A new non-polynomial solution to multivariate Hermite-Birkhoff interpolation

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Abstract: - A new solution to the multivariate Hermite-Birkhoff interpolation problem is presented. The classical approach to this problem consists in constructing the minimum degree polynomial, which coincides with the prescribed function and derivative values at the sample points. Here the interpolant is represented as a truncated Multipoint Taylor (MT) series. A MT series can be regarded as an extension to multiple points of the ordinary (one point) Taylor series. The constructed MT series converges when (i) the number of derivatives tends to infinity, while the number of sample points remains finite, or (ii) when the spacing between sample points tends to zero, while the number of derivatives remains finite. The MT series region of convergence depends on the support of its basis functions and the location of the singularities of the function represented.

Examples are given of the construction of interpolants in one and two dimensions and for different samples grid types. A comparison between the numerical results achieved with this new method and those obtained with one-dimensional polynomial Hermite-Birkhoff interpolation is made. Our interpolant does not suffer from the ill conditioning seen with the classical polynomial solution for certain data configurations. It is always numerically stable in its region of convergence. In addition, it is computationally much more efficient than the polynomial approach, because no basis function construction phase is required. Finally, the presented construction is applicable to interpolation in any number of dimensions and could also be regarded as a finite element data representation, which uses both functional values as well as derivative values.

Keywords: - Hermite-Birkhoff interpolation, Multivariate interpolation, Multipoint Taylor series, Finite elements.

1 Introduction

We consider the general Hermite-Birkhoff interpolation problem on rectangular and irregular samples grids in \( \mathbb{R}^D \) \([6, 18, 19]\). The problem consists in finding an interpolant that passes through given function values and matches a number of (partial) derivatives prescribed at the sample points. The classical approach to this problem consists in constructing a polynomial, which coincides with the prescribed functional sample data.

This problem was first solved in 1878 by Hermite on \( \mathbb{R} \), for function values and first derivatives given at the sample points. In 1906 Birkhoff considered a more general problem, involving arbitrary higher order derivatives. For an overview of Hermite-Birkhoff interpolation, see e.g., \([18, 19]\). More recently, several authors have derived polynomial solutions for the multivariate case en for various data configurations \([1, 3, 5, 11, 13, 15, 23, 25]\).

It turns out that the polynomial Hermite-Birkhoff interpolation problem is very difficult. It is basically equivalent to solving a linear system, which can have small eigenvalues or even be singular (depending on the specified derivatives and the position of the sample points), and it is in general not easy to determine the solvability of the problem \([12, 20]\). Even should the solvability of the problem be known, it remains difficult to obtain an explicit form for the solution. In cases where an explicit representation of the solution can be found, it is usually a complicated expression and thus hard to study. All this explains why at present no numerical stable algorithms are available to solve the polynomial Hermite-Birkhoff problem in its full generality.

To circumvent these problems an alternative method is presented that uses a non-polynomial approach. It is based on a new concept, called a Multipoint Taylor (MT) series. This series expresses the value of a real-analytic function \( f : \Omega \subset \mathbb{R} \rightarrow \mathbb{R} \) at some point, in terms of the function and derivative values at a finite set of sample points, thus constituting a generalization of the ordinary (one point) Taylor series. It was proven in \([8]\) that a MT series converges when (i) the number of derivatives tends to infinity, while the number of sample points remains finite, or (ii) when the spacing between sample points tends to zero, while the number of derivatives remains finite. The MT series region of convergence depends on the support of its basis functions and the location of the singularities of the function represented.

In contrast with the ordinary Taylor series, the basis functions of a MT series are not unique. This degree of freedom gives the algorithm designer additional control
over the behavior of the interpolant, but without
loosing any of the basic properties of the method. Here,
we use for the basis functions simple, compact support,
transcendental functions. These functions must satisfy
certain conditions, as will be explained in Section 2.

The Hermite-Birkhoff interpolation solution is then
most naturally formulated as a truncated MT series.
The so obtained interpolant is infinitely differentiable
everywhere, has a computational cost that is linear with
the number of sample points and is always numerical
stable in its region of convergence.

It has the additional property that, at the sample
points, all non-prescribed derivatives are automatically
zero. In this it also differs from polynomial Hermite-
Birkhoff interpolation, where certain higher order
derivatives are completely determined, as a conse-
quence of specifying lower order ones [8].

In the next section, the approximation of a function
by a truncated MT series is summarized. In Section 3
the construction of basis functions is explained for
randomly scattered data (irregular samples grid) and for
the special case when the samples lie on a rectangular
(product) grid. In Section 4, a numerical comparison is
made between the MT method and polynomial
Hermite-Birkhoff interpolation for \( D = 1 \). For \( D = 2 \),
a numerical example is treated for a product grid and
for an irregular grid.

2 Multipoint Taylor series

Let \( \mathbf{x} = (x_1, \ldots , x_D) \in \mathbb{R}^D \) denote an evaluation point and
\( S = \{ \mathbf{x}_n = (x_{1n}, \ldots , x_{Dn}) \in \mathbb{R}^D , \ 1 \leq n \leq N \} \) a given set
of sample points of a real-analytic function \( f : \Omega_f \subset \mathbb{R}^D \to \mathbb{R} \), \( \mathbf{x} \mapsto f(\mathbf{x}) \). A MT series is
defined inside its region of convergence \( \Omega_c \subset \mathbb{R}^D \) as
\[
  f(\mathbf{x}) = \sum_{n=1}^{N} \sum_{k_1=0}^{K_1} \ldots \sum_{k_D=0}^{K_D} f_{k_1\ldots k_D}^{n} U_{k_1\ldots k_D}^{n} (\mathbf{x}) ,
\]
where \( f_{k_1\ldots k_D}^{n} \in \mathbb{R} \) stands for the \( (k_1, \ldots , k_D) \) -partial
derivative at the sample point \( \mathbf{x}_n \)
\[
  f_{k_1\ldots k_D}^{n} = \frac{\partial^{k_1+\ldots+k_D} f(x_1, \ldots , x_D)}{\partial x_1^{k_1} \ldots \partial x_D^{k_D}} \bigg|_{\mathbf{x}=\mathbf{x}_n}.
\]
\( U_{k_1\ldots k_D}^{n} \in C^{\infty} \) are basis functions \( \Omega_{U_n} \subset \mathbb{R}^D \to \mathbb{R} \),
\( \mathbf{x} \mapsto U_{k_1\ldots k_D}^{n} (\mathbf{x}) \) and must satisfy
\[
  \frac{\partial^{k_1+\ldots+k_D} U_{k_1\ldots k_D}^{n} (x_1, \ldots , x_D)}{\partial x_1^{k_1} \ldots \partial x_D^{k_D}} \bigg|_{\mathbf{x}=\mathbf{x}_n} = \delta_{n} \delta_{k_1} \ldots \delta_{k_D},
\]
\( \forall \ k_1 + \ldots + k_D \geq 0 , \ l_1 + \ldots + l_D \geq 0 \) and \( 1 \leq n,m \leq N \).

For reasons explained elsewhere [8], we relate
higher order basis functions \( U_{k_1\ldots k_D}^{n} (\mathbf{x}) \) for all
\( k_1 + \ldots + k_D > 0 \) to zero order functions \( U_{k_1\ldots k_D}^{0} (\mathbf{x}) \) by
\[
  U_{k_1\ldots k_D}^{n} (\mathbf{x}) = \frac{(x_1-x_{1n})^{k_1} \ldots (x_D-x_{Dn})^{k_D}}{k_1! \ldots k_D!} U_{k_1\ldots k_D}^{0} (\mathbf{x}) .
\]
Conditions (3) then imply that the zero order basis functions \( U_{k_1\ldots k_D}^{0} (\mathbf{x}) \) should satisfy, for \( 1 \leq n,m \leq N \),
\[
  U_{k_1\ldots k_D}^{0} (\mathbf{x}) \big|_{\mathbf{x}=\mathbf{x}_n} = \delta_{nm},
\]
\[
  \frac{\partial^{k_1+\ldots+k_D} U_{k_1\ldots k_D}^{0} (x_1, \ldots , x_D)}{\partial x_1^{k_1} \ldots \partial x_D^{k_D}} \bigg|_{\mathbf{x}=\mathbf{x}_n} = 0 , \ k_1 + \ldots + k_D > 0
\]
(5). Using (4) the MT series for \( \mathbf{x} \in \Omega \) can be written as
\[
  f(\mathbf{x}) = f(\mathbf{x}) \left( \sum_{n=1}^{N} U_{k_1\ldots k_D}^{n} (\mathbf{x}) \right) .
\]
Because we assumed real-analyticity of the function \( f(\mathbf{x}) \) in \( \Omega \subset \Omega_f \), the \( N \) ordinary Taylor series in
(6), evaluated about each of the \( N \) points, converge for
any interpolation point \( \mathbf{x} \in \Omega \), to the same value
\( f(\mathbf{x}) \). A sufficient condition for convergence of a MT
series is therefore that \( \Omega \subset \Omega_{U_n} \cap \ldots \cap \Omega_{U_N} \), with \( U_n \)
the region of convergence of the \( n \)-th ordinary Taylor
series. This condition can be relaxed however if the
basis functions have local support (\( \Omega_{U_n} \subset \Omega \),
\( 1 \leq n \leq N \), see Section 3).

From (6) follows that, to ensure convergence to the
correct function, the zero order functions \( U_{k_1\ldots k_D}^{0} (\mathbf{x}) \)
must form a partition of unity over the domain \( \Omega \),
\[
  \sum_{n=1}^{N} U_{k_1\ldots k_D}^{0} (\mathbf{x}) \equiv 1 .
\]
We will call the truncated MT series
\[
  \tilde{f}(\mathbf{x}) = \sum_{n=1}^{N} \sum_{k_1=0}^{K_1} \ldots \sum_{k_D=0}^{K_D} f_{k_1\ldots k_D}^{n} U_{k_1\ldots k_D}^{n} (\mathbf{x}) ,
\]
a MT interpolant for \( f(\mathbf{x}) \) with functional data
\( F \triangleq \{ f_{k_1\ldots k_D}^{n} , \ 0 \leq k_i \leq K_i , \ 1 \leq i \leq D, 1 \leq n \leq N \} \) at \( S \).
It is readily clear that (8) is a natural solution to the
Hermite-Birkhoff interpolation problem.

3 Construction of the basis functions

3.1 Random grid

For practical computations, it is desirable to have a
local interpolation algorithm. Then the computational
cost of the algorithm becomes linear with the number
of sample points. This is achieved by using compact
support basis functions.

To this end, we introduce a decomposition of the
interpolation domain $\Omega$ by some tessellation method [21]. A particular convenient tessellation for scattered sample points $S$ in two dimensions and used here, is Delaunay triangulation [2, 7, 14]. It has the property that within the circumscribed circle of a spherical triangle, having three sample points as vertices, no other sample point is found. For $D > 2$ a similar higher dimensional decomposition in ‘(D+1)-hedra’ can be used.

First, consider the case $D = 2$. As support area for the basis functions we use the local polygon around each sample point, as follows. We define the neighboring sample points of a given sample point $x_i$, the ordered set $X_i \triangleq \{x_{m(i)}, i = 1, N_i \}$, such that the Delaunay triangles $H_{m(i)} \triangleq H(x_i, x_{m(i)}, x_{m(i+1)})$ for $i = 1, N_i$ (with $x_{m(N_i+1)} \triangleq x_{m(1)}$), have $x_{m(i)}$ as their common vertex. The notation $m(i)$ stands for the map that converts the local numbering (over the vertices of the local polygon) to the global sample point numbering. Then $P_i \triangleq H_{i-n} \cup \ldots \cup H_{N_i}$ is called the local polygon associated with the point $x_i$. This definition is readily extended to higher dimensions by replacing ‘triangle’ by ‘(D+1)-hedron’ and ‘local polygon’ by ‘local polytope’.

Conditions (3) do not uniquely determine the zero order basis functions. This non-uniqueness is no problem in the full MT series, because the exact form of the basis functions drops out if the summation is taken over all derivatives up to infinity, due to the partition of unity condition (7). But for the interpolant it is an additional degree of freedom, which the algorithm designer can use to his advantage. We will give here one particular form, which was found to be of computational convenience.

For the zero order basis function $U_{0,\ldots,0}^i(x)$, associated with the sample point $x_i$, we use the following radial construction. Put

$$U_{0,\ldots,0}^i(x) \triangleq \frac{\Psi_n(x)}{\sum_{j=0}^{D} \Psi_{m(j)}(x)}.$$  \hspace{1cm} (9)

The summation in the denominator runs over the vertices of the (D+1)-hedron that contains the evaluation point $x$. The form (9) assures that the partition of unity condition (7) will be satisfied. The function $\Psi_n(x)$ is defined by

$$\Psi_n(x) \triangleq \begin{cases} \Phi(r/r_b(\theta_1, \ldots, \theta_{D-1})), & x \in P_i \\ 0, & x \notin P_i \end{cases},$$  \hspace{1cm} (10)

where

$$\Phi(\omega) \triangleq \frac{1}{2} \left[ 1 + \tanh \left( \frac{1}{4} \left( \frac{1}{\omega} - \frac{1}{1-\omega} \right) \right) \right],$$  \hspace{1cm} (11)

for $0 \leq \omega \leq 1$ and an evaluation point $x(\theta_1, \ldots, \theta_{D-1}) = x_i + ru(\theta_1, \ldots, \theta_{D-1})$. Here, $u(\theta_1, \ldots, \theta_{D-1})$ denotes the unit vector from $x_i$ to $x$, with direction specified by the Euler angles $\theta_1, \ldots, \theta_{D-1}$, so $r = |x(\theta_1, \ldots, \theta_{D-1}) - x_i|$, i.e., the distance from the central sample point $x_i$ to its local polytope boundary $\partial P_i$ along the line through $x_i$ and $x$.

All the derivatives of the function $\Phi(\omega)$ tend to zero when $\omega \rightarrow 0, 1$, and this makes that $U_{0,\ldots,0}^i(x)$ will satisfy the derivative conditions (5). $U_{0,\ldots,0}^i(x)$ is non-negative, equals 1 at the interior sample point and decreases monotonically to 0 on the local polytope boundary $\partial P_i$. In addition, it has all its partial derivatives zero at $x_i$, as well as on $\partial P_i$. Values of higher order basis functions are obtained from (4). Notice that, due to the compact support of the $\Psi_n(x)$, only the D+1 sample points that form the vertices of the (D+1)-hedron, containing the evaluation point, contribute to the interpolation value at that evaluation point. It is therefore sufficient to sum only over these D+1 $\Psi_n(x)$ functions in the denominator of (9) to satisfy (7). Notice also that, for univariate interpolation, (7) is satisfied when the denominator in (9) is 1, because $\Phi(\omega) + \Phi(1-\omega) = 1$. Due to the compact support of the basis functions the first sum in the interpolant (8) will also consists of only D+1 non-vanishing terms. Our representation is thus similar to a finite element construction, which not only takes function values at the vertices, but also a finite set of partial derivative values, to approximate a function inside a (D+1)-hedron element.

For basis functions, having their local polytope as support, the necessary and sufficient condition for convergence of the MT series is now that $H(x_{m(1)}, \ldots, x_{m(D+1)}) \subset \Omega(1) \cap \cdots \cap \Omega_{m(D+1)}$, i.e., any (D+1)-hedron should lie in the intersection of the convergence regions of the D+1 ordinary Taylor series about its vertices $x_{m(1)}, \ldots, x_{m(D+1)}$.

To numerically evaluate the basis functions one should take some minor precautions to prevent overflow and underflow when evaluating the $\tanh$
function in (11). Apart from this, it will be clear from (9)-(11) that the evaluation of the basis functions poses no numerical problems. Hence, our interpolant (8) avoids the computational errors that are inherent in the classical polynomial approach.

3.2 Product grid
An important simplification arises when the functional data is given on the knots of a product grid $S = S_1 \times \ldots \times S_D$, $S_i = \{x_n, 1 \leq n \leq N_i\}$, $N = N_1 \ldots N_D$.

In this case, we do not need a decomposition of the interpolation domain, but just use separation of variables. We now take as zero order basis functions

$$U_{n}^{0} = \begin{cases} \Phi \left( \frac{x_n - x_i}{x_n - x_{n+1}} \right), & x_{n+1} \leq x_i \leq x_n \\ \Phi \left( \frac{x - x_m}{x_{m+1} - x_m} \right), & x_m \leq x_i \leq x_{m+1} \\ 0, & x_i < x_{m-1} \& x_{m+1} < x_i \end{cases},$$

and herein $\Phi(\omega)$ is again given by (11). Because $\Phi(\omega) + \Phi(1-\omega) \equiv 1$ and due to the compact support of the partial basis functions $U_{n}^{0}(x_i)$, it is readily seen that the partition of unity condition (7) is satisfied. Also the conditions (5) will hold. The first sum in the interpolant (8) will now consists of $2^D$ terms, which makes the interpolant computation time proportional to the number of interpolation points times the number of vertex points per grid cell (i.e., $2^D$).

4 Numerical examples

4.1 Comparison for $D=1$
We use our method to interpolate an arbitrary test function and compare the results with a classical polynomial Hermite-Birkhoff interpolation algorithm, for various combinations of the number of sample points and number of derivatives.

The polynomial Hermite-Birkhoff interpolation algorithm used to compare against is the one implemented in the NAG Fortran Library (Mark 17), by routines 'E01AEF' and 'E02AKF' [16, 17, 20]. The interpolation interval is $\Omega = [-1,+1]$. The derivative values of the test function at the sample points up to $K_n = K$, $1 \leq n \leq N$ were computed analytically and fed to both algorithms. The relative root mean square (RMS) error is defined as

$$\eta_{\text{RMS}} = \sqrt{\frac{\int_{-1}^{1} \left( f(x) - f(x) \right)^2 \, dx}{\int_{-1}^{1} f(x)^2 \, dx}}. \quad (14)$$

All computations used double precision (16 digits).

Table 1 shows $\eta_{\text{RMS}}$ obtained by our method when applied to the arbitrary test function, for $x \in \Omega$,

$$f(x) = \frac{0.75}{(x+0.85)^2 + (0.11)^2} - \frac{2.75}{(x+0.2)^2 + (0.15)^2} + \frac{1.6}{x^2 + (0.11)^2} + \frac{2.0}{(x-0.6)^2 + (0.2)^2}.$$

It shows that the presented method is capable of solving the Hermite-Birkhoff problem for a wide variety of number of sample points and number of derivatives and that the interpolation error converges to machine precision.

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Table 1. Relative RMS error for test function (15) for truncated MT interpolation.

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Table 2. Relative RMS error for test function (15) for polynomial Hermite-Birkhoff interpolation.
interpolation. It is striking to see that the NAG routine was unable to interpolate this function satisfactorily for none of the considered samplings or number of derivatives. A dash entry in the table indicates no output from the routine. The polynomial routine also fails for non-equidistant samplings, as was demonstrated in [8]. For a more detailed numerical comparison between one-dimensional polynomial and truncated MT interpolation, see [8].

### 4.2 Examples for D=2

Due to the absence of any publicly available polynomial Hermite-Birkhoff interpolation codes in two dimensions, we only show the numerical results obtained with the MT algorithm. The RMS interpolation error \( \eta_{RMS} \) as obtained for the test function

\[
f(x, y) = \cos(\pi x) \exp(-4y^2),
\]

over \(-1 \leq x \leq 1\) and \(-1 \leq y \leq 1\), is shown (i) in Tables 3 and 4 for a product samples grid and (ii) in Tables 5 and 6 for a random samples grid.

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<td>5 x 5</td>
<td>2.10^{-2}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 3</td>
<td>10 x 10</td>
<td>1.10^{-3}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 3</td>
<td>20 x 20</td>
<td>5.10^{-5}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 3</td>
<td>40 x 40</td>
<td>3.10^{-6}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 3</td>
<td>80 x 80</td>
<td>2.10^{-7}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 3</td>
<td>160 x 160</td>
<td>1.10^{-8}</td>
</tr>
</tbody>
</table>

Table 3. Relative RMS error for function (16) for varying grid size.

<table>
<thead>
<tr>
<th>Derivatives</th>
<th>Grid size</th>
<th>RMS error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 \leq k, l \leq 1</td>
<td>10 x 10</td>
<td>5.10^{-2}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 2</td>
<td>10 x 10</td>
<td>4.10^{-3}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 3</td>
<td>10 x 10</td>
<td>1.10^{-3}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 4</td>
<td>10 x 10</td>
<td>1.10^{-4}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 5</td>
<td>10 x 10</td>
<td>2.10^{-5}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 6</td>
<td>10 x 10</td>
<td>4.10^{-6}</td>
</tr>
</tbody>
</table>

Table 4. Relative RMS error for function (16) for varying number of derivatives.

Both numerical examples illustrate the practical usefulness of the truncated MT algorithm for two-dimensional Hermite-Birkhoff interpolation.

As an application of the MT series representation, a compression algorithm was developed for scientific data sets [9].

### 5 Conclusions

A new algorithm for Hermite-Birkhoff interpolation on \( \mathbb{R}^D \) was presented. The interpolant has the form of a truncated Multipoint Taylor series. It is infinitely differentiable everywhere, has a computational cost that is linear with the number of evaluation points and is always numerical stable in its region of convergence.

Numerical examples were given that demonstrate our algorithm superiority over the classical polynomial solution. The classical solution has a number of drawbacks. First, it is very difficult, if not impossible, to prove in general, from the construction of such a polynomial, whether it converges to a given function when the number of prescribed derivatives at each sample point tend to infinity. Because certain higher order derivatives are completely determined as a consequence of specifying lower order ones [8], the polynomial interpolant may (and often does) show unwanted oscillatory behavior. It is not clear what effect this (unwanted) property of polynomial interpolation has on the convergence of the interpolant. Secondly, it is well known (for the univariate case) that calculating a Hermite-Birkhoff polynomial interpolant is an inherently unstable process for certain functional data (even in exact arithmetic) and is especially critical for equidistant samples [20]. Thirdly, for \( D > 1 \) there

<table>
<thead>
<tr>
<th>Derivatives</th>
<th>Grid size</th>
<th>RMS error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 \leq k, l \leq 3</td>
<td>600</td>
<td>3.10^{-5}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 3</td>
<td>6400</td>
<td>3.10^{-6}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 3</td>
<td>25600</td>
<td>2.10^{-7}</td>
</tr>
</tbody>
</table>

Table 5. Relative RMS error for function (16) for varying grid size.

<table>
<thead>
<tr>
<th>Derivatives</th>
<th>Grid size</th>
<th>RMS error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 \leq k, l \leq 1</td>
<td>100</td>
<td>8.10^{-2}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 2</td>
<td>100</td>
<td>1.10^{-2}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 3</td>
<td>100</td>
<td>4.10^{-3}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 4</td>
<td>100</td>
<td>1.10^{-3}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 5</td>
<td>100</td>
<td>5.10^{-4}</td>
</tr>
<tr>
<td>0 \leq k, l \leq 6</td>
<td>100</td>
<td>1.10^{-4}</td>
</tr>
</tbody>
</table>

Table 6. Relative RMS error for function (16) for varying number of derivatives.
exists sample point distributions for which no polynomial solution exists to the Hermite-Birkhoff problem [12, 22, 24].

The MT interpolant, as constructed here, can also be regarded as a finite element data representation, which takes besides functional values also derivative values at the sample points into account [4]. This makes it a natural technique for the implementation of mixed FEM’s, and so avoids the stability issues associated with the classical polynomial element interpolation schemes, see e.g., [10]. In addition, our interpolant has the desirable property that derivatives and (indefinite) integrals of the represented function can be approximated with the same functional data, in a similar way as the function itself, thus avoiding having to compute derivatives numerically from the interpolant.

References: