W_{\infty}(\gamma_z))}^2 - \epsilon' W_{\infty}(\gamma_z) \right\}. \tag{7.1}
\]

Now, the assumption \( \lim_{z \to z^*} |W_{\infty}^* \Delta W_{\infty}(\gamma_z)| = 0 \) clearly implies the two following limits: \( \lim_{z \to z^*} |\Phi_\infty(\cdot, z)|_{L^2(W_{\infty}(\gamma_z))} = |\Phi_\infty(\cdot, z^*)|_{L^2(W_{\infty}^*)} \) and \( \lim_{z \to z^*} \epsilon' W_{\infty}(\gamma_z) = \epsilon'(W_{\infty}^*) \). Then, from the last inequality in (7.1), we obtain:

\[
\lim_{z \to z^*} F_{\tilde{\Omega}(z^*, r_z)}(p_z^\epsilon) \geq \frac{k}{2\omega \rho_0} \left[ |\Phi_\infty(\cdot, z^*)|_{L^2(W_{\infty}^*)}^2 - \epsilon'(W_{\infty}^*) \right]. \tag{7.2}
\]

Now, by hypothesis, the area of \( W_{\infty}^* \) is positive and then we have \( |\Phi_\infty(\cdot, z^*)|_{L^2(W_{\infty}^*)}^2 > 0 \). Accordingly, we can take \( \epsilon > 0 \) so small that the right-hand side of (7.2) is positive. More precisely, remembering the explicit expressions of \( \Phi_\infty(\hat{x}, z^*) = \frac{1}{4\pi} e^{-ik\hat{x} \cdot z^*} \) and of \( \epsilon'(W_{\infty}^*) \), as given below (4.17), it suffices to choose \( \epsilon \) such that

\[
0 < \epsilon < (\sqrt{2} - 1) |\Phi_\infty(\cdot, z^*)|_{L^2(W_{\infty}^*)} = (\sqrt{2} - 1) \frac{\sqrt{|W_{\infty}^*|}}{4\pi}. \tag{7.3}
\]

With this choice of \( \epsilon \), the flux \( F_{\tilde{\Omega}(z^*, r_z)}(p_z^\epsilon) \) is bounded from below as \( z \to z^* \); then, we can apply Theorem 5.1 to conclude that \( \lim_{z \to z^*} \|g_z^\epsilon\|_{L^2(\Omega)} = \infty \), in agreement with limit (2.5)(a). $\Box$

Remark 7.1. By relying on the numerical experiments performed in Section 6, we can give a lower bound for the asymptotic angular width \( |W_{\infty}^*| \) determining the upper bound (7.3). Indeed, for \( z \in D \) (and even on \( \partial D \)) the flow lines of \( p_z^\epsilon \) resemble those of \( \Phi(\cdot, z) \) in that they are essentially radial with respect to \( z \). Hence, \( |W_{\infty}^*| \) will
not be less than the lower limit amplitude $\beta_\alpha$ of the angle $\tilde{z}(z^*, r_z)$ as $z \to z^*$, i.e., as shown soon before (5.1), $2\pi \left(1 - \frac{r_0^2}{2}\right)$.

Remark 7.2. Theorem 7.1 deals with generic $\epsilon$-approximate solutions of the far-field equation. However, in the implementation of the LSM, Tikhonov regularized solutions $g^2_\alpha$ play a major role: then, it is interesting to specify the previous analysis to them. To this end, Theorem 7.1 can be paraphrased as follows. For each $z \in D$, consider the discrepancy $d_{\alpha^*(z)} := \|F g_{\alpha^*(z)} - \Phi_\infty(z, z)\|_{L^2(\Omega)}$, where $\alpha^*(z)$ is chosen by means of some optimality criterion. Assume that the condition $d_{\alpha^*(z)} \leq \epsilon$ holds, where $\epsilon > 0$ fulfills bound (7.3): then, if the flow tubes of $p^z_{2^\alpha^*}$ satisfy the specified regularity requirements, we find that $\lim_{z \to z^*} \|g_{\alpha^*(z)}\|_{L^2(\Omega)} = \infty$, which can only happen if $\lim_{z \to z^*} \alpha^*(z) = 0$. The latter limit shows that, although we do not assume $\alpha \to 0^+$ in our framework, we find such a result, but only for $z$ approaching $z^* \in \partial D$: this is in qualitative agreement with numerical simulations, which highlight [13] a strong decrease of $\alpha^*(z)$ as $z \to z^*$.

Remark 7.3. The aspect-limited case can be dealt with in an analogous way, by restricting the set of incidence directions to $\Omega^\alpha$. However, an additional hypothesis is required in the statement of Theorem 7.1: the flow tubes under exam should reach the portion of the far-field region where the receiving antennas are placed, i.e., it should hold that $W_\infty \subset \Omega^\alpha$. Indeed, outside $\Omega^\alpha$ the modified far-field equation imposes no constraint whatsoever (cf. also the end of Section 6 in [2]).

We conclude this section by observing that, although redundant, a similar physical approach could be pursued also for the FM, with a notable simplification: since, for $z \in D$, the radiating field $p^z[g_z]$ is equal to $\Phi_2$ in $\mathbb{R}^3 \setminus D$, the flow lines of interest are portions of half-lines outgoing from $z$, and the flow tubes are portions of cones with vertex at $z$. Notably, should a flow tube impinge on a portion of $\partial D$, this would not affect the value of the power flux carried by it beyond this obstruction, since the flux of $p^z[g_z]$ is always and exactly equal to that of $\Phi_2$. Hence, in the FM, the requirements stated in Definition 4.1 would become partly useless, partly trivial, and no regularity issue would arise.

8. The case of a sampling point outside the scatterer. Soon after the formulation of the Cauchy problem (4.12), we noticed that an initial point $x_0 \in \partial D$ may be a ramification point, i.e., such that several flow lines may start from it. In particular, it is even possible that a flow tube originates from $x_0$: in other terms, $x_0$ might become a degenerate transverse section of a flow tube. Actually, the numerical simulations presented in Section 6 confirm this possibility. Indeed, the (approximately) radial behavior of the flow lines of the scattered field with respect to the sampling point is observed not only for $z \in D$, but also for $z \in \partial D$, as shown in Figure 6.4: then, in the latter case, $z$ seems to be just a ramification point.

This introduces us to the analysis of what happens when the sampling point is chosen outside the scatterer. Interestingly, the numerical simulations performed in this case highlight that at least one ramification point shows up on $\partial D$. For example, with reference to the same two scattering experiments considered in Section 6, we now choose the exterior sampling point $z = (0, 1, 1)$. In the full-view case, a ramification point can be apparently detected at $z_0 = (-0.3, 0, 0)$, as shown by the three plane sections $x_1 = -0.3$, $x_2 = 0$ and $x_3 = 0$ represented in panels (a), (b) and (c) of Figure 8.1. In the aspect-limited case, a good candidate to be a ramification point is $z_0 = (0, 0.212, -0.212)$, as suggested by the three plane sections $x_1 = 0$, $x_2 = 0.212$.
and \(x_3 = -0.212\) plotted in panels (d), (e) and (f) of Figure 8.1. Note that, although the exterior sampling point is the same in the full-view and in the aspect-limited configurations, there is a priori no reason why the ramification point (or points) should be the same for both cases.

In our framework, the role played by ramification points can be formalized as follows.

**Definition 8.1.** Let \(u \in C^1,\alpha(\mathbb{R}^3 \setminus D)\) be a radiating solution of the Helmholtz equation in \(\mathbb{R}^3 \setminus D\), and \(z_0 \in \partial D\) a ramification point for the flow lines of \(u\). Let \(I \subset \mathbb{R}\) be a set of indices, and denote a flow line of \(u\) starting from \(z_0\) as \(C_{z_0}(\tau)\), for \(i \in I\) (and \(\tau \geq 0\)). Consider the set (of images) of all flow lines \(\{C_{z_0}(\tau)\}_{\tau \in I}\) and the ball \(B(z_0, r) := \{x \in \mathbb{R}^3 : |x - z_0| < r\}, \) with \(r > 0\). Assume that there exists \(I' \subset I\) such that \(T_{z_0}(I') := \{C_{z_0}(\tau)\}_{\tau \in I'}\) is a regular flow tube with degeneracy in \(z_0\), i.e., \(T_{z_0}(I') \setminus [T_{z_0}(I') \cap B(z_0, r)]\) is regular (in the sense of Definition 4.1) for each \(r < r_0\), being \(r_0 > 0\) small enough. Then, the field \(u\) is called ‘partially pseudo-radial’ with respect to \(z_0\). Moreover, let \(|W_{\infty}(z_0, I')| > 0\) be the asymptotic angular width of each possible \(T_{z_0}(I')\): then, the quantity \(|W_{\infty}(z_0)| := \sup_{\tau'} |W_{\infty}(z_0, I')|\) is called the ‘asymptotic angular width’ of the beam of flow lines starting from \(z_0\).

We now have all the elements to pursue our physical approach to the LSM in the case of \(z \notin D\). The first step consists in a preliminary result, analogous to Theorem 5.1, but much simpler to be proved.

**Theorem 8.2.** Given \(z^* \in \partial D\), define \(\Omega(z^*, r) := \{x \in \mathbb{R}^3 : |x - z^*| = r\} \) and \(\hat{\Omega}(z^*, r) := \Omega(z^*, r) \cap (\mathbb{R}^3 \setminus D)\); moreover, let \(\hat{\Omega}'(z^*, r)\) be an open and connected subset of \(\hat{\Omega}(z^*, r)\). Then, there cannot exist \(g \in L^2(\Omega)\) such that the flux across \(\hat{\Omega}'(z^*, r)\) of the corresponding scattered field \(p^*_g\), defined analogously to (2.3), is bounded from below as \(r \to 0\), i.e., such that \(\lim_{r \to 0} \frac{\int_{\hat{\Omega}'(z^*, r)} |p^*_g(x)|^2 \, d\sigma(x)}{2\omega_0} = c > 0\).

**Proof.** Assume, by contradiction, that such a \(g \in L^2(\Omega)\) exists. Similarly to the proof of Theorem 5.1 (cf. relation (5.9)) and by applying the mean-value theorem for integration, we can bound the flux under exam as follows:

\[
|F_{\hat{\Omega}'(z^*, r)}(p^*_g)| \leq \frac{1}{2\omega_0} \int_{\hat{\Omega}'(z^*, r)} \left| \frac{\partial p^*_g}{\partial \nu}(x) \right| d\sigma(x) = \left| \frac{\hat{\Omega}'(z^*, r)}{2\omega_0} \right| \left| \frac{\partial p^*_g}{\partial \nu}(\bar{x}_r) \right|.
\]

where \(\bar{x}_r\) is an appropriate point of \(\hat{\Omega}'(z^*, r)\). It is clear that \(\left| \hat{\Omega}'(z^*, r) \right| \to 0\) and \(|\bar{x}_r - z^*| \to 0\) as \(r \to 0\); then, the continuity of \(p^*_g\) and \(\frac{\partial p^*_g}{\partial \nu}\) in \(\mathbb{R}^3 \setminus D\) implies that the flux vanishes, i.e., \(\lim_{r \to 0} F_{\hat{\Omega}'(z^*, r)}(p^*_g) = 0\), in contradiction with the positive lower boundedness condition for \(F_{\hat{\Omega}'(z^*, r)}(p^*_g)\).

Like Theorem 5.1, also Theorem 8.2 holds for the aspect-limited case, provided that a function \(g \in L^2(\Omega)\) is considered and the integration domain \(\Omega\) in (2.3) is replaced by \(\Omega\).

In analogy with Theorem 7.1, the second step consists in showing how the far-field equation and the pseudo-radiality of the scattered field allow applying Theorem 8.2.

**Theorem 8.3.** Let \(z \in \mathbb{R}^3 \setminus D\): if \(\epsilon > 0\) is small enough, there cannot exist \(g^*_\epsilon \in L^2(\Omega)\) such that \(\|F g^*_\epsilon - \Phi_{\infty}(z, \cdot)\|_{L^2(\Omega)} \leq \epsilon\) and the scattered field \(p^*_\epsilon\) is partially pseudo-radial with respect to some point \(z_0 \in \partial D\).

**Proof.** Assume, by contradiction, that such a \(g^*_\epsilon \in L^2(\Omega)\) exists, and define \(\Omega(z_0, r) := \{x \in \mathbb{R}^3 : |x - z_0| < r\}, \Omega(z_0, r) := \Omega(z_0, r) \cap (\mathbb{R}^3 \setminus D)\) and \(\hat{\Omega}'(z_0, r) := \hat{\Omega}(z_0, r) \cap T_{z_0}(I')\), where \(I'\) and \(T_{z_0}(I')\) are as in Definition 8.1. By the same argument...
Fig. 8.1. Qualitative behavior of the vector field $\mathbf{S}$ for a sampling point $z = (0, 1, 1)$ (red bullet) outside the sphere (black contour); a ramification point (red square box) is detectable on the boundary of the scatterer. Full-view configuration of antennas: (a) plane section $x_1 = -0.3$; (b) plane section $x_2 = 0$; (c) plane section $x_3 = 0$. Aspect-limited configuration of antennas: (d) plane section $x_1 = 0$; (e) plane section $x_2 = 0.212$; (f) plane section $x_3 = -0.212$. 

(a) (d)

(b) (e)

(c) (f)
justifying relations (7.1), we have

\[ F_{\tilde{\Omega}}(z_0, r) \left( p^x, r \right) = \frac{1}{4i\omega \rho_0} \int_{\tilde{\Omega}} \left[ \frac{p^x}{\partial \nu} - p^x \frac{\partial p^x}{\partial \nu} \right] (x) d\sigma(x) \geq \]

\[ \geq \frac{k}{2i\omega \rho_0} \left\{ \| \Phi_\infty(\cdot, z) \|^2_{L^2[|W_\infty(z_0, I')] - \epsilon'[|W_\infty(z_0, I')] \right\}, \quad (8.1) \]

whence

\[ \liminf_{r \to 0} F_{\tilde{\Omega}}(z_0, r) \left( p^x, r \right) \geq \frac{k}{2i\omega \rho_0} \left\{ \| \Phi_\infty(\cdot, z) \|^2_{L^2[|W_\infty(z_0, I')] - \epsilon'[|W_\infty(z_0, I')] \right\}. \quad (8.2) \]

Since \(|W_\infty(z_0, I')| \geq 0\), we can proceed as from (7.2) to (7.3), and make the right-hand side of (8.2) positive by choosing \(\epsilon > 0\) such that

\[ 0 < \epsilon < (\sqrt{2} - 1) \| \Phi_\infty(\cdot, z^*) \|_{L^2[|W_\infty(z_0, I')] = (\sqrt{2} - 1) \sqrt{|W_\infty(z_0, I')| \frac{4\pi}{\pi}}. \quad (8.3) \]

Incidentally, we note that, according to Definition 8.1, an upper bound for \(|W_\infty(z_0, I')|\) is given by the asymptotic angular width \(|W_\infty(z_0)|\). We can now apply Theorem 8.2 to conclude the proof with the desired contradiction. \(\square\)

**Remark 8.1.** As in Remark 7.2, let us now focus on Tikhonov regularized solutions. In this case, Theorem 8.2 states that, for \(z \notin D\), the pseudo-radiality of the scattered field is incompatible with a discrepancy \(d_\alpha^\ast(z) \leq \epsilon\), if \(\epsilon\) verifies bound (8.3). This incompatibility means that \(g_\alpha^\ast(z)\) cannot belong to \(L^2(\Omega)\), i.e., \(\| g_\alpha^\ast(z) \|_{L^2(\Omega)} \to \infty\), which can only happen if \(\alpha^\ast(z) \to 0^+\), in agreement with limit (2.6)(a). Again, this is consistent with numerical simulations [13], which show that the values of \(\alpha^\ast(z)\) are much smaller for \(z \notin D\) than for \(z \in D\).

**Remark 8.2.** As in Remark 7.3, we point out that Theorem 8.3 can be easily adapted to the aspect-limited case, by requiring, in particular, that \(|W_\infty(z_0, I')| \subset \Omega^\circ\).

**9. Conclusions and further developments.** We have presented an approach to the LSM based on the conservation of energy inside the flow tubes of the scattered field. In this framework, a) the far-field equation expresses a constraint on the power flux of the scattered field in the far-field region; b) under appropriate assumptions (suggested by numerical simulations) on the flow tubes carrying this flux, the information contained in the far-field constraint is propagated throughout the background up to the boundary of the scatterer; c) the approximate fulfillment of this constraint forces the \(L^2\)-norm of any \(\epsilon\)-approximate solution of the far-field equation to behave as a good indicator function for visualizing the shape and location of the scatterer.

Possible future developments of our approach are concerned with both theoretical and numerical issues. From the former viewpoint, we can mention the case of lossy and inhomogeneous backgrounds with near-field measurements, but the key-problem is clearly to predict the behavior of flow lines starting from the knowledge of the geometrical and material properties of the scatterer: achieving such a goal is the only way for making our physical approach a mathematical justification of the LSM, and seems to require tools and results in topological dynamics [1]. As far as numerics is concerned, it would be interesting to observe the behavior of flow lines for more complex scattering situations than those considered here: a starting point is, of course, a 3D direct code for penetrable and impenetrable scatterers.
Appendix A. A minimum property for cones inscribed in a half-sphere.

As usual, we denote by \((x_1, x_2, x_3)\) and \((r, \theta, \varphi)\) the Cartesian and spherical coordinates in \(\mathbb{R}^3\), respectively.

**Theorem A.1.** Let \(\tilde{B} := \{x \in \mathbb{R}^3 : |x| \leq 1\}\), \(\tau := \{x \in \mathbb{R}^3 : x_3 = 0\}\) and \(C := \tilde{B} \cap \tau\). Let \(P\) be a point in \(\Omega = \partial \tilde{B}\) and let us consider the solid angle \(\hat{P}\) identified by the cone of vertex \(P\) and basis \(C\). Then, the amplitude \(|\hat{P}|\) of the solid angle \(\hat{P}\) is minimum when the cone is right, i.e., when \(x_3(P) = 1\).

**Proof.** In general, the amplitude \(|\hat{P}_\Sigma|\) of the solid angle \(\hat{P}_\Sigma\) subtended by a surface \(\Sigma\) when observed by a point \(P \notin \Sigma\) is given by the formula (see e.g. [19]):

\[
|\hat{P}_\Sigma| = \int_\Sigma \frac{\mathbf{s} \cdot \hat{n}}{s^3} d\sigma,
\]

where, if \(Q\) denotes a generic point of \(\Sigma\), \(\mathbf{s} := \hat{P}Q\), \(s = |\mathbf{s}|\), \(\hat{n} = \hat{n}(Q)\) is the unit normal in \(Q\) to \(\Sigma\) and \(d\sigma = d\sigma(Q)\) is the surface element on \(\Sigma\). In our case, the most natural choice for \(\Sigma\) would be \(C\); however, analytical computations are simpler by choosing the lower half sphere, i.e., \(\Sigma = \Omega^- := \{x \in \mathbb{R}^3 : |x| = 1 \land x_3 \leq 0\}\).

Of course, we have \(r(P) = 1\), while, by symmetry, it is not restrictive to assume that \(\theta(P) \in [0, \pi/2]\) and \(\varphi(P) = 0\) (the case \(\theta(P) = \pi/2\) will be discussed separately).

In the following, we shall set \(\theta_P := \theta(P)\) and denote with \((r, \theta, \varphi)\) the spherical coordinates of the generic point \(Q \in \Omega^-\). Then, the vectors \(\mathbf{s}\) and \(\hat{n}\) can be expressed in Cartesian components as \(\mathbf{s} = (\sin \theta \cos \varphi - \sin \varphi \cos \theta, \sin \theta \sin \varphi, \cos \theta)\) and \(\hat{n} = (\sin \theta \cos \varphi, \sin \varphi, \cos \theta)\). Inserting these representations into (A.1), with the identifications \(\Sigma = \Omega^-\) and \(\hat{P} = \hat{P}_{\Omega^-}\), we find

\[
f(\theta_P) := |\hat{P}|(\theta_P) = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin \theta \left[ \int_0^{2\pi} \frac{d\varphi}{\sqrt{1 - \sin \theta \cos \varphi \sin \theta_P - \cos \theta \cos \theta_P}} \right] d\theta.
\]

It is easy to verify that \(f(0) = 2\pi \left(1 - \frac{\sqrt{2}}{2}\right)\), a well-known result concerning right cones. We now want to show that \(f(0)\) is the minimum amplitude of the solid angle \(\hat{P}\). To this end, we shall compute \(f'\) and show that \(f'(\theta_P) \geq 0\) \(\forall \theta_P \in [0, \pi/2]\).

Since it is possible to prove that the derivation operator \(\frac{d}{d\theta_P}\) can be brought inside the integrals (as a consequence of Lebesgue’s dominated convergence theorem and Lagrange’s mean value theorem), we have

\[
f'(\theta_P) = \frac{1}{4\sqrt{2}} \int_0^{\pi/2} \sin \theta \left[ \int_0^{2\pi} \frac{\sin \theta \cos \varphi \cos \theta_P - \cos \theta \sin \theta_P}{\sqrt{1 - \sin \theta \cos \varphi \sin \theta_P - \cos \theta \cos \theta_P}^3} d\varphi \right] d\theta.
\]

Then, it suffices to prove that the internal integral in \(d\varphi\) is non-negative for each \(\theta_P \in [0, \pi/2]\) and \(\theta \in [\pi/2, \pi]\). To this end, we put \(a := \sin \theta \cos \theta_P \geq 0\), \(b := -\cos \theta \sin \theta_P \geq 0\), \(c := 1 - \cos \theta \cos \theta_P \geq 1\) and \(d := \sin \theta \sin \theta_P \geq 0\), where the inequalities follows from the ranges of \(\theta_P\) and \(\theta\). With these notations, the integrand function can be written as

\[
g(\varphi) := \frac{a \cos \varphi + b}{\sqrt{(c - d \cos \varphi)^3}},
\]

and we want to prove that \(\int_0^{2\pi} g(\varphi) d\varphi \geq 0\).

If \(a > 0\), we put \(\varphi_1 := \arccos(b/a), \varphi_2 := \arccos(-b/a), \varphi_3 := 0 + \pi\) and \(\varphi_4 := \varphi_2 + \pi\). Then, easy computations show that if \(a = 0\), then \(g(\varphi) \geq 0\) \(\forall \varphi \in [0, 2\pi]\);
instead, if \( a > 0 \), we have \( g(\varphi) \geq 0 \) \( \forall \varphi \in [0, \varphi_2] \cup [\varphi_3, 2\pi] \). Moreover, for \( a > 0 \), the following relations hold:

1) \( g(\varphi) \geq |g(\varphi + \pi)| \) \( \forall \varphi \in [0, \varphi_1] \implies \int_0^{\varphi_1} g(\varphi) d\varphi - \int_{\pi}^{\varphi_1} |g(\varphi)| d\varphi \geq 0; \)

2) \( g(\varphi) \geq |g(\varphi - \pi)| \) \( \forall \varphi \in [\varphi_2, 2\pi] \implies \int_{\varphi_2}^{2\pi} g(\varphi) d\varphi - \int_{\varphi_2}^{\varphi_4} |g(\varphi)| d\varphi \geq 0 \).

By summing the two last inequalities and remembering that \( g(\varphi) \leq 0 \) \( \forall \varphi \in [\varphi_2, \varphi_3] \), we find that

\[
\int_0^{\varphi_1} g(\varphi) d\varphi + \int_{\varphi_2}^{\varphi_3} g(\varphi) d\varphi + \int_{\varphi_4}^{2\pi} g(\varphi) d\varphi \geq 0. \tag{A.5}
\]

On the other hand, we already know that \( g(\varphi) \geq 0 \) \( \forall \varphi \in [\varphi_1, \varphi_2] \cup [\varphi_3, \varphi_4] \), and then

\[
\int_{\varphi_1}^{\varphi_2} g(\varphi) d\varphi + \int_{\varphi_3}^{\varphi_4} g(\varphi) d\varphi \geq 0. \tag{A.6}
\]

Inequalities (A.5) and (A.6) together allows concluding that \( \int_0^{2\pi} g(\varphi) d\varphi \geq 0 \), as claimed. Summing up, we have proved that \( f'(\theta_P) \geq 0 \) \( \forall \theta_P \in [0, \pi/2] \): in particular, by putting \( \theta_P = 0 \) in (A.3), we get \( f'(0) = 0 \), as it needs to be by symmetry.

In the case \( \theta_P = \pi/2 \), the cone of basis \( C \) and vertex \( P \) is clearly degenerate. We shall not discuss how the solid angle \( \hat{P} \) can be defined in this case: for our purposes, it suffices to observe that any meaningful (i.e., not merely conventional) definition should be obtained by continuity as a limit case for \( \phi_P \rightarrow \pi/2 \); in particular, it must hold

\[
f(\pi/2) = \lim_{\phi_P \rightarrow (\pi/2)^-} f(\theta_P).\]

Since \( f \) is non-decreasing in \([0, \pi/2]\), we can conclude that \( f(0) \leq f(\theta_P) \) \( \forall \theta_P \in [0, \pi/2] \).

Although unnecessary for our purposes, we finally observe that the previous proof can be easily refined in order to prove that \( f'(\theta_P) > 0 \) for \( \theta_P \in (0, \pi/2) \); hence \( \theta_P = 0 \) is a proper minimum point for \( f \). \( \square \)

REFERENCES


