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Complete analysis of 3-point binary subdivision scheme

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Abstract

In this paper, a new 3-point approximating subdivision scheme with shape parameter is presented with its complete analysis. It generates C^3-continuous curve and its Hölder regularity is four. Its limit function has a support width over the interval [-5/2, 5/2] and it reproduces the polynomial of degree two. It has interpolator behavior when the shape parameter is set to -1/16. Moreover, its tensor product version for the generation of regular surfaces has been presented in this paper along with its analysis. Furthermore, visual performances in curve and regular surface cases have also been presented.

Keywords: Subdivision scheme; Primal and dual schemes; Smoothness; Continuity; Polynomial reproduction; Polynomial generation.

Introduction

Geometric modeling is the heart of Computer Graphics and Computer Aided Geometric Design (CAGD) and covers a long range of applications. A powerful pattern for the design of objects in the field of CAGD is the subdivision. Subdivision algorithms are preferably suited to computer applications. Chaikin (1974) and Rham (1956) are regarded as the pioneers. Rham a French mathematician in 1956, presented recursively corner cutting piecewise linear approximation method to obtain a limiting curve of C^0 continuity. In 1974, Chaikin, developed another recursively corner cutting piecewise linear approximation method to create a limiting curve of C^1continuity. Dyn *et al*. (1987) developed a general form of 4-point C^1 interpolatory subdivision with `$w$` as tension parameter and for a particular value of parameter w=1/16, the scheme behaves exactly as that of Deslauriers and Dubuc (1989). Deslauriers and Dubuc (1989) introduced a 4-point C^1 interpolatory subdivision scheme.

Dyn and Levin (2002) indicated a number of possible extensions and generalization of the uniform binary subdivision. Zhang *et al*. (2004) developed a four-point approximating subdivision scheme for a quadrilateral net. The limiting surface was claimed to be C^3. Tang *et al*. (2005) proved the scheme developed by Dyn *et al*. (1987) to be C^1 using the Laurent polynomial method. The scheme was proved to be C^2 for a certain range of tension parameter. Dyn *et al*. (2005) presented a 4-point approximating subdivision scheme that generates C^2 curves.

Choi *et al*. (2006) presented a new class of stationary subdivision schemes with a tension parameter reproducing polynomials, which not only unify the well-known Deslauriers and Dubuc (1989) but also quadratic and cubic B-spline schemes. Ko *et al*. (2007) rebuilt the masks of some interpolatory symmetric subdivision schemes. Siddiqi and Ahmad (2006; 2008) introduced a set of (i.e. 3-point, 4-point, 5-point and 6-point) approximating subdivision schemes. Hormann and Sabin (2008) introduced the family of subdivision schemes with cubic precision and determined how the support, the Hölder regularity, the precision set, the degree of polynomials spanned by the limit curves.

Mustafa and Rahman (2010) constructed (2b+2)-point *n*-ary interpolating as well as approximating subdivision scheme. Cashman *et al*. (2009) presented a univariate refine and smooth subdivision algorithm which is symmetric, non-uniform and has similar properties to the uniform Lane Riesenfeld refine and smooth construction. Romani (2009) constructed three novel families of approximating subdivision schemes that generate piecewise exponential polynomials and a method to convert these into interpolating schemes of great interest in curve design. Siddiqi and Rehan (2010) presented improved form of the binary 4-point approximating subdivision scheme using the global tension parameter.

Mustafa *et al*. (2011) proposed a three point approximating subdivision scheme, with three shape parameters that unify three different existing 3- point approximating schemes. Ghaffar *et al*. (2012) proposed and analyzed a subdivision scheme which unifies 3-point approximating subdivision schemes of any arity in its compact form. The usefulness of the scheme is illustrated by considering different examples along with its comparison with the established subdivision schemes. Ghaffar *et al*. (2013) proposed and analyzed a tensor product subdivision scheme which is the extension of three point binary scheme for curve modeling. Khan *et al*. (2013) also presented tensor product schemes. In this paper, we are presenting new 3-point binary scheme along with its complete analysis.

1.1 Preliminaries

Theorem 1.1. Hormann (2012) If the coarse polygon $f^k = \{f_i^k\}_{i\in\mathbb{Z}}$ be refined by the scheme into refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in\mathbb{Z}}$ and if each f_i^k is attached with parameter $t_i^k = -\tau + (i+\tau)/2^k$, where $\tau = \frac{\alpha'(1)}{2}$, then scheme has primal parameterization if $t_i^k = \frac{i}{2^k}$ and it has dual parameterization if $t_i^k = \frac{1}{2} + (i - \frac{1}{2})/2^k$, here $\alpha'(1)$ denotes the first derivative of Laurent polynomial $\alpha(z)$ of the scheme at $z = 1$.

Theorem 1.2. Hormann (2012) The subdivision scheme with symbol $\alpha(z)$ reproduces polynomials of degree d with respect to the parameterizations with $\tau = \alpha'(1)/2$, if and only if

$$\alpha^{(k)}(-1) = 0, \quad \alpha^{(k)}(1) = 2\prod_{j=0}^{k-1}(\tau - j), \quad k = 0,1,...,d.$$

Polynomial reproduction of degree d requires polynomial generation of degree d .

Theorem 1.3. Dyn (2002) A convergent subdivision scheme that reproduces polynomial π_n (set of polynomials S at most degree n) has an approximation order of $n+1$.

Theorem 1.4. Hormann (2012) If the scheme S_b converges, then the limit curves of the scheme S_a with symbol $a(z) = (\frac{1+z}{2})^m b(z)$ are C^m continuous.

Definition 1.1. According to Dyn and Levin (2002) and Rioul (1992), the Hölder regularity of subdivision scheme with symbol $\alpha(z)$ can be computed in the following way. Let $\alpha(z) = \left(\frac{1+z}{2}\right)^l b(z)$, without loss of generality we can assume $b_0,\ldots,b_q$ to be the non-zero coefficients of $b(z)$ and let A_0, A_1 be the $q\times q$ matrices with elements

$$(A_0)_{ij} = b_{q+i-2j},$$
$$(A_1)_{ij} = b_{q+i-2j+1},$$

for $i, j = 0,\ldots,q$. Then the Hölder regularity is given by $r = l - \log_2(\eta)$, where η is the joint spectral radius of the matrices A_0, A_1,

$$\eta = \rho(A_0, A_1) = \limsup_{\kappa\to\infty}\left(\max\left\{\| A_{i_\kappa}\ldots A_{i_2}A_{i_1}\|_\infty^{1/\kappa} : i_\kappa \in \{0,1\}\right\}\right),$$

and

$$\max\{\rho(A_0), \rho(A_1)\} \le \rho(A_0, A_1) \le \max\{\| A_0\|_\infty, \| A_1\|_\infty\}.$$

Since η is bounded from below by the spectral radii and from above by the norm of the metrics A_0, A_1 then $\max\{\rho(A_0), \rho(A_1)\} \le \eta \le \max\{\| A_0\|_\infty, \| A_1\|_\infty\}$.

The paper is furnished as follows: In Section 2, we first introduce the 3-point binary approximating scheme, then check the convergence, continuity, Hölder regularity, polynomial generation, polynomial reproduction, support of the scheme and limit stencil of the schemes. We end this section with visual performance of the scheme. Section 3 deals with the construction of tensor product 3-point binary approximating scheme for the generation of regular surface. The continuity analysis and visual performance is also presented. We end this paper with some concluding remarks.

2. A 3-point binary approximating scheme

In this section, we present and analyze a 3-point binary approximating subdivision scheme. We analysis the scheme by discussing the important features of the scheme such as: continuity, Hölder exponent, polynomial generation, polynomial reproduction, support width of the scheme and limiting behaviour of the curve generated by 3-point scheme. Following is a proposed 3-point binary approximating subdivision scheme with one shape parameter:

$$f_{2i}^{j+1}=\left(\frac{4+w}{16}\right)f_{i-1}^{j}+\left(\frac{12-2w}{16}\right)f_i^{j}+\left(\frac{w}{16}\right)f_{i+1}^{j},$$
$$f_{2i+1}^{j+1}=\left(\frac{w}{16}\right)f_{i-1}^{j}+\left(\frac{12-2w}{16}\right)f_i^{j}+\left(\frac{4+w}{16}\right)f_{i+1}^{j},\quad (2.1)$$

where w is shape parameter.

Remark 2.1. Following is the particular cases of the above scheme:

- It reduces to two point scheme of Chaikins (1974) (i.e. quadratic B-spline), when $w=0$.
- It becomes Hassan *et al*. (2002) (i.e. quadratic B-spline), when we take $w=1$.
- For $w=\frac{-3}{2}$, it coincides with the scheme of Hormann and Sabin (2008).
- It coincides with the scheme of Siddiqi and Ahmad (2007), at $w=\frac{1}{2}$.

When we deal with open initial polygon $f^0=\{f_i^0 : i=0,...,N\}$, it is not easy to improve the first and last edges by rules (2.1). For the sake of simplicity, we describe auxiliary point $f_{-1}^0=2f_0^0-f_1^0$ as extrapolatory rule in the initial polygon f^0 then the first edge $f_0^k f_1^k$ of the non-refined polygon $\{f_i^k : i=0,...,4^k N\}$ can be refined by the following rules

$$f_0^{k+1}=\left(\frac{5}{4}\right)f_0^k+\left(\frac{-1}{4}\right)f_1^k,$$
$$f_1^{k+1}=\left(\frac{12-w}{16}\right)f_0^k+\left(\frac{1}{4}\right)f_1^k.$$

2.1. Analysis of the scheme

The mask of the scheme (2.1) is

$$\alpha=\{w,4+w,12-2w,12-2w,4+w,w\}/16.\quad (2.2)$$

The odd and even stencil of the scheme are $[w,12-2w,4+w]/16$ and $[4+w,12-2w,w]/16$ respectively. The sum of coefficients of even/odd, mask/stencil sum to one i.e.

$$\{w+(12-2w)+(4+w)\}/16=1,$$
$$\{(4+w)+(12-2w)+w\}/16=1.\quad (2.3)$$

The Laurent polynomial $\alpha(z)$ of (2.1) is

$$\alpha(z)=\{wz^{-3}+(4+w)z^{-2}+(12-2w)z^{-1}+(12-2w)z^{0}+(4+w)z^{1}+wz^{2}\}/16.\quad (2.4)$$

This implies $\alpha(z)=(1+z)b(z),\quad (2.5)$

Where

$b(z)=\{wz^{-3}+4z^{-2}+(8-2w)z^{-1}+4z^{0}+wz^{1}\}/16.$

By (2.3) and (2.4), we have $\alpha(-1)=0$, $\alpha(1)=2$, this implies $\alpha(z)=(1+z)b(z)$, and $b(1)=1$. So the necessary conditions for convergence of proposed scheme are satisfied.

Now we discuss the continuity of the proposed scheme (2.1) by repeated application of Theorem 1.4.

Theorem 2.1. The scheme S_α defined by (2.1) with mask α is C^1 continuous over the parametric interval $-2<w<6$ and C^2 continuous for $0<w<2$. Furthermore, it is C^3 continuous at $w=1$.

Proof. Laurent polynomial (2.5) can be written as

$\alpha(z)=\left(\frac{1+z}{2}\right)b_1(z)$, where

$b_1(z)=\{wz^{-3}+4z^{-2}+(8-2w)z^{-1}+4z^{0}+wz^{1}\}/8.$

For C^1 continuity of the scheme S_α corresponding to $\alpha(z)$, we need to show that $b_1(z)$ is convergent. For this we make a difference scheme S_{c_1} corresponding to $c_1(z)$ obtain from $b_1(z)$, where $c_1(z)=\{wz^{-3}+(4-w)z^{-2}+(4-w)z^{-1}+wz^{0}\}/8.$ If scheme S_{c_1} is contractive then scheme S_{b_1} will be convergent and scheme S_α will be C^1 continuous. For contractiveness of the scheme, we see that

$$\|S_{c1}\|_\infty = \max\left\{\left|\frac{w}{8}\right| + \left|\frac{4-w}{8}\right|, \left|\frac{4-w}{8}\right| + \left|\frac{w}{8}\right|\right\} < 1, \text{ for}$$

$-2 < w < 6.$ This implies that S_{c_1} is contractive, S_{b_1} is convergent, and S_α is C^1-continuous. For C^2 continuity of the scheme we re-write (2.5) as $\alpha(z) = \left(\frac{1+z}{2}\right)^2 b_2(z),$ where

$$b_2(z) = \{wz^{-3} + (4-w)z^{-2} + (4-w)z^{-1} + wz^0\}/4.$$

For C^2 continuity of the scheme S_α corresponding to $\alpha(z)$, we need to show that $b_2(z)$ is convergent. For this we make a difference scheme S_{c_2} corresponding to $c_2(z)$ obtain from $b_2(z)$, where $c_2(z) = \{wz^{-3} + (4-2w)z^{-2} + wz^{-1}\}/4.$ If scheme S_{c_2} is contractive then scheme S_{b_2} will be convergent and scheme S_α will be C^2 continuous. For contractiveness of the scheme, we see that

$$\|S_{c_2}\|_\infty = \max\left\{\left|\frac{w}{4}\right| + \left|\frac{w}{4}\right|, \left|\frac{4-2w}{4}\right|\right\} < 1, \text{ for } 0 < w < 2$$

.So scheme S_{c_2} is contractive, S_{b_2} is convergent and S_α is C^2 continuous. For C^3 continuity of the scheme we re-write (2.5) as $\alpha(z) = \left(\frac{1+z}{2}\right)^3 b_3(z),$ where $b_3(z) = \{wz^{-3} + (4-2w)z^{-2} + wz^{-1}\}/2.$ Now at $w = 1$, $b_3(z) = (z^{-3} + 2z^{-2} + z^{-1})/2.$

For C^3 continuity of the scheme S_α corresponding to $\alpha(z)$, we need to show that $b_3(z)$ is convergent. For this we make a difference scheme S_{c_3} corresponding to $c_3(z)$ obtain from $b_3(z)$, where $c_3(z) = (z^{-3} + z^{-2})/2.$ If scheme S_{c_3} is contractive then scheme S_{b_3} will be convergent and scheme S_α will be C^3 continuous. For contractiveness of the scheme, we see that

$$\|S_{c_3}\|_\infty = \max\left\{\left|\frac{1}{2}\right|, \left|\frac{1}{2}\right|\right\} < 1.$$

So at $w = 1,$ scheme S_{c_3} is contractive, S_{b_3} is convergent and S_α is C^3 continuous. This completes the proof.

Now we compute the Hölder regularity of the 3-point binary approximating scheme.

Theorem 2.2. The Hölder regularity of the scheme (2.1) is 4.

Proof. The Laurent polynomial (2.4) at $w = 1$ of the scheme (2.1) can be written as

$$\alpha(z) = \left(\frac{1+z}{2}\right)^5 b(z), \qquad (2.6)$$

where $$b(z) = 2z^{-4}. \qquad (2.7)$$

From (2.6) and (2.7), $b_0 = 2$

(i.e. non zero coefficient of z in $b(z)$), $l = 5$ (no. of factors in $\alpha(z)$), $q = 0$ (i.e. number of non-zero coefficient of z in $b(z)$ start counting from 0) and then A_0 and A_1 are the matrices with elements

$$(A_0)_{ij} = b_{\text{q+i-2j}}$$
$$(A_1)_{ij} = b_{\text{q+i-2j+1}}$$

where $i, j = 0$. This implies only one matrix $A_0 = (b_0) = (2)$. Since $\max\{\rho(A_0)\} \le \eta \le \max\{\| A_0 \|_\infty\},$

then this implies $\rho(A_0) = 2$ and $\| A_0 \|_\infty = 2$. So the spectral radius is $\eta = 2$. Thus by Definition 1.1, the Hölder regularity of the scheme is $r = 5 - \log_2(2) = 5 - 1 = 4$.

2.2. Properties of the scheme

Here we discuss the ability of polynomial generation, polynomial reproduction and approximation order of our scheme.

Theorem 2.3. The degree of polynomial generation of the scheme (2.1) is 2 for parameter w but it is 4 at parametric value $w = 1$.

Proof. The Laurent polynomial $\alpha(z)$ of the scheme (2.1) is

$$\alpha(z)=\{wz^{-3}+(4+w)z^{-2}+(12-2w)z^{-1}+(12-2w)z^{0}+(4+w)z^{1}+wz^{2}\}/16.$$

By making its factors, we get $\alpha(z)=(1+z)^{2+1}b(z),$

where $b(z)=\frac{1}{16z^{4}}\{\omega+(-2\omega+4)z+\omega z^{2}\}.$

Then degree of polynomial generation is 2 for parameter w. But at $w=1,\ \alpha(z)=(1+z)^{4+1}\frac{1}{16z^{4}}$, so degree of polynomial generation is 4.

Theorem 2.4. A scheme (2.1) has dual parameterization.

Proof. Taking derivative of (2.4) with respect to z and substituting $z=1$, we get $\alpha'(1)=-1$. This implies $\tau=\alpha'(1)/2=-1/2$. This further implies $t_i^k=-\tau+\frac{i+\tau}{2^k}=\frac{1}{2}+\frac{i-1/2}{2^k}$. Thus by Theorem 1.1, proposed scheme has dual parameterization.

Theorem 2.5. A 3-point binary subdivision scheme (2.1) reproduces polynomials of degree 2 at parametric value $w=-3/2$ with respect to the parameterizations $t_i^k=\frac{1}{2}+\frac{i-1/2}{2^k}$ if

$$\alpha^{(k)}(1)=2\prod_{l=0}^{k-1}\left(-\frac{1}{2}-l\right)\quad\text{and}\quad \alpha^{(k)}(-1)=0,\quad k=0,1,2.$$

Proof. The Laurent polynomial (2.4) of the scheme (2.1) and its derivatives with respect to z are

$$\alpha(z)=\alpha^{(0)}(z)=\{wz^{-3}+(4+w)z^{-2}+(12-2w)z^{-1}+(12-2w)z^{0}+(4+w)z^{1}+wz^{2}\}/16,$$

$$\alpha^{(1)}(z)=\{-3wz^{-4}-2(4+w)z^{-3}-(12-2w)z^{-2}+4+w+2wz\}/16,$$

and

$$\alpha^{(2)}(z)=\{12wz^{-5}+6(4+w)z^{-4}+2(12-2w)z^{-3}+2w\}/16.$$

Taking, $z=-1$, we get $\alpha^{(k)}(-1)=0,\ k=0,1,2.$ It is easy to see that

$\alpha^{(0)}(1)=2,\alpha^{(1)}(1)=-1,\alpha^{(2)}(1)=3+\omega.$ This implies

$$\alpha^{(0)}(1)=2,\alpha^{(1)}(1)=-1,\alpha^{(2)}(1)=\frac{3}{2},\ \text{at}\ w=-\frac{3}{2}.$$

This further implies

$$\alpha^{(0)}(1)=2,\ \alpha^{(1)}(1)=2\prod_{l=0}^{1-1}\left(-\frac{1}{2}-l\right),\ \alpha^{(2)}(1)=2\prod_{l=0}^{2-1}\left(-\frac{1}{2}-l\right).$$

Thus

$$\alpha^{(k)}(1)=2\prod_{l=0}^{k-1}\left(-\frac{1}{2}-l\right),\quad k=0,\ldots,2.$$

This implies by Theorem1.2 that subdivision scheme (2.1) reproduces polynomials of degree 2 at parameter $w=-3/2$ with respect to the dual parameterization.

Theorem 2.6. A 3-point binary approximating scheme has approximation order 3.

Proof. Since proposed 3-point binary approximating scheme is convergent and its degree of polynomial reproduction is two so by Theorem 1.3 scheme has approximation order 3.

2.3. Support of the scheme

The basic limit function of a subdivision scheme is the limit function of proposed scheme for the data: $f_i^0=1$ for $i=0$ and $f_i^0=0$ for $i\neq 0$. The basic limit function of the proposed scheme is shown in Figure 1. Following theorem can be easily proved by adapting similar methodology of Mustafa *et al* (2015).

Theorem 2.7. The basic limit function of proposed scheme has support width $s=5$ which implies that it vanishes outside the interval which implies that it vanishes outside the interval $\left[-\frac{5}{2},\frac{5}{2}\right]$.

2.4. Limit stencil of 3-point binary scheme

In this section, by using local analysis, we find the limit stencil of binary three point scheme. We can find the limit position of control point of initial control polygon on the limit curve with the help of limit stencil. For this by considering the scheme (2.1) for $i=-1,0,$ we get

$$f_{-2}^{j+1}=\frac{4+w}{16}f_{-2}^{j}+\frac{12-2w}{16}f_{-1}^{j}+\frac{w}{16}f_{0}^{j},$$
$$f_{-1}^{j+1}=\frac{w}{16}f_{-2}^{j}+\frac{12-2w}{16}f_{-1}^{j}+\frac{4+w}{16}f_{0}^{j},$$
$$f_{0}^{j+1}=\frac{4+w}{16}f_{-1}^{j}+\frac{12-2w}{16}f_{0}^{j}+\frac{w}{16}f_{1}^{j}$$

$$f_1^{j+1}=\frac{w}{16}f_{-1}^{j}+\frac{12-2w}{16}f_0^{j}+\frac{4+w}{16}f_1^{j}$$

This implies

$$\begin{pmatrix} f_{-2}^{j+1}\\ f_{-1}^{j+1}\\ f_{0}^{j+1}\\ f_{1}^{j+1}\end{pmatrix}=\begin{pmatrix}\frac{4+w}{16} & \frac{12-2w}{16} & \frac{w}{16} & 0\\ \frac{w}{16} & \frac{12-2w}{16} & \frac{4+w}{16} & 0\\ 0 & \frac{4+w}{16} & \frac{12-2w}{16} & \frac{w}{16}\\ 0 & \frac{w}{16} & \frac{12-2w}{16} & \frac{4+w}{16}\end{pmatrix}\begin{pmatrix} f_{-2}^{j}\\ f_{-1}^{j}\\ f_{0}^{j}\\ f_{1}^{j}\end{pmatrix}$$

Again implies $f^{j+1}=Sf^{j}$, where $f^{j+1}=\{f_i^{j+1}\}_{i=1}^{-2}$ $f^{j}=\{f_i^{j}\}_{i=1}^{-2}$ column matrices and local subdivision matrix S is defined below are

$$S=\begin{pmatrix}\frac{4+w}{16} & \frac{12-2w}{16} & \frac{w}{16} & 0\\ \frac{w}{16} & \frac{12-2w}{16} & \frac{4+w}{16} & 0\\ 0 & \frac{4+w}{16} & \frac{12-2w}{16} & \frac{w}{16}\\ 0 & \frac{w}{16} & \frac{12-2w}{16} & \frac{4+w}{16}\end{pmatrix}$$

has invariant neighbourhood size of four. Eigenvalues of the matrix S are $\lambda=1,1/2,1/4,\frac{1}{4}-\frac{1}{8}w.$ Eigenvectors corresponding to these eigenvalues are

$$\xi_0=(1,1,1,1)^T,$$

$$\xi_1=(-1,\frac{-1}{3},\frac{1}{3},1)^T,$$

$$\xi_2=(1,\frac{w}{-12+w},\frac{w}{-12+w},1)^T,$$

$$\xi_3=(-1,\frac{-w}{-4+w},\frac{w}{-4+w},1)^T,$$

respectively. Therefore we can define diagonal matrix $\wedge$ and non-singular matrix Q as

$$\wedge=\begin{pmatrix}1&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{4}&0\\0&0&0&\frac{1}{4}-\frac{1}{8}w\end{pmatrix},\quad Q=\begin{pmatrix}1&-1&1&-1\\1&\frac{-1}{3}&\frac{w}{-12+w}&\frac{-w}{-4+w}\\1&\frac{1}{3}&\frac{w}{-12+w}&\frac{w}{-4+w}\\1&1&1&1\end{pmatrix}.$$

By diagonalization of matrix S, we get $S=Q\wedge Q^{-1}$ where

$$Q^{-1}=\begin{pmatrix}\frac{1}{24}w & \frac{1}{2}-\frac{1}{24}w & \frac{1}{2}-\frac{1}{24}w & \frac{1}{24}w\\ \frac{-3}{4}\frac{w}{w+2} & \frac{3}{4}\frac{-4+w}{w+2} & \frac{-3}{4}\frac{-4+w}{w+2} & \frac{3}{4}\frac{w}{w+2}\\ \frac{1}{2}-\frac{1}{24}w & \frac{-1}{2}+\frac{1}{24}w & \frac{-1}{2}+\frac{1}{24}w & \frac{1}{2}-\frac{1}{24}w\\ \frac{1}{4}\frac{-4+w}{w+2} & \frac{-3}{4}\frac{-4+w}{w+2} & \frac{3}{4}\frac{-4+w}{w+2} & \frac{-1}{4}\frac{-4+w}{w+2}\end{pmatrix}.$$

Since $S=Q\wedge Q^{-1}$. This can be proved by induction on j. Since $\wedge$ is a diagonal matrix and for any diagonal matrix $\wedge^2$ means square of diagonal entries and so on. Therefore

$$\wedge^j=\begin{pmatrix}(1)^j&0&0&0\\0&\left(\frac{1}{2}\right)^j&0&0\\0&0&\left(\frac{1}{4}\right)^j&0\\0&0&0&\left(\frac{1}{4}-\frac{1}{8}w\right)^j\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1/2^j&0&0\\0&0&1/4^j&0\\0&0&0&1/4^j-1/8^j w\end{pmatrix}.$$

This implies

$$\lim_{j\to\infty}\wedge^j=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Since $f^{j+1}=Sf^{j}=S(Sf^{j-1})=S^{2}f^{j-2}=\cdots S^{j}f^{0}$,
Then $f^{j+1}=(Q\wedge^{j}Q^{-1})f^{0}$. Taking limit, we get

$f^{\infty}=Q(\lim_{j\to\infty}\wedge^{j})Q^{-1}f^{0}$. This implies

$$\begin{pmatrix} f_{-2}^{\infty} \\ f_{-1}^{\infty} \\ f_{0}^{\infty} \\ f_{1}^{\infty} \end{pmatrix} = \begin{pmatrix} \frac{1}{24}w & \frac{1}{2}-\frac{1}{24}w & \frac{1}{2}-\frac{1}{24}w & \frac{1}{24}w \\ \frac{1}{24}w & \frac{1}{2}-\frac{1}{24}w & \frac{1}{2}-\frac{1}{24}w & \frac{1}{24}w \\ \frac{1}{24}w & \frac{1}{2}-\frac{1}{24}w & \frac{1}{2}-\frac{1}{24}w & \frac{1}{24}w \\ \frac{1}{24}w & \frac{1}{2}-\frac{1}{24}w & \frac{1}{2}-\frac{1}{24}w & \frac{1}{24}w \end{pmatrix} \begin{pmatrix} f_{-2}^{0} \\ f_{-1}^{0} \\ f_{0}^{0} \\ f_{1}^{0} \end{pmatrix}.$$

Thus the limit stencil is $\left\{\frac{w}{24},\frac{1}{2}-\frac{1}{24}w,\frac{1}{2}-\frac{1}{24}w,\frac{w}{24}\right\}$. This limit stencil can be used to find the limiting position of the control point f_0^0 on the limit curve.

2.5. Applications

In this section, we present the performance of proposed scheme (2.1), by applying it on closed polygon. Figure 2 shows the initial closed polygon and the results after 1st, 2nd and 3rd subdivision levels by choosing $\omega=1/16$. Figure $2(i)$ is initial polygon which is the sketch of a Chinese character "tha" while figure 2 (ii) (iii) and (iv) are the results after 1st, 2nd and 3rd subdivision levels. Figure 3 shows the initial closed polygon and the results after 1st, 2nd and 3rd subdivision levels by choosing $\omega=1/16$. Figure $3(i)$ is initial polygon which is the sketch of rabbit while 3 (ii) (iii) and (iv) are the results after 1st, 2nd and 3rd subdivision levels.

3. A 3-point tensor product binary scheme

In this section, we will present a 3-point tensor product binary approximating scheme. Laurent polynomial of tensor product scheme can be obtained by the following rule

$$\alpha(\mathbf{z})=\alpha(z_1,z_2)=\alpha(z_1)\alpha(z_2), \qquad (3.1)$$

where $\alpha(z_1)$ and $\alpha(z_2)$ are the Laurent polynomials of 3-point binary scheme. A general compact form of binary subdivision scheme S which maps a polygon $\mathbf{f}^{k}=\{f_{i,j}^{k}\}_{i,j\in\mathbb{Z}}$ to a refined polygon $\mathbf{f}^{k+1}=\{f_{i,j}^{k+1}\}_{i,j\in\mathbb{Z}}$ is defined by

$$f_{2i,2j}^{k+1}=\sum_{r,s\in\mathbb{Z}}\alpha_{2r,2s}f_{i+r,j+s}^{k}, \qquad f_{2i+1,2j}^{k+1}=\sum_{r,s\in\mathbb{Z}}\alpha_{2r+1,2s}f_{i+r,j+s}^{k},$$
$$f_{2i,2j+1}^{k+1}=\sum_{r,s\in\mathbb{Z}}\alpha_{2r,2s+1}f_{i+r,j+s}^{k}, \quad f_{2i+1,2j+1}^{k+1}=\sum_{r,s\in\mathbb{Z}}\alpha_{2r+1,2s+1}f_{i+r,j+s}^{k}. \qquad (3.2)$$

A necessary condition for uniform convergence of scheme (3.2) is given in the following theorem.

Theorem 3.1. Let $\alpha(\mathbf{z})=\alpha(z_1,z_2)=\sum_{i,j}\alpha_{i,j}z_1^{i}z_2^{j}$ be the symbol or Laurent polynomial of bivariate subdivision scheme S, which is defined on quad-meshes. Then a necessary condition for the convergence of S is

$$\sum_{\beta\in\mathbb{Z}^2}\alpha_{a-2\beta}=1,\ a\in\{(0,0),(0,1),(1,0),(1,1)\}, \qquad (3.3)$$

$$\alpha(1,1)=4,\ \alpha(-1,1)=\alpha(1,-1)=\alpha(-1,-1)=0. \qquad (3.4)$$

Theorem 3.2. Dyn (2002) Suppose the schemes with symbols

$$\alpha^{[1]}(\mathbf{z})=\frac{\alpha(\mathbf{z})}{1+z_1}=(1+z_2)b(\mathbf{z}),$$

$$\alpha^{[2]}(\mathbf{z})=\frac{\alpha(\mathbf{z})}{1+z_2}=(1+z_1)b(\mathbf{z}),$$ are both contractive, namely

$\lim_{k\to\infty}(S_{\alpha^{[1]}})^{k}f^{0}=0,\quad \lim_{k\to\infty}(S_{\alpha^{[2]}})^{k}f^{0}=0,$ for any initial data f^{0} then the scheme S_{α} with the symbol $\alpha(\mathbf{z})=(1+z_1)(1+z_2)b(\mathbf{z}),\quad \mathbf{z}=(z_1,z_2),$ is convergent. Conversely, if S_{α} is convergent then $S_{\alpha^{[1]}}$ and $S_{\alpha^{[2]}}$ are contractive.

Theorem 3.3. Dyn (2002) Let $\alpha(z_1,z_2)=(1+z_1)^{m}(1+z_2)^{m}b(\mathbf{z})$. If the schemes with the masks

$$\alpha_{i,j}(z_1,z_2)=\frac{2^{i+j}\alpha(z_1,z_2)}{(1+z_1)^i(1+z_2)^j},\quad i,j=0,...,m$$
(3.5)

are convergent, then S_α generate C^m function.

Remark 3.1. Thus convergence is checked in this case by checking the contractivity of two subdivision schemes $S_{\alpha_1}, S_{\alpha_2}$. If in (3.5), $b(z_1,z_2)=b(z_2,z_1)$, which is typical for schemes having the symmetry of the square grid, then $\alpha(z_1,z_2)=\alpha(z_2,z_1)$, and the contractivity of only one scheme has to be checked.

$$\alpha(z_1,z_2)=(1+z_1)^m(1+z_2)^m b(\mathbf{z}). \quad (3.6)$$

If the schemes with the masks

$$\alpha_{i,j}(z_1,z_2)=\frac{2^{i+j}\alpha(z_1,z_2)}{(1+z_1)^i(1+z_2)^j},\quad i,j=0,...,m.$$
(3.7)

are convergent, then S_α generate C^m function.

Remark 3.2. For C^m continuity of S_α, we have to show that the subdivision schemes $S\_\{i,j\}$, corresponding to masks $\alpha_{i,j}(z_1,z_2)$ for $i,j=0,1,...,m.$ are convergent and it is equivalent to checking whether schemes $\{S_{i,j}^{[1]}\}$ and $\{S_{i,j}^{[2]}\}$ corresponding to the masks

$$\alpha_{i,j}^{[1]}(z_1,z_2)=\frac{2\alpha_{i,j}(z_1,z_2)}{1+z_1}$$ and

$$\alpha_{i,j}^{[2]}(z_1,z_2)=\frac{2\alpha_{i,j}(z_1,z_2)}{1+z_2}$$ are contractive, which is equivalent to checking whether $\left\|\left(\frac{1}{2}S_{i,j}^{[1]}\right)^L\right\|_\infty<1$ and $\left\|\left(\frac{1}{2}S_{i,j}^{[2]}\right)^L\right\|_\infty<1,$ for some integer $L>0.$ Since there are four rules for computing the values at next refinement level, we define the norm for $k=1,2$

$$\left\|\frac{1}{2}S_{i,j}^{[k]}\right\|_\infty=\frac{1}{2}\max\left\{\sum_{k,l\in\mathbb{Z}}|\alpha_{2k,2l}^{[k]}|,\sum_{k,l\in\mathbb{Z}}|\alpha_{2k+1,2l}^{[k]}|,\sum_{k,l\in\mathbb{Z}}|\alpha_{2k,2l+1}^{[k]}|,\sum_{k,l\in\mathbb{Z}}|\alpha_{2k+1,2l+1}^{[k]}|\right\}. \quad (3.8)$$

3.1. Construction of 3-point tensor product scheme

In this section, we present C^3 continuous tensor product binary approximating scheme. For this we consider 3-point binary univariate subdivision scheme proposed in Section 2, for particular value of $\omega=1$ that is

$$\begin{aligned} f_{2i}^{k+1}&=\frac{5}{16}f_{i-1}^k+\frac{10}{16}f_i^k+\frac{1}{6}f_{i+1}^k,\\ f_{2i+1}^{k+1}&=\frac{1}{16}f_{i-1}^k+\frac{10}{16}f_i^k+\frac{5}{16}f_{i+1}^k.\end{aligned} \quad (3.9)$$

Its Laurent polynomial is given as $\alpha(z)=z^{-3}\left\{\frac{1}{16}z^0+\frac{5}{16}z+\frac{10}{16}z^2+\frac{10}{16}z^3+\frac{5}{16}z^4+\frac{1}{16}z^5\right\}.$
This implies

$$\alpha(z_1)=z_1^{-3}\left\{\frac{1}{16}z_1^0+\frac{5}{16}z_1+\frac{10}{16}z_1^2+\frac{10}{16}z_1^3+\frac{5}{16}z_1^4+\frac{1}{16}z_1^5\right\},$$
$$\alpha(z_2)=z_2^{-3}\left\{\frac{1}{6}z_2^0+\frac{5}{16}z_2+\frac{10}{16}z_2^2+\frac{10}{16}z_2^3+\frac{5}{16}z_2^4+\frac{1}{16}z_2^5\right\}.$$

This further implies

$\alpha(z_1)=\frac{z_1^{-3}}{16}(1+z_1)^5,\quad \alpha(z_2)=\frac{z_2^{-3}}{16}(1+z_2)^5.$ Since $\alpha(z_1,z_2)=\alpha(z_1)a(z_2)$ then we have following Laurent polynomial of 3-point tensor product binary approximating scheme S_α

$$\alpha(z_1,z_2)=\frac{z_1^{-3}z_2^{-3}}{256}(1+z_1^5z_2^5+100z_1^3z_2^2+10z_1^5z_2^3+10z_1^5z_2^2+100z_1^2z_2^2$$
$$+100z_1^2z_2^3+50z_1^2z_2^4+10z_1^2z_2^5+10z_2^2+25z_1z_2+100z_1^3z_2^3$$
$$+50z_1z_2^3+25z_1z_2^4+5z_1z_2^5+50z_1^2z_2+5z_1+5z_2+50z_1^3z_2$$
$$+5z_1^5z_2+25z_1^4z_2^4+10z_1^3+50z_1^4z_2^3+25z_1^4z_2+z_1^5+5z_1^4$$

$$+ 50z_1^4z_2^2 + 10z_1^3z_2^5 + 10z_2^3 + 5z_1^5z_2^4 + 50z_1^3z_2^4 + 50z_1z_2^2$$

$$+5z_1^4z_2^5 + 5z_2^4 + z_2^5 + 10z_1^2),$$

$$= \frac{z_1^{-3}z_2^{-3}}{256}(1+z_1)^5(1+z_2)^5.$$

(3.10)

By taking the tensor product of the scheme (3.8) or from (3.9) and changing notations, we suggest following 3-point tensor product binary approximating scheme

$$f_{2i,2j}^{k+1} = \frac{25}{256} f_{i-1,j-1}^k + \frac{25}{128} f_{i,j-1}^k + \frac{5}{256} f_{i+1,j-1}^k + \frac{25}{128} f_{i-1,j}^k + \frac{25}{64} f_{i,j}^k + \frac{5}{128} f_{i+1,j}^k + \frac{5}{256} f_{i-1,j+1}^k + \frac{5}{128} f_{i,j+1}^k + \frac{1}{256} f_{i+1,j+1}^k,$$

$$f_{2i+1,2j}^{k+1} = \frac{5}{256} f_{i-1,j-1}^k + \frac{25}{128} f_{i,j-1}^k + \frac{25}{256} f_{i+1,j-1}^k + \frac{5}{128} f_{i-1,j}^k + \frac{25}{64} f_{i,j}^k + \frac{25}{128} f_{i+1,j}^k + \frac{1}{256} f_{i-1,j+1}^k + \frac{5}{128} f_{i,j+1}^k + \frac{5}{256} f_{i+1,j+1}^k,$$

$$f_{2i,2j+1}^{k+1} = \frac{5}{256} f_{i-1,j-1}^k + \frac{5}{128} f_{i,j-1}^k + \frac{1}{256} f_{i+1,j-1}^k + \frac{25}{128} f_{i-1,j}^k + \frac{25}{64} f_{i,j}^k + \frac{5}{128} f_{i+1,j}^k + \frac{25}{256} f_{i-1,j+1}^k + \frac{25}{128} f_{i,j+1}^k + \frac{5}{256} f_{i+1,j+1}^k,$$

$$f_{2i+1,2j+1}^{k+1} = \frac{1}{256} f_{i-1,j-1}^k + \frac{5}{128} f_{i,j-1}^k + \frac{5}{256} f_{i+1,j-1}^k + \frac{5}{128} f_{i-1,j}^k + \frac{25}{64} f_{i,j}^k + \frac{25}{128} f_{i+1,j}^k + \frac{5}{256} f_{i-1,j+1}^k + \frac{25}{128} f_{i,j+1}^k + \frac{25}{256} f_{i+1,j+1}^k.$$

(3.11)

The Laurent polynomial of the scheme (3.11) is given in (3.10). The Laurent polynomial (3.10) satisfy the necessary condition of convergence given in (3.4). To check the continuity of the 3-point tensor product scheme (3.11), we apply similar analysis tools given in the beginning of this section. Since $\{S_{i,j}^{[k]};\ k=1,2\quad i,j=0,1,2,3\}$ are contractive, so by Theorem 3.2, the subdivision schemes $S_{i,j}$, corresponding to masks $\alpha_{i,j}(z_1,z_2)$, for $i,j=0,1,2,3$ are convergent. Hence by Theorem 3.3 the proposed scheme S_α is C^3 continuous.

The visual performance of our 3-point tensor product binary approximating scheme is shown in Figure 4. In this figure models are constructing at different subdivision levels to show the visual performance of our proposed scheme.

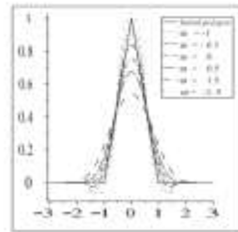

Figure 1: *The basic limit function of proposed scheme.*

Figure 2: *(i) Initial closed polygon. After applying 1st, 2nd and 3rd subdivision steps, we get (ii), (iii) and (iv).*

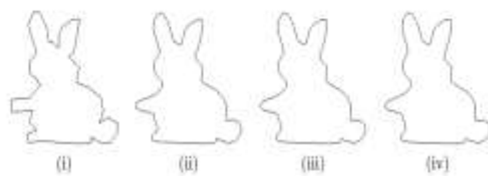

Figure 3: *(i) Initial closed polygon. After applying 1st, 2nd and 3rd subdivision steps, we get (ii), (iii) and (iv).*

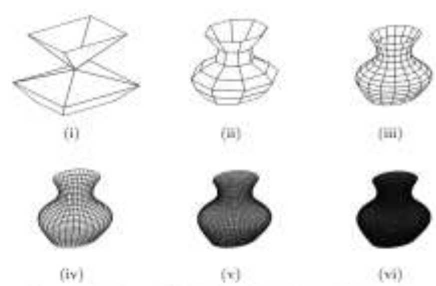

Figure 4: *(i) initial polygon while (ii), (iii), (iv), (v) and (vi) are the results after 1st, 2nd, 3rd, 4th and 5th subdivision steps.*

4. Conclusion: In this paper, we have offered 3-point binary approximating subdivision scheme with shape parameter. The proposed scheme is good for modeling of curves and surfaces. It generates smooth models. The important properties of the scheme have been also discussed.

References

Cashman, T.J., N.A. Dodgson and M.A. Sabin. 2009. A symmetric, non-uniform, and smooth subdivision algorithm for general degree b-splines. *Computer Aided Geometric Design*, 26: 94-104.

Chaikin, G.M. 1974. An algorithm for high speed curve generation. *Computer Graphics and Image Processing,* 3: 346-349.

Choi, S.W., B.G. Lee, Y.J. Lee and J. Yoon. 2006. Stationary subdivision schemes reproducing polynomials. *Computer Aided Geometric Design*, 23: 351-360.

Deslauriers G. and S. Dubuc. 1989. Symmetric iterative interpolation processes. *Constructive Approximation*, 5: 49-68.

Dyn, N. 2002. Tutorial on multi resolution in geometric modelling summer school lecture notes series, Mathematics and Visualization, Springer, ISBN: 3-540-43639-1.

Dyn, N. and D. Levin. 2002. Subdivision scheme in the geometric modling. *Actaumerica*, 11: 73-144.

Dyn, N., J.A. Gregory and D. Levin. 1987. A 4-points interpolatory subdivision scheme for curve design. *Computer Aided Geometric Design*, 4: 257-268.

Dyn, N., M.S. Floater and K. Hormann. 2005. A C^2 four-point subdivision scheme with fourth order accuracy and its extensions, In: Mathematical methods for curves and surfaces: troms 2004, Nashboro Press, Brentwood.

Ghaffar, A., G. Mustafa and K. Qin. 2012. Unification and application of 3-point approximating subdivision schemes of varying arity. *Open Journal of Applied Sciences*, 2: 48-52.

Ghaffar, A., G. Mustafa and K. Qin. 2013. Construction and application of 3-point tensor product scheme. *Applied Mathematics*, 4: 477-485.

Hassan, M.F., M.F. Ivrissimitzis, N.A. Dodgson and M.A. Sabin. 2002. An interpolating 4-points C^2 ternary stationary subdivision scheme. *Computer Aided Geometric Design* 19:1-18.

Hormann, K. 2012. Subdivision schemes for geometric modelling: a hands-on tutorial Dolomites research notes on approximation. *Publisher University of Verona*, 5: 2035-680.

Hormann, K. and M.A. Sabin. 2008. A family of subdivision schemes with cubic precision. *Computer Aided Geometric Design*, 25: 41-52.

Khan, F., I. Mukhtar and N. Batool. 2013. A unified interpolating subdivision scheme for curve/surface by using Newton interpolating polynomial. *Open Journal of Applied Sciences*, 3: 263-269.

Ko, K.P., B.G. Lee and G.J. Yoon. 2007. A study on the mask of interpolatory symmetric subdivision schemes. *Applied Mathematics and Computation*, 187: 609-621.

Mustafa, G. and N.A. Rahman. 2010. The mask of (2b+4)-point n-ary subdivision scheme. *Computing*, 90: 1-14.

Mustafa, G., F. Khan, M.S. Hashmi and M.Z. Afzal. 2011. A unified three point approximating subdivision scheme. *Analysis in Theory and Applications*, 27: 10-20.

Mustafa, G., L. Hao, J. Zhang and J. Deng. 2015. L1-regression subdivision schemes for noisy data. *Computer aided Geometric Design*, 58: 189-199.

Rham, D.G. 1956. Sur unecourbe plane. *Journal de Math Pures and Applied*, 35: 25-42.

Rioul, O. 1992. Simple regularity for subdivision schemes. *SIAM Journal on Mathematical Analysis,* 23: 1544-1576.

Romani, L. 2009. From approximating subdivision schemes for exponential splines to high performance interpolating algorithms. *Journal of Computational and Applied Mathematics*, 224: 383-396.

Siddiqi, S.S. and K. Rehan. 2010. Improved binary four point subdivision scheme and new corner cutting scheme. *Computers and*

Mathematics with Applications, 59: 2647 - 2657.

Siddiqi, S.S. and N. Ahmad. 2006. An approximating C^2 stationary subdivision scheme. *European Journal of Scientific Research*, 20: 97-102.

Siddiqi, S.S. and N. Ahmad. 2007. A C^5 approximating subdivision scheme. *Applied Mathematics Letter*, 21: 722-728.

Siddiqi, S.S. and N. Ahmad. 2008. A new five-point approximating C^5 subdivision. *International Journal of Computer Mathematics*, 85: 65-72.

Tang, Y., K.P. Ko and B.G. Lee. 2005. A new proof of the smoothness of 4-point Deslauriers-Dubuc Scheme. *Journal of Applied Mathematics and Computing*, 18: 553-562.

Zhang, L., Z. Luo and W. Liu. 2004. An approximating C^3 stationary subdivision scheme. *Journal of Information and Computing Sciences*, 1: 157-161.