Probabilistically Reliable Default Reasoning

Gerhard Schurz, University of Salzburg

Abstract

This paper superimposes an assumption generation mechanism (AGM) and a lower bound propagation mechanism (LBM) on default reasoning. AGM generates minimal probabilistic assumptions which are needed to derive default conclusions safely, and LBM supplies (approximately) tight lower probability bounds of the conclusions. Together both mechanisms make default reasoning probabilistically reliable. The underlying logical framework is an extension of Adams’ probability logic \( P_\epsilon \) by irrelevance and contraposition assumptions, called the system \( P_\epsilon DP \). There is an exact correspondence between Poole-extensions and \( P_\epsilon DP \)-extensions. The procedure for \( P_\epsilon DP \)-entailment is comparable in complexity with Poole’s procedure.

1 Introduction and Motivation

Default reasoning is reasoning from uncertain laws, like “Normally, birds can fly”. We call these laws default laws and formalize them as \( B \Rightarrow F \) (where \( \Rightarrow \) stands for uncertain implication and \( B, F \) are associated with an invisible variable \( x: Bx, Fx \)). Default laws have two characteristics. First, they have exceptions (e.g., penguins don’t fly), and second, we do not know the conditional probability of flying animals among birds, but we assume that this probability is high – otherwise we would not be justified in calling this the normal case. Hence a necessary condition for the acceptability of a default law \( A \Rightarrow B \) is that \( p(B/A) \) is ‘sufficiently’ high (higher than a minimal acceptability threshold). We call \( p(B/A) \) the conditional probability associated with \( A \Rightarrow B \).
The intuitive principle which underlies reasoning with default laws is that one may detach $Ga$ from $F \Rightarrow G$ and $Fa$ as long as $\neg Ga$ is *not* implied by one’s knowledge base. This principle is ambiguous, and so, several different systems of reasoning with default laws (or rules) have been developed (e.g., [19, 29, 20, 26, 25]). How are these systems justified? In this respect, clear criteria are often missing. Such criteria should be explicated in terms of *truth preservation conditions*, for it is TRUTH which connects statements with the real world.

In contrast to the inferences of deductive (monotonic) logic, nonmonotonic inferences are uncertain: they do not preserve truth (from premises to conclusion) for all of their individual instances. Still, they should be reliable – that is, they should at least preserve truth in most of their individual instances. Assume a knowledge base, consisting of default laws and facts, predicts (i.e. infers) a conclusion $C$ about an individual $a$. Then by the *law of large numbers*, the frequency of true predictions among all predictions of $C$ for individuals $x$ which share all factual information with $a$ will approximate the conditional statistical probability of $C$ given all factual knowledge about $a$ (with $a$ replaced by $x$; cf. §2.1). Since the law premises of the inference are true iff their associated conditional probabilities are high, it follows that the requirement of reliability will be satisfied iff the following preservation condition for conditional probabilities is satisfied:

- *(Reliability:)* If the conditional probabilities associated with the default laws used as premises in the derivation of a conclusion are high, then the conditional probability of the conclusion given all factual knowledge is high, too.

The probability concept ‘$p(\cdot)$’ which underlies this approach is statistical (as opposed to subjective) probability – this is essential for the *tie* between reliability and a high predictive success rate. It is also important that the premises need not be identical with the default laws contained in the knowledge base, but may be derived from them with help of probabilistic *default assumptions* (irrelevance and contraposition assumptions).

Standard default logic violates the requirement of reliability. E.g., this becomes apparent in the *Conjunction Problem*. Given a set of default laws with the same antecedent $A \Rightarrow B_1, \ldots, A \Rightarrow B_n$ (and without conflicting laws), all standard default logics will reason from $A$ to the conjunction $\wedge_{1 \leq i \leq n} B_i$. 

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But the high probabilities of the conjuncts may multiply into a very improbable conjunction. For example, assume a (default) bus driver reasons that on normal days, his tires won’t blow out, so (by forming conjunctions) this will never happen during his life whence he won’t need spare tires. Though his premises are justified, his conclusion is rather unrecommendable.

Like Pearl et al. [23, 24, 14] I base my investigations on Adams’ probability logic [1, 2, 3, 4], but I will develop it into a new and different direction. I try to achieve reliability by superimposing two mechanisms on the inferential reasoning of standard default logics:

- A mechanism of generating the minimal probabilistic default assumptions which must be made in order to derive the conclusion safely – this is called the assumption generation mechanism.

- A mechanism of propagating approximately tight lower probability bounds from the premises to the conclusion – this is called the lower bound propagation mechanism.

Assumption generation provides transparency to the user, and lower bound propagation supplies risk information. Both together guarantee reliability. For they ensure that given the premises of the inference are true, then the conclusion’s probability (given the facts are true) must be greater or equal than a certain value coming along with the conclusion. The decision whether or not this value is high enough to accept the conclusion is dependent on the pragmatic context and hence better left to the user instead of being apriorily decided by some system-inbuilt acceptability threshold. Of course, the lower probability bounds associated with the conclusion should be as tight as possible – within the restrictions of computational feasibility.

The mechanism of assumption generation is the key for embedding standard default logic into probabilistic default logic. It enables a certain reduction. We reduce default inferences ($\vdash_{d}$) from the knowledge base $\mathcal{K}$ to $\epsilon$-entailments ($\vdash_{\epsilon}$) from $\mathcal{K}$ enriched by the generated default assumptions.

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1A more sophisticated kind of the conjunction problem is the lottery paradox and its variants ([28], [25, p.395], [10, p.225]). Related to the conjunction problem is the Chaining Problem: given $A_1 \Rightarrow A_2, \ldots, A_{n-1} \Rightarrow A_n$ (without any conflicting laws), standard default logics will reason from $A_1$ to $A_n$; but the small uncertainties involved in each default law may sum up into a very improbable conclusion.
Hence we will prove theorems of the form: (for any $Ca$) $K \not\models_d Ca$ iff $K \cup \text{Ass} \not\models_\epsilon Ca$. The described reduction makes default reasoning transparent, because $\epsilon$-entailment, in contrast to default-inference, is monotonic w.r.t. the law premises and it preserves high probability for all statistical probability distributions. In contrast, standard default inference systems preserve high probability only for certain but not for other probability distributions [6, 7].

So far we have motivated the advantage of probabilistic default reasoning as against standard default reasoning. Its advantage as compared to straightforward probabilistic reasoning in Bayesian networks [22, 23, 16] is that it enables us to reason efficiently in a situation of partial and even minimal probabilistic information. First of all, we do not presuppose knowledge of the exact conditional probability values but only of some lower bounds of them. Second, probabilistic default reasoning assumes only partial probabilistic knowledge about the relevant possibility space. If the Bayesian reasoner knows that both $A$ and $B$ influence $C$, then she must know all the six conditional probabilities $p((\pm C/\pm A \land \pm B)$ ( $\pm$ for ‘unnegated or negated’), plus the truth values or prior probabilities of the four statements $\pm Aa, \pm Ba$ for the given instance $a$. In contrast, the default reasoner can reason also in a situation where she knows, e.g., only a lower bound of $p(C/A)$ and the truth of $Aa$, or alternatively the falsity of $Ba$ – namely by the mechanism of generating default assumptions. This mechanism constructs partial reasoning nets from partial information. It is also the main advantage of probabilistic default reasoning as compared to probabilistic ‘anytime’ deduction systems [21, 11].

2 Probability logic: The systems $P_\epsilon$

2.1 The principle of total evidence

Our basic formal language $B\text{Lang}$ is a ‘first order variant’ of the propositional language (with $\neg, \land, \lor, \text{and } \rightarrow$ for material implication). Its formulas are built up from a finite set of monadic predicates denoted by $F, G \ldots$, a

\begin{footnotesize}
\footnote{Probability logic is weaker than classical logic; it needs a strengthening by probabilistic default assumptions to achieve the strength of standard default logics.}
\footnote{Also rules and propagation mechanism of these systems are very different; cf. §2.3.}
\end{footnotesize}
single individual constant \(a\) and a single variable \(x\) (by standard formation rules). We omit the variable \(x\) in open formulas, hence \(F,G\ldots\) abbreviate \(Fx,Gx\ldots\); \(A,B,\ldots\) (possibly indexed) range over arbitrary open formulas of \(BLang\), and \(Aa,Ba,\ldots\) are the associated closed formulas with \(x\) replaced by \(a\). If these closed formulas are known, they are called facts; if they are derived, they are called singular conclusions. Default laws have the form \(A \Rightarrow B\). The symbol \(\Rightarrow\) does not belong to the basic but to the extended formal language \(ELang \supset BLang\). A knowledge base \(K\) is a pair \(\langle L,F \rangle\) with \(L,F\) finite sets of default laws and facts, respectively. \(\vdash\) stands for classical propositional inference (defined over \(BLang\)). \(\vdash_{\epsilon}\) stands for (monotonic) probabilistic entailment between default laws, and \(\models\) stands for indexed notions of nonmonotonic inference. \(\top\) (verum) and \(\bot\) (falsum) abbreviate \(p \lor \neg p\) and \(p \land \neg p\), respectively. For reasons of simplicity we do not introduce a separate set of deterministic laws in \(K\).\(^4\)

In the following we will assume that all laws \(L \in L\) are associated with (reasonably determined) lower probability bounds \(b(L)\). We write \(A \Rightarrow_{r} B\) to indicate that the lower bound of \(p(B/A)\) is \(r\). \(K \models_{\epsilon} [Ca,r]\) abbreviates that \(Ca\) is \(\epsilon\)-entailed by \(K\) with lower bound \(r\). The \(\epsilon\)-entailment of closed formulas from default laws and facts is reduced to the \(\epsilon\)-entailment of default laws from default laws by the principle of total evidence, which goes back to Carnap [9, p.211]. Let \(Fx\) denote the conjunction of all facts with the constant \(a\) replaced by \(x\). Then this principle says (cf. [23, ch.10.2.2],[5, p.149],[6, p.563]):

\[
\text{Principle of total evidence} \\
\langle L,F \rangle \models_{\epsilon} [Ca,r] \text{ iff } L \vdash_{\epsilon} Fx \Rightarrow C
\]

Hence the subjective probability which \(\langle L,F \rangle\) conveys to a singular statement \(Ca\) is identified with the statistical probability of \(C\) conditional to all facts (with \(a\) replaced by \(x\)) which is \(\epsilon\)-entailed by \(L\). This principle applies in the same way to the system \(P,DP\) (§3) where \(L\) gets enriched by default assumptions.

Probability semantics for unrestricted 1st order languages involves various complications [5, 15]. All work on (Adams’) probability logic has been carried out in the propositional language; the extension to the general 1st

\(^4\)The simplest way to account for them is to treat their ground instances \(Aa \rightarrow Ba\) as facts.
order case is an open problem. For this reason we work in an essentially propositional framework: our basic language, if restricted to open formulas, is isomorphic to a purely propositional language.\(^5\) The difference is only one of interpretation. Take the Bird (\(B\)) - Canfly (\(F\)) - Tweety (\(a\)) example. Statistical probabilities apply to open formulas (extensionally to classes, like ‘being a bird’), while subjective probabilities apply to closed formulas (extensionally to individual states of affairs, like ‘Tweety is a bird’). Statistical probability is defined on the partition of possible states of a variable individual: \(Bx \land Fx, Bx \land \neg Fx, \neg Bx \land Fx, \neg Bx \land \neg Fx\). In contrast, subjective probability is defined on the partition of possible worlds: \(Ba \land Fa, Ba \land \neg Fa, \neg Ba \land Fa, \neg Ba \land \neg Fa\). Possible states can be (equivalently) identified with truth assignments to the open atomic formulas of \(BLang\). Possible world semantics applies in the same way, except that we have possible states instead of possible worlds, and our probabilities are the frequencies (or frequency limits) of these states in the REAL world.

### 2.2 Probability semantics and the calculus \(P\_\epsilon\)

Some further terminology. \(\mathcal{L}\) ranges over finite sets of default laws \(L \in \mathcal{L}\). \(\Omega\) denotes the finite set of truth assignments \(\langle u, v \ldots \rangle\) to the atomic open formulas of \(BLang\); they are also called possible states. For a given probability function \(p\) on \(\Omega\) and default law \(L := A \Rightarrow B\), \(p(L) := p(B/A)\) denotes the (conditional) probability associated with \(L\) and \(u(L) := 1 - p(B/A)\) the (conditional) uncertainty associated with \(A \Rightarrow B\). We put \(p(B/A) = 1\) if \(p(A) = 0\). \(p\) is called proper for \(\mathcal{L}\) if \(p(A) > 0\) holds for all default laws \(A \Rightarrow B \in \mathcal{L}\) with logically consistent antecedent \(A\). \(\Pi(\mathcal{L})\) denotes the set of all probability functions on \(\Omega\) which are proper for \(\mathcal{L}\). \(\Pi(\mathcal{L}, L)\) abbreviates \(\Pi(\mathcal{L} \cup \{L\})\).

Adams [2, p.57] has developed a semantics and a calculus for a probabilistic entailment relation between sets of default laws and default laws which Pearl [23, ch.10.2] calls ‘\(\epsilon\)-entailment’ and which is defined as follows:

**Definition 1 (\(\epsilon\)-entailment)** \(\mathcal{L}\) \(\epsilon\)-entails \(L\) iff for every (small) \(\delta > 0\) there exists an \(\epsilon > 0\) such that for each probability function \(p \in \Pi(\mathcal{L}, L)\): if \(p(L') \geq \)

\(^5\)The isomorphism is defined by dropping the unique variable \(x\) and treating predicates as propositional variables. This isomorphism preserves all logical relations (this follows from the theorem of uniform substitution for predicate letters, cf. [31]).
1 − ε for all L′ ∈ L, then p(L) ≥ 1 − δ.

This infinitesimal probabilistic entailment condition implies that the conclusion probability can be forced to get arbitrarily close to 1 by making the premise probabilities sufficiently close to 1. But it does not imply any nontrivial lower bounds for the conclusion probability if the premise probabilities are high but nonextreme (say ≥ 0.95). So, reliability in our sense is not per se guaranteed by infinitesimal probabilistic entailment. But fortunately, infinitesimal probabilistic entailment is equivalent with the following inequality for nonextreme probability values: the uncertainty of the conclusion is always smaller or equal than the sum of the uncertainties of the premises (th. 1.2). This inequality is the basis of our propagation mechanism for lower probability bounds.

The calculus Pe is correct and complete for ε-entailment (th. 1.3) and consists of the following rules:

The calculus Pe

(CC) A ⇒ B, A ∧ B ⇒ C / A ⇒ C (Cautious Cut)
(CM) A ⇒ B, A ⇒ C / (A ∧ B) ⇒ C (Cautious Monotonicity)
(Or) A ⇒ C, B ⇒ C / (A ∨ B) ⇒ C (Disjunction)
(SC) ⊢ A → B / A ⇒ B (Supraclassicality)
(εEFQ) ⊬ ¬A, A ⇒ ⊥ / B ⇒ C (Ex ε-Falso Quodlibet)

L ⊢ε L if L is derivable from L by using these five rules. Although ⊢ε is monotonic7, ⇒ is nonmonotonic (only ‘cautiously monotonic’), whence \(\vdash_{\epsilon} (\text{ε-entailment between knowledge bases and singular statements})\) is semimonotonic: monotonic w.r.t. the law premises, but nonmonotonic w.r.t. fact-premises. E.g., \(\{A \Rightarrow B\}, \{Aa\} \vdash_{\epsilon} Ba\), but \(\{A \Rightarrow B\}, \{Aa, Ca\} \not\vdash_{\epsilon} Ba\), because \(A \Rightarrow B \not\vdash_{\epsilon} (A ∧ C) ⇒ B\). The following theorem is proved in [2]:

Theorem 1 (System Pe) For every \(\mathcal{L}\) and L, the following conditions are equivalent:

(1) \(\mathcal{L}\) ε-entails L
(2) For every probability distribution \(p \in \Pi(\mathcal{L}, L)\): \(u(L) \leq \sum_{L' \in \mathcal{L}} u(L')\)
(3) \(\mathcal{L} \vdash_{\epsilon} L\) is derivable in the calculus Pe.

\(\epsilon\)-entailment is noncompact and thus restricted to finite premise sets [2, p.52].
A fourth equivalent representation of $\epsilon$-entailment which is omitted here is in terms of normal world (ranked model) semantics. It was first noticed by Adams [3] and further elaborated by Pearl [24] and Lehmann and Magidor [17]. Hence the concept of probabilistic entailment unifies four different perspectives: (1) infinitesimal probabilistic entailment (th.1.1), (2) nonextreme probability propagation (th.1.2), (3) logical calculus (th.1.3) and (4) normal world semantics. While the work of Pearl et al. [24, 12, 14] concentrates on the first and fourth perspective, this approach focuses on the second and third perspective.

Important is the notion of $\epsilon$-consistency: $L$ is $\epsilon$-consistent iff for all $\epsilon > 0$ there exists a probability function $p \in \Pi(L)$ such that $p(L) \geq 1 - \epsilon$ for all $L \in L$. It is well-known that $L$ $\epsilon$-entails $A \Rightarrow B$ iff $L \cup \{A \Rightarrow \neg B\}$ is $\epsilon$-inconsistent [2, p.58]. Among the various $P_\epsilon$-theorems (cf. Adams [2, p.60-65]) are (And): $A \Rightarrow B, A \Rightarrow C \vdash_\epsilon A \Rightarrow B \land C$, (LLE, left logical equivalence): $A \vdash B / A \Rightarrow C \vdash_\epsilon B \Rightarrow C$, (RW, right weakening): $B \vdash C / A \Rightarrow A \Rightarrow C$. In the presence of (CC), the rule (SC) is equivalent with (RW) + (R): $A \Rightarrow A$. A slightly weaker version of the system $P_\epsilon$, called the system $P$, is obtained if one drops the rule $\epsilon$EFQ (which is only needed for $P_\epsilon$-derivations from $\epsilon$-inconsistent premise sets) and admits also improper probability functions. This system was proposed by Adams [1, 3], and it is equivalent with the calculus of preferential entailment after [17, p.5f].

2.3 Tightness of bounds

The computed lower bounds should not only be safe, i.e. be satisfied for all probability distributions, but they should also be tight, i.e. there should not exist greater lower bounds which are also safe. As an example, take the rule (CM) $A \Rightarrow B, A \Rightarrow C / A \land B \Rightarrow C$, and let $f(r_1, r_2)$ be a function which determines the lower bound of $p(A \land B \Rightarrow C)$ in dependency of the lower bounds $r_1 \leq p(A \Rightarrow B)$ and $r_2 \leq p(A \Rightarrow C)$. The function which underlies our lower bound propagation is $f(r_1, r_2) = 1 - (1 - r_1) - (1 - r_2)$ (th.1.2). Even if the function $f$ is tight for the most general instances of this rule (i.e., instances where different schematic letters are replaced by different atomic formulas), it will not be tight for all instances of this rule. For example, no safe function $f$ is tight for special (CM)-instances where $A \vdash B$ holds, for

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8Except that the latter one is not restricted to finite premise sets.
in this case the greatest lower bound for \( p(A \land B \Rightarrow C) \) is \( r_1 \leq p(A \Rightarrow B) \).
The best what can be expected from a function determining lower bounds is that it is tight for the most general instances of a rule. In [11, p.109], this property is called quasi-tightness.\(^9\)

In contrast to the rules of [11], not all of our rules are quasi-tight, if furnished with the lower bound function of th.1.2. The rules (CM) and (SC) indeed are quasi-tight, but the rules (CC) and (Or) are not: it can be proved that the quasi-tight bound for (CC) is \( 1 - \epsilon_1 - \epsilon_2 + \epsilon_1 \epsilon_2 \) (cf. [11, p.103], rule vii; the \( \epsilon \)'s are the upper uncertainty bounds of the two premises), and the quasi-tight bound for (Or) is \( 1 - \frac{\epsilon_1 + \epsilon_2 - 2 \epsilon_1 \epsilon_2}{1 - \epsilon_1 \epsilon_2} \). In both cases, the loss is in the second power of the \( \epsilon \)'s and thus small.

We argue in a twofold way. First, the small loss is insignificant. For quasi-tightness is not preserved through the concatenation of rules. Thus, if the lower bounds are computed stepwise along the inference-chain, as it is done in [11], then the computed lower bound for the entire inference might be not quasi-tight, even if all rules are quasi-tight. Second, the small loss of our method is compensated by a significant advantage. Based on theorem 1.2, we compute the lower bound of the entire inference in one step. Our lower bounds depend only on the lower bounds of the premises and are independent from the way in which the conclusion has been proved from the premises.\(^10\) In several cases this produces a better bound than a stepwise calculation. An example is the \( \epsilon \)-derivation of (iii): \( A \land D \Rightarrow C \) from (i): \( A \xrightarrow{1-\epsilon_1} B \land D \) and (ii): \( A \land B \land D \xrightarrow{1-\epsilon_2} C \). It is proved by first applying (CC) to (i + ii), obtaining (iv): \( A \Rightarrow C \), then applying (RW) to (i), obtaining (v): \( A \Rightarrow D \), and finally applying (CM) to (iv + v), obtaining (iii). If we compute the lower bounds stepwise, we obtain \( 1 - \epsilon_1 - \epsilon_2 \) as the lower bound for (iv), \( 1 - \epsilon_1 \) as the lower bound for (v), and thus \( 1 - 2\epsilon_1 - \epsilon_2 \) as the lower bound for the conclusion. With our one-step method we obtain a better lower bound, namely \( 1 - \epsilon_1 - \epsilon_2 \).

Therefore we think that our method of computing lower bounds is a good approximation to tightness. Moreover, its additional computational complexity is small. All what has to be achieved in order to compute good lower bounds by our method is to prove the conclusion from a premise set

\(^9\)The definition in [11, p.109] is given for intervals, while we are only concerned with lower bounds. It is provably equivalent with our formulation.

\(^10\)On this reason the lower bounds are not part of our rules.
which is as small as possible, and which has lower bounds as high as possible (cf. §3.4).

3 Default logic based on $P_\epsilon$

3.1 Irrelevance assumptions

Epsilon-entailment is too weak as a basis for default reasoning, because it does not sanction the principle of default detachment which is characteristic for nonmonotonic reasoning: to infer $Ba$ from $Aa \in \mathcal{F}$ and $(A \implies B) \in \mathcal{L}$ as long as $K$ does not entail $\neg Ba$. For example, $K = \langle \{\text{Bird} \implies \text{CanFly}\}, \{\text{Bird(tweety)}, \text{Female(tweety)}\} \rangle$ default-implies $\text{CanFly(tweety)}$. But $K$ does not $\epsilon$-entail $\text{CanFly(tweety)}$ because $A \implies B$ does not $\epsilon$-entail $A \land C \implies B$. In order to give a probabilistic foundation of default reasoning, a probabilistic account of these default detachments is needed.

In the maximal entropy approach ([23, ch.10.2.3], [13, 6]), this is done by an additional constraint on the probability functions: they should have maximal entropy. In the system $Z$ and its extension $Z^+$ [24, 14, 17] it is achieved by an additional constraint on the ranked models: they should minimize the rank (i.e., the abnormality) of worlds. Both approaches give maximal justifications of default reasoning in the sense that except from the dependencies stated by the default laws, the world has maximal entropy, or is maximally normal.

This approach intends to give a minimal justification of default detachment, and it does this by additional syntactic assumptions. In order to derive $\text{CanFly(a)}$ from $\text{Bird} \implies \text{CanFly}$ and $\text{Bird(a)} \land \text{Female(a)}$ in a probabilistically safe way we only have to assume that $\text{Female(a)}$ is irrelevant for $\text{Canfly(a)}$, given $\text{Bird(a)}$, and need not take care about the entropy or normality of ‘the rest of the world’. The irrelevance assumption means probabilistically that $p(\text{CanFly}/\text{Bird})$ and $p(\text{CanFly}/\text{Bird} \land \text{Female})$ are (approximately) equal.

So, the principle which lies behind default detachment is to assume that the remainder facts are irrelevant for the conclusion as long as the knowledge base does not tell the contrary.\textsuperscript{11} Whenever a default reasoning system

\textsuperscript{11}The idea is similar to the relevance-based approach of [12], but this approach remains syntactical. Probabilistic irrelevance assumptions have also been suggested by Bacchus [5,
makes a default detachment, thereby inferring \( Ba \) from \( A \Rightarrow B \in \mathcal{L} \) and \( Aa \), it has (implicitly) assumed that the (remainder) facts in \( \mathcal{F} \) are probabilistically irrelevant for \( A \Rightarrow B \). We denote this irrelevance assumption by \( \text{Irr}(\mathcal{F} : A \Rightarrow B) \) and say that it has been generated by the corresponding default detachment. Since we only deal with lower and not with upper probability bounds, we can ‘minimalize’ the irrelevance assumptions to negative irrelevance assumptions, saying that the remainder facts do not lower the probability of \( B \) given \( A \). Hence \( \text{Irr}(\mathcal{F} : A \Rightarrow B) \) means \( p(B/\mathcal{F} \land A) \geq p(B/A) \).

We include irrelevance assumptions into our extended language \( \text{ELang} \) and add the following rule to \( \text{P}_\epsilon \):

\[
(Irr) \quad A \Rightarrow B, \text{Irr}(C : A \Rightarrow B) / C \land A \Rightarrow B \quad \text{(Irrelevance updating)}
\]

Whenever \( (A \Rightarrow B) \in \mathcal{L} \) and \( \text{Irr}(\mathcal{F} : A \Rightarrow B) \) has been generated, the so-called updated default law \( (\mathcal{F} \land A) \Rightarrow B \) is also generated. Hence in the rule \( (Irr) \), the same lower bound is transferred from the law to the updated law. For better readability, conjuncts of \( \mathcal{F} \) which are already contained as conjuncts in \( A \) get cancelled in the irrelevance assumption; e.g., \( \text{Irr}(C \land A : A \Rightarrow B) \) is written as \( \text{Irr}(C : A \Rightarrow B) \), so that \( C \) contains only facts which are distinct from \( A \). If \( \mathcal{F} \) coincides with \( A \), the ‘reduced’ irrelevance assumption \( \text{Irr}( : A \Rightarrow B) \) becomes trivial; throughout the following we also assume that all trivial irrelevance assumptions get cancelled.

\( I \) ranges over irrelevance assumptions and \( \mathcal{I} \) over sets of them; \( D \) over updated default laws and \( \mathcal{D} \) over sets of them. \( \mathcal{I} \) is for \( \mathcal{L} \) iff each \( I \in \mathcal{I} \) has the form \( \text{Irr}(C : A \Rightarrow B) \) for some \( A \Rightarrow B \in \mathcal{L} \). A probability function \( p \) is called proper for \( \text{Irr}(C : A \Rightarrow B) \) iff \( p(C \land A) > 0 \). \( \text{Irr}(C : A \Rightarrow B) \) is said to be \( \epsilon \)-satisfied by \( p \) iff \( p(B/A) - p(B/C \land A) \leq \epsilon \). The notion of \( \epsilon \)-entailment is extended to the entailment from sets of default laws \( \mathcal{L} \) enriched with irrelevance assumptions \( \mathcal{I} \) for \( \mathcal{L} \) as follows: \( \mathcal{L} \cup \mathcal{I} \) \( \epsilon \)-entails \( \mathcal{L} \) iff \( \forall \delta > 0, \exists \epsilon > 0, \forall p \in \Pi(\mathcal{L}, \mathcal{I}, L) : p(L) \geq 1 - \epsilon \) for all \( L \in \mathcal{L} \) and \( p \) \( \epsilon \)-satisfies all \( I \in \mathcal{I} \), then \( p(L) \geq 1 - \delta \). Th. 2(i,ii) states that the \( \text{P}_\epsilon \)-calculus extended by the rule \( (Irr) \) is correct and complete for this extended notion of \( \epsilon \)-entailment, if we restrict it to entailments from sets of default laws plus irrelevance assumptions for them. Given \( \mathcal{L} \) and \( \mathcal{I} \) for \( \mathcal{L} \), we define \( \mathcal{D}(\mathcal{L}, \mathcal{I}) := \{ C \land A \Rightarrow B \mid \text{Irr}(C : A \Rightarrow B) \in \mathcal{I} \} \) as the set of updated

ch.5; his framework is an (incomplete) numerical 1st order formalization of probability theory.
default laws derivable from \( \mathcal{L} \cup \mathcal{I} \) by the rule (Irr). Th. 2(i,iii) shows that the extension of the system \( P_\epsilon \) by irrelevance assumptions is conservative: irrelevance assumptions and the corresponding updated default laws \( \epsilon \)-entail the same default laws (in the presence of a given \( \mathcal{L} \)).

**Theorem 2 (Extension of \( P_\epsilon \) by Irr)** For every \( L, \mathcal{L} \) and \( \mathcal{I} \) for \( \mathcal{L} \): (i) \( \mathcal{L} \cup \mathcal{I} \models \epsilon L \) iff (ii) \( \mathcal{L} \cup \mathcal{I} \) \( \epsilon \)-entails \( L \) iff (iii) \( \mathcal{L} \cup \mathcal{D}(\mathcal{L}, \mathcal{I}) \models \epsilon L \).

Default laws \( A \Rightarrow B \) correspond to Reiter’s normal defaults of the form \( (A : MB/B) \) [29, p.95]. In [30] it has been shown that \( P_\epsilon \) extended by irrelevance assumptions supplies probabilistic reliability for Reiter-style default logic. Th. 5 and cor. 2 of [30, p.256] together with theorem 2 imply the following: Ca is in some Reiter-extension of \( \langle \mathcal{L}, \mathcal{F} \rangle \) iff \( \mathcal{L} \cup \mathcal{I} \models \epsilon \mathcal{F} \xrightarrow{\epsilon} C \), where \( \mathcal{I} \) is a set of irrelevance assumptions for \( \mathcal{L} \) and \( r = 1 - \sum \{1 - b(L) \mid Irr(\mathcal{F}_x : L) \in \mathcal{I} \} \) (hence only those laws for which irrelevance assumptions are generated contribute to Ca’s lower bound). Here is an example.

**Example 1 (Irrelevance updating)**

**Default Laws** (\( \mathcal{L} \)): Student \( \mathcal{L} \\Rightarrow \) Adult, Adult \( \mathcal{L} \\Rightarrow \) HasCar.

**Facts:** Student(a), Female(a).

**Conclusion:** [HasCar(a), 0.85].

**Default detachments:** Student(a) \( \vdash \) Adult(a), Adult(a) \( \vdash \) HasCar(a).

**Irrelevance assumptions generated** (\( \mathcal{I} \)): \( Irr(Female: \forall \mathcal{L}) \Rightarrow Adult \), \( Irr(Student \land \forall \mathcal{I}) \Rightarrow HasCar \).

**Updated default laws** (\( \mathcal{D} \)): Student \( \land \forall \mathcal{L} \Rightarrow \) Adult, Adult \( \land \forall \mathcal{L} \Rightarrow \) HasCar.

\( \mathcal{L} \cup \mathcal{I} \models \epsilon \mathcal{D} \models \epsilon \) Student \( \land \forall \mathcal{L} \Rightarrow \) HasCar \( [1 - (1 - 0.95) - (1 - 0.9) = 0.85] \).

In contrast to the system \( Z \) [24, p.128], irrelevance-based default reasoning has no problems with property-inheritance from classes to exceptional subclasses: if we add the two additional laws \{ Student \( \Rightarrow \) Employed, Adult \( \Rightarrow \) Employed \} which make students to exceptional adults, the same conclusion follows.

In the above example the addition of irrelevance assumptions is without problems. This is not so in the case of conflicting default laws, as in the famous Nixon-example: \( K = \{ \text{Quaker} \Rightarrow \text{Pacifist}, \text{Republican} \Rightarrow \neg \text{Pacifist} \} \),

\(^{12}\)In [30] it was not known how to include irrelevance assumptions into the object language; th.2 as well as all other results of this paper are new.
Here one cannot generate both irrelevance assumptions $\text{Irr}(\text{Republican}: \text{Quaker} \Rightarrow \text{Pacifist})$ and $\text{Irr}(\text{Quaker}: \text{Republican} \Rightarrow \neg \text{Pacifist})$ without making $\mathcal{L} \cup \mathcal{I}$ $\epsilon$-inconsistent and the set of resulting singular conclusions logically inconsistent. The question which irrelevance assumptions should be added in the case of conflicting laws is a decisive crossroad for different systems (multiple extensions [29, 26], sceptical approach [19], rule priorities [26, 8], specificity-based approaches [24, 25]). Our approach is extremely flexible in this respect. There is a spectrum of possibilities between (1.) irrelevance assumptions completely guided by the user, which corresponds formally to a multiple-extension approach, and (2.) irrelevance assumptions completely determined by system-immanent rationality principles, which corresponds formally to a specificity-based approach.

3.2 Poole-extensions and maximal irrelevance-updates

In this paper we focus on the most liberal default logics which admit multiple extensions, for it is here where the fundamentals of probabilistic default logic have to be laid. We restrict the rationality principles for irrelevance assumptions to those premises (facts and updated default laws) which must be used to derive a given extension (or part of it). Even under this restriction irrelevance assumptions need not always be reasonable. For Reiter-extensions it was proved that they satisfy at least the minimal rationality requirement of the $\epsilon$-consistency of the updated default laws [30, 32]. This is different for the stronger system of default logic proposed by Poole [26, p.29f], on which we concentrate in this paper. The reason for this difference is that Reiter-style default reasoning is $\text{MP-sequential}$ – all default consequences follow by a series of Modus Ponens steps applied to default laws (plus classical consequences) – while Poole’s default logic may also apply contraposition and other classically valid steps to default laws.¹³

$\mathcal{L}^- := \{ A \rightarrow B \mid (A \Rightarrow B) \in \mathcal{L} \}$ denotes the set of material counterparts of laws in $\mathcal{L}$. A Poole-extension of $\langle \mathcal{L}, \mathcal{F} \rangle$ is obtained by adding a maximal set of material counterparts of laws in $\mathcal{L}$ to $\mathcal{F}$ which preserves consistency.

¹³Poole [26, p.32] proves that a Poole-extension corresponds to a Reiter-extension if every $A \Rightarrow B$ is replaced by $\top \Rightarrow (A \rightarrow B)$. Though logically valid, this technique is probabilistically invalid: uncertain laws express high conditional probabilities, which are much stronger than high probabilities of the material conditional (cf. [2]). Compare, e.g., ‘most humans are drunken’ with ‘most entities are non-humans or drunken’.
and then forming the classical closure.

**Definition 2 (Poole-extensions)**

1. A materialization of \( \langle L, F \rangle \) is a set \( M \subseteq L \) such that \( M \cup F \) is consistent if \( F \) is so. A materialization \( M \) is maximal if no proper superset of \( M \) is a materialization.

2. \( E \) is a Poole-extension of \( \langle L, F \rangle \) iff \( E = Cn(M \cup F) \) for some maximal materialization \( M \) of \( \langle L, F \rangle \). \( POOLE(K) \) denotes the set of Poole-extensions of \( K \).

Poole speaks of scenarios; they are the unions of consistent \( F \)'s with materializations in our sense. Based on def.2, an extension-dependent default inference relation \( [K | \sim_P Aa] \) iff \( Aa \in E \in POOLE(K) \), an extension-independent choice default inference relation \( [K | \sim_P Aa] \) iff \( Aa \in E \) for some \( E \in POOLE(K) \), and an extension-independent sceptical default inference relation \( [K | \sim_S Aa] \) iff \( Aa \in E \) for all \( E \in POOLE(K) \) can be given.

The maximal \( F \)-irrelevance set of \( L \) is defined as \( I_F(L) := \{Irr(Fx : A \Rightarrow B) \mid (A \Rightarrow B) \in L\} \), and the corresponding maximal \( F \)-update of \( L \) is \( D_F(L) := \{Fx \land A \Rightarrow B \mid (A \Rightarrow B) \in L\} \). Obviously, \( L \cup I_F(L) \vdash \epsilon \) by the rule (Irr). In the following, \( M \) ranges over sets of material implications. \( M^\rightarrow := \{A \Rightarrow B \mid (A \rightarrow B) \in M\} \) is the default counterpart of \( M \). Theorem 3.1 shows that classical consequence from \( F \cup L \) implies \( \epsilon \)-entailment from \( \langle D_F(L), F \rangle \), and th. 3.2 supplements that the other direction holds if \( D_F(L) \) is \( \epsilon \)-consistent.

**Theorem 3 (Poole-extensions)** For every \( L \) and \( F \):

1. If \( L^- \cup F \vdash Aa \), then \( D_F(L) \vdash \epsilon Fx \Rightarrow A \).
2. If \( D_F(L) \) is \( \epsilon \)-consistent and \( D_F(L) \vdash \epsilon Fx \Rightarrow A \), then \( L^- \cup F \vdash Aa \).

Unfortunately, we cannot make good sense of th. 3 so far. For, the irrelevance assumptions \( I_F(L) \), which together with \( L \) \( \epsilon \)-entail \( D_F(L) \), will not always be reasonable, due to Modus Tollens (contraposition) and related rules. For example, assume \( L = \{A \Rightarrow B\}, F = \{\neg Ba\} \). Then the (single) Poole-extension of this base contains \( \neg Aa \), and the corresponding \( F \)-updated law is \( A \land \neg B \Rightarrow B \). This law indeed \( \epsilon \)-entails \( \neg B \Rightarrow \neg A \), but \( \neg B \) lowers the conditional probability of \( B \) to 0 (provided \( p(\neg) \) is proper). Hence, \( \neg B \) is

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14 Poole excludes inconsistent \( F \)'s ([26], p.29); we include them for systematic reasons.

15 Lemma 5.21 of [17, p.41] follows as the special case of our th.3.1 where \( F = \top \).
certainly not irrelevant for \( A \Rightarrow B \). This is reflected in the fact that the updated law \( A \land \neg B \Rightarrow B \) is \( \epsilon \)-inconsistent.

The \( \epsilon \)-consistency of the updated default laws following from the irrelevance assumptions is a minimal rationality condition. For if \( D_F(L) \) is \( \epsilon \)-inconsistent, then for every \( D := (F \land A \Rightarrow B) \in D_F(L) \), \( D_F(L \setminus \{D\}) \vdash \epsilon \), \( F \land A \Rightarrow \neg B \) (recall §2.2), hence the remainder updated default laws give reason to suppose that \( F \) is not irrelevant for \( A \Rightarrow B \). However, even if \( D_F(L) \) is \( \epsilon \)-consistent, the corresponding irrelevance assumptions may be doubtful – namely if the law-consequent is a disjunction and \( D_F(L) \cup F \) \( \epsilon \)-entails the negation of one disjunct. Consider the irrelevance assumption \( Irr(\neg B : A \Rightarrow B \lor C) \) (where \( L = \{A \Rightarrow B\} \) and \( F = \{\neg Ba\} \)). It is doubtful whether \( B \lor C \) (given \( A \)) remains highly probable under the additional condition that \( B \) is false. Here is a counterexample: most employed people earn either less than 25,000 Dollars or more than 100,000 Dollars a year, but certainly not most employed people who earn more than 25,000 Dollars earn more than 100,000 Dollars a year. Only in one case \( D_F(L) \cup F \not\vdash \epsilon \neg B \) is not a reason to doubt \( Irr(F \land A \Rightarrow B \lor C) \), namely when \( D_F(L) \cup F \) at the same time \( \epsilon \)-entails the entire disjunction \( B \lor C \). Then we know that most \( A \)'s as well as most \( F \)'s are \( (B \lor C) \)'s, which is, without further knowledge, a reason to believe in the irrelevance assumption. An example is the irrelevance assumption \( Irr(\neg Fish : Water-Living \land Big \Rightarrow Fish \lor Mammal) \), given the background-knowledge \( \epsilon \)-entails \( \neg Fish \land Big \Rightarrow Mammal \).

Every default law can be \( \epsilon \)-equivalently decomposed into a set of default laws having a disjunction of literals as consequent (by prop. logic and (And), (RW)). Given such a decomposition, we summarize our considerations in the following rationality principle (P) on the maximal \( F \)-updates \( D_F(L) \):

\((P)\) For every \( (A \Rightarrow B) \in L \) and elementary disjunct \( D \) of \( B \), \( D_F(L) \not\vdash \epsilon \neg D \) unless \( D_F(L) \vdash \epsilon \neg F \Rightarrow B \)

We will guarantee condition (P) by ensuring the satisfaction of the following slightly stronger but computationally simpler condition on \( L \):

\((P^*)\) For every \( (A \Rightarrow B) \in L \) and elementary disjunct \( D \) of \( B \), (a) \( L^* \cup F \not\vdash \neg Da \) unless \( L^* \vdash Ba \), and (b) \( L^* \cup F \not\vdash \neg Aa \).

By th. 3.2, \((P^*b)\) is a sufficient condition for \((P)\) if \( D_F(L) \) is \( \epsilon \)-consistent, and \((P^*a)\) will be proved to imply that \( D_F(L) \) is \( \epsilon \)-consistent.
To guarantee that the irrelevance-updating satisfies \((P^*)\) we propose the following simple method. Assume \((A \Rightarrow B) \in \mathcal{L}\). If \(\mathcal{L}^- \cup \mathcal{F}\) logically implies \(\neg A\alpha\) but not \(B\alpha\), we replace \(A \Rightarrow B\) by its contraposition \(\neg B \Rightarrow \neg A\); and if \(\mathcal{L}^- \cup \mathcal{F}\) logically implies \(\neg A\alpha\) and \(B\alpha\), we replace \(A \Rightarrow B\) by its forward-transposition \(\top \Rightarrow \neg A \lor B\). If we apply the irrelevance updating to the set \(\mathcal{L}'_F\) of default laws transposed in this way, the resulting maximal \(\mathcal{F}\)-update \(\mathcal{D}_F(\mathcal{L}'_F)\) will satisfy principle \((P^*)\), and because \((\mathcal{L}'_F)^{-}\) is logically equivalent with \(\mathcal{L}^-\), th. 3 remains still intact for \(\mathcal{D}_F(\mathcal{L}'_F)\). There are just two problems.

First, forward-transpositions are probabilistically valid, but contrapositions are probabilistically invalid. For example, most non-drunken entities are non-humans, which does not entail that most humans are drunken. So what is needed is a justification of contraposition steps by additional probabilistic default assumptions. This is done in the next section.

Second, the method does not always work if the consequents of laws in \(\mathcal{L}\) are disjunctions of literals. For example, if \(\mathcal{L}^- \cup \mathcal{F}\) logically implies \(A\alpha\) and \(\neg F\alpha\) but not \(G\alpha\), then \((P^*)\) is violated for both the law \(A \Rightarrow F \lor G\) and its contraposition \(\neg F \land \neg G \Rightarrow \neg A\). To handle this case a backwack transposition would be required which transforms the law into \(A \land \neg F \Rightarrow G\). In [32] it is proposed to justify backward transpositions by negated default laws: this method is more general than that described here but also more complicated, and it may yield worse lower bounds. From now on we presuppose that the consequents of the laws in \(\mathcal{L}\) are single literals. For this case, which very often can be assumed, the method will generally work.

### 3.3 Contraposition assumptions

For all probability functions \(p\) proper for \(A \Rightarrow B\) and \(\neg B \Rightarrow \neg A\) it holds that:

\[
u(\neg B \Rightarrow \neg A) = u(A \Rightarrow B) \frac{p(A)}{p(\neg B)}
\]

This is an easy consequence of probability calculus.\(^{16}\) Hence the minimal probabilistic justification of a contraposition step from \(A \Rightarrow B\) to \(\neg B \Rightarrow \neg A\), where the same lower bound is transferred from the law to its contraposi-

\(^{16}\)Proof: \(u(\neg B \Rightarrow \neg A) = u(\neg A/\neg B) = p(A/\neg B)\) [since \(p(\neg B) > 0\)] = \(p(\neg B/A) \frac{p(A)}{p(\neg B)}\)

[since \(p(A) > 0\)] = \(u(B/A) \frac{p(A)}{p(\neg B)}\) [since \(p(A) > 0\)] = \(u(A \Rightarrow B) \frac{p(A)}{p(\neg B)}\).
tion, is the default assumption $p(\neg B) \geq p(A)$. We call it a contraposition assumption.

Contraposition assumptions compare prior probabilities. To generate such assumptions by default may seem inappropriate, because common-sense knowledge about prior probabilities is rather weak. This argument does not apply to contraposition assumptions, because they make a very specific claim: A’s probability is smaller or equal than that of the negation of B. Normally, the antecedent and the consequent of a default law will describe rather specific properties (or situations). Hence even if we know nothing else about $p(A)$ and $p(B)$, we normally know that $p(A) < p(\neg A)$ and $p(B) < p(\neg B)$, and thus that $p(A) \ [< 0.5] < p(\neg B)$, i.e. the contraposition assumption is known to be true (cf. example 2 in the next section). Only if either A or B describe purely negative properties, the contraposition assumption will not be entailed by common sense knowledge. In such a case the user may simply reject it.

We add contraposition assumptions of the form $p(\neg B) \geq p(A)$ to our extended language $ELang$ and add the following rule to $P_\epsilon$:

(Cont) $A \Rightarrow B, \; p(\neg B) \geq p(A) / \neg B \Rightarrow \neg A$ (Contraposition)

The same lower bound is transferred from $A \Rightarrow B$ to $\neg B \Rightarrow \neg A$. $C$ ranges over contraposition assumptions $C \in C$. $C$ is for $\mathcal{L}$ iff each $C \in C$ has the form $p(\neg B) \geq p(A)$ for some $A \Rightarrow B$. A contraposition assumption $p(\neg B) \geq p(A)$ is said to be $\epsilon$-satisfied by $p(\neg)$ iff $p(\neg B) \geq 1 - \epsilon$. We extend the notion of $\epsilon$-entailment to sets of default laws enriched by irrelevance and contraposition assumptions for $\mathcal{L}$ as follows: $\mathcal{L} \cup I \cup C$ $\epsilon$-entails $L$ iff $\forall \delta > 0 \exists \epsilon > 0 \forall p \in \Pi(\mathcal{L}, I, L)$: if $p(L) \geq 1 - \epsilon$ for all $L \in \mathcal{L}$, and all $I \in I$ and $C \in C$ are $\epsilon$-satisfied by $p$, then $p(L) \geq 1 - \delta$. $T(\mathcal{L}, C)$ denotes the set of transformed (contraposited) laws following from $\mathcal{L} \cup C$ by the rule (Cont). We can prove only correctness but not completeness for the calculus $P_\epsilon$ extended by (Cont) [and (Irr)] – which is the content of th. 4(ii). This is not a significant drawback, because th. 4(ii) tells us that $P_\epsilon$ extended by (Cont) [and (Irr)] is a conservative extension of $P_\epsilon$ w.r.t. the $\epsilon$-entailment of default laws, which enables lower bound propagation according to th. 1.2.

**Theorem 4 (Extension of $P_\epsilon$ by Irr and Cont)** For every $L$, $\mathcal{L}$ and $C$, $I$ for $\mathcal{L}$: (i) if $\mathcal{L} \cup C \cup I \vdash_\epsilon L$, then $\mathcal{L} \cup C \cup I$ $\epsilon$-entails $L$, and (ii) $\mathcal{L} \cup C \cup I \vdash_\epsilon L$ iff $\mathcal{L} \cup T(\mathcal{L}, C) \cup I \vdash_\epsilon L$. 17
We illustrate our technique of generating contraposition and irrelevance assumptions for Poolean default conclusions at hand of a simple example:

Example 2 (Contraposition-based irrelevance updating)

Default laws $\mathcal{L}$: Flue $\Rightarrow$ Fever.

Facts $\mathcal{F}$: $\neg$Fever(a), Female(a).

$\mathcal{L} \cup \mathcal{F} \vdash \neg$Flue(a).

Contraposition assumption generated ($\mathcal{C}$): $p(\neg$ Fever) $\geq p$(Flue).

Contraposed laws: $\neg$Fever $\Rightarrow \neg$Flue.

Irrelevance assumptions generated ($\mathcal{I}$): $\text{Irr}(\text{Female}: \neg$Fever $\Rightarrow \neg$Flue).

Updated transformed default laws ($\mathcal{D}$): $\neg$Fever $\land$ Female $\Rightarrow \neg$Flue.

Conclusion $\mathcal{C}a$: $\{\neg$Flue(a), 0.95$\}.

$\mathcal{L} \cup \mathcal{C} \cup \mathcal{I} \vdash \epsilon D \vdash _{\epsilon} Fx 0.95 \Rightarrow C$.

3.4 The system $P_e DP$

$P_e DP$ stands short for $P_e$-based Default logic in Poole-style. Def.3.1+2 defines the transformed law set $\mathcal{L}'_F$ and the contraposition assumptions $\mathcal{C}_F(\mathcal{L})$ which together with $\mathcal{L}$ $\epsilon$-entail $\mathcal{L}'_F$ by applications of (Cont). Def.3.3+4 defines $P_e DP$-extensions in terms of maximal Poolean $F$-updates.

Definition 3 (System $P^e DP$) Let $\mathcal{M}$ be a materialization of $\langle \mathcal{L}^*, \mathcal{F} \rangle$ and $\mathcal{L} := \mathcal{M}^\circ$.

1. $\mathcal{L}^a := \{(A \Rightarrow B) \in \mathcal{L} \mid \mathcal{M} \cup \mathcal{F} \not\vdash \neg Aa\}.$

2. $\mathcal{L}^c := \{\neg B \Rightarrow \neg A \mid (A \Rightarrow B) \in \mathcal{L}, \mathcal{M} \cup \mathcal{F} \vdash \neg Aa, \mathcal{M} \cup \mathcal{F} \not\vdash Ba\}.$

3. $\mathcal{L}'_F$ (the $\mathcal{F}$-transformation of $\mathcal{L}$) $= \mathcal{L}^a \cup \mathcal{L}^c \cup \mathcal{L}^f$.

4. $\mathcal{C}_F(\mathcal{L})$ (the $\mathcal{F}$-contraposition set of $\mathcal{L}$) $= \{p(\neg B) \geq p(A) \mid (\neg B \Rightarrow \neg A) \in \mathcal{L}^c\}.$

5. A Poolean $\mathcal{F}$-update of $\mathcal{L}^*$ is any set $\mathcal{D}_F(\mathcal{L}'_F)$ for $\mathcal{L} \subseteq \mathcal{L}^*$ which is $\epsilon$-consistent. A Poolean $\mathcal{F}$-update $\mathcal{D}_F(\mathcal{L}'_F)$ is maximal if there exists no proper extension $\mathcal{L}' \supset \mathcal{L}$ with $\mathcal{L}' \subseteq \mathcal{L}^*$ such that $\mathcal{D}_F(\mathcal{L}'_F)$ is $\epsilon$-consistent.

6. $\mathcal{E}$ is a $P_e DP$-extension of $\langle \mathcal{L}^*, \mathcal{F} \rangle$ iff $\mathcal{E} = \mathcal{E}(\mathcal{D}) := \{Ca \mid \mathcal{D} \vdash _{\epsilon} Fx \Rightarrow C\}$ for some maximal Poolean $\mathcal{F}$-update $\mathcal{D}$ of $\mathcal{L}^*$. $P_e DP(\mathcal{K})$ is the set of all $P_e DP$-extensions of $\mathcal{K}$.

Theorem 5 states the fundamental facts about the system $P_e DP$. 18
Theorem 5 For given \( \langle \mathcal{L}^*, \mathcal{F} \rangle \) with consistent \( \mathcal{F} \), assume \( \mathcal{L} \subseteq \mathcal{L}^* \) and \( \mathcal{M} = \mathcal{L}^\sim \). For claims 2, 3 and 5, assume in addition that \( \mathcal{M} \) is a materialization of \( \langle \mathcal{L}, \mathcal{F} \rangle \).

1. \( \mathcal{M} \) is a materialization of \( \langle \mathcal{L}^*, \mathcal{F} \rangle \) iff \( \mathcal{D}_\mathcal{F}(\mathcal{L}_\mathcal{F}) \) is a Poolean \( \mathcal{F} \)-update of \( \mathcal{L}^* \).

2. (a) \( \mathcal{L}_\mathcal{F} \) satisfies the principle \((P^*)\) and (b) \( \mathcal{D}_\mathcal{F}(\mathcal{L}_\mathcal{F}) \) satisfies the principle \((P)\).

3. For any \( A \in \mathcal{B}_{\mathcal{L}_\mathcal{F}} \): \( A \in \mathcal{C}_\mathcal{F}(\mathcal{M} \cup \mathcal{F}) \) iff \( \mathcal{D}_\mathcal{F}(\mathcal{L}_\mathcal{F}) \vdash _F Fx \Rightarrow A \).

4. For any \( E \in \mathcal{B}_{\mathcal{L}_\mathcal{F}} \): \( E \in \mathcal{P}_{\mathcal{E}} \mathcal{D}_\mathcal{F}(\langle \mathcal{L}^*, \mathcal{F} \rangle) \) iff \( E \in \mathcal{P}_{\mathcal{E}} \mathcal{D}_\mathcal{F}(\langle \mathcal{L}^*, \mathcal{F} \rangle) \).

5. For any \( A \in \mathcal{B}_{\mathcal{L}_\mathcal{F}} \): (a) \( \mathcal{L}_\mathcal{F} \cup \mathcal{C}_\mathcal{F}(\mathcal{L}_\mathcal{F}) \cup \mathcal{I}_\mathcal{F}(\mathcal{L}_\mathcal{F}) \vdash _F Fx \Rightarrow A \) iff \( \mathcal{D}_\mathcal{F}(\mathcal{L}_\mathcal{F}) \vdash _F Fx \Rightarrow A \), then for all \( p \) on \( \Omega \) satisfying \( \mathcal{C}_\mathcal{F}(\mathcal{L}) \) and \( \mathcal{I}_\mathcal{F}(\mathcal{L}_\mathcal{F}) \): \( u(Fx \Rightarrow A) \leq \sum_{L \in \mathcal{L}} u(L) \).

Th. 5.1 says, in other words, that the consistency of Poolean scenarios has its exact counterpart in the \( \epsilon \)-consistency of the corresponding \( \mathcal{F} \)-transformed default laws. Concerning Th. 5.2, note that principle \((P)\), though satisfied for \( \mathcal{M} \), need not be satisfied if one passes to proper \( \mathcal{F} \)-consistent extensions of \( \mathcal{M} \). This is an unavoidable consequence of the nonmonotonic character of default reasoning. It follows that the transformed set \( \mathcal{L}_\mathcal{F} \) has to be computed newly if the materialization \( \mathcal{M} \) found for one query is extended when asking further queries. Th.5.3-4 tells us what is expected, namely that the \( \mathcal{F} \)-updated transformed laws together with \( \mathcal{F} \) \( \epsilon \)-entail the same singular conclusions as the corresponding Poolean scenario, and that Poolean extensions of a knowledge base are exactly \( \mathcal{P}_{\mathcal{E}} \mathcal{D}_\mathcal{F}(\langle \mathcal{L}^*, \mathcal{F} \rangle) \)-extensions. Th. 5.5a says that from \( \mathcal{L}_\mathcal{F}(\mathcal{L}) \cup \mathcal{I}_\mathcal{F}(\mathcal{L}_\mathcal{F}) \) not more laws with \( \mathcal{F} \)-antecedent are \( \epsilon \)-derivable than from \( \mathcal{D}_\mathcal{F}(\mathcal{L}_\mathcal{F}) \) alone. This immediately entails lower bound propagation according to th. 5.5b.\(^\text{17}\)

We finally present a procedure for \( \mathcal{P}_{\mathcal{E}} \mathcal{D}_\mathcal{F} \) default entailment.

Procedure for \( \mathcal{P}_{\mathcal{E}} \mathcal{D}_\mathcal{F} \)

Input: A knowledge base \( \langle \mathcal{L}^*, \mathcal{F} \rangle \) with consistent \( \mathcal{F} \) and a singular query \( Q \).

Output: If \( \langle \mathcal{L}^*, \mathcal{F} \rangle \vdash _{\mathcal{P}_{\mathcal{E}}} Q \), then a (good) lower probability bound \( b(Q) \) and (small) sets \( \mathcal{L}(Q), \mathcal{C}(Q) \) and \( \mathcal{I}(Q) \) such that \( \mathcal{L}(Q) \) is a materialization of \( \langle \mathcal{L}^*, \mathcal{F} \rangle \), and \( \langle \mathcal{L}(Q) \cup \mathcal{C}(Q) \cup \mathcal{I}(Q), \mathcal{F} \rangle \vdash _{\mathcal{P}_{\mathcal{E}}} [Q, b(Q)] \). – Else: FAIL.

1. Subprocedure Poole: Find a set \( \mathcal{L}(Q) \subseteq \mathcal{L}^* \) such that (a) \( \mathcal{L}(Q) \) is a materialization of \( \langle \mathcal{L}^*, \mathcal{F} \rangle \cup \mathcal{F} \) and

\(^{17}\)The restriction to \( p \)'s proper for \( \mathcal{L} \cup \mathcal{I}_\mathcal{F}(\mathcal{L}_\mathcal{F}) \) is not necessary because \( \mathcal{D}_\mathcal{F}(\mathcal{L}_\mathcal{F}) \) is \( \epsilon \)-consistent. Th. 5.5(a) implies also that \( \vdash _{\mathcal{P}_{\mathcal{E}}} \) is a proper extension of \( \vdash _{\mathcal{E}} \).
\(-Q\) resolves to \(\bot\) and (b) \(\mathcal{L}(Q)^- \cup \mathcal{F}\) is consistent. If not found: FAIL.

2. Subprocedure Transform: Based on the search tree created in step 1b, determine for each \((A \Rightarrow B) \in \mathcal{L}(Q)\) whether (a) \(\mathcal{L}(Q)^- \cup \mathcal{F} \vdash \neg Aa\), and if yes, whether (b) \(\mathcal{L}(Q)^- \cup \mathcal{F} \vdash Ba\).

3. Subprocedure Generate: Based on 2., generate \(\mathcal{L}_F, \mathcal{C}(Q) := \mathcal{C}_F(\mathcal{L}(Q))\) (according to def. 3.1-2), \(\mathcal{I}(Q) := \mathcal{I}_F(\langle \mathcal{L}(Q) \rangle_F)\) and \(b(Q) := 1 - \sum_{L \in \mathcal{L}(Q)}(1 - b(L))\). Output \(\mathcal{L}(Q), \mathcal{C}(Q), \mathcal{I}(Q)\) and \(b(Q)\).

The procedure is a correct and complete decision procedure, and its complexity is not significantly greater than that of subprocedure Poole. For subprocedure Transform does not require any new consistency tests, since the search tree created in step 1b contains every clause provable from \(\mathcal{L}(Q)^- \cup \mathcal{F}\) by resolution. It is sufficient to check for each \((A \Rightarrow B) \in \mathcal{L}(Q)^-\) whether \(\text{Cl}(\neg Aa)\) is subsumed by some node in this search tree, and if yes, to perform the same check for \(\text{Cl}(Ba)\) (thereby, \(\text{Cl}(X)\) is the set of clauses in which \(X\) has been decomposed during step 1).

Recall the considerations on approximating tight bounds at the end of §2.3. Subprocedure Poole has been implemented in [27]. It is based on a PROLOG-variant of linear resolution [28, p.10] and thus never encounters premises which are derivationally irrelevant for \(Q\) (cf. [18, p.139]). We furthermore impose an ordering on the scanning procedure involved in step 1a where the facts come before the laws (cf. [29, p.111]). Usually, this will be sufficient for finding a minimal set of law premises \(\mathcal{L}(Q)\). By assuming that the default laws are ordered with decreasing lower probability bound we take care of the requirement of finding law premises with bounds as high as possible. Implemented in this way, the procedure will generally find good approximations of tight bounds. Of course, a guarantee that an optimal premise set has been found can only be given by backtracking to all alternative subsets of law premises. This is not recommendable for knowledge bases containing many default laws.

To implement the procedure in a user-interactive fashion has significant advantages. By refuting certain default assumptions in \(\mathcal{C}(Q)\) and \(\mathcal{I}(Q)\) the user can control which extensions are preferred among mutually incompatible extensions. By asking bounded queries \([Q, r_i]\) (‘prove \(Q\) with lower bound \(\geq r_i\)’) with successively increasing bounds \(r_i\), the user can help to approximate a proof of \(Q\) with an optimal bound.
Appendix: Proofs of the theorems

Proof of theorem 2:

Let \( D := \mathcal{D}(L, I) \). We first prove (1) \( L \cup I \vdash_r L \) iff (2) \( L \cup D \vdash_r L \). The right-to-left-direction holds because \( L \cup I \vdash_r D \) by applications of the rule (Irr). The left-to-right direction holds because (Irr) is the only rule applying to elements of \( I \), and the conclusions of its applications are elements of \( D \).

If we replace every \( A \Rightarrow B \) in the proof of (1) to which rule (Irr) has been applied by \( F x \land A \Rightarrow B \), and remove all \( I \)-premises from this proof, we obtain a proof of (2).

Next we prove that (3) \( L \cup I \vDash \varepsilon \)-entails \( L \) iff (4) \( L \cup D \vDash \varepsilon \)-entails \( L \). For the left-to-right-direction, take some \( \delta > 0 \). To prove (4) we must show that there exists \( \epsilon > 0 \) such that for all \( p \in \Pi(L, D, L) \), if (a) \( \forall L \in L \left[ p(L) \geq 1 - \epsilon \right] \) and (b) \( \forall D \in D \left[ p(D) \geq 1 - \epsilon \right] \), then (c) \( p(L) \geq 1 - \delta \). By (3), there exists \( \epsilon' \) such that for all \( p \in \Pi(L, I, L) \), if (a') \( \forall L \in L \left[ p(L) \geq 1 - \epsilon' \right] \) and (d') \( \forall D \in D \left[ (p(L^D) - p(D)) \leq \epsilon' \right] \), then (c) – where \( L^D \) is the unique law \( (A \Rightarrow B) \in L \) such that Irr(C : A ⇒ B) ∈ I and \( D = (C \land A \Rightarrow B) \).

\( \Pi(L, I, L) = \Pi(L, D, L) \), since (by def.) \( p \) is proper for \( \text{Irr}(C : A \Rightarrow B) \) iff \( p \) is proper for \( C \land A \Rightarrow B \). We put \( \epsilon = \epsilon' \). Then the assumption that (a)+(b) hold for some given \( p \) implies that (a') and (d') hold for \( p \). Hence (c) follows by (3); which proves (4). – For the right-to-left-direction, take again some \( \delta > 0 \). To prove (3) we must show that there is \( \epsilon > 0 \) such that for all \( p \in \Pi(L, I, L) \), if (a) holds and (d) \( \forall D \in D \left[ (p(L^D) - p(D)) \leq \epsilon \right] \), then (c). By (4) there is \( \epsilon' > 0 \) such that for all \( p \in \Pi(L, D, L) \), if (a') holds and (b') \( \forall D \in D \left[ p(D) \geq 1 - \epsilon' \right] \), then (c). Again \( \Pi(L, I, L) = \Pi(L, D, L) \). (a) and (d) imply that \( p(D) \geq 1 - 2\epsilon \) for all \( D \in D \). We put \( \epsilon = \frac{\epsilon'}{2} \). Then the assumption that (a) and (d) hold for given \( p \) implies that (a') and (b') hold for \( p \). Hence (c) follows by (4), which proves (3).

By th.1.1+3, (2) is equivalent with (4), so (1) is equivalent with (3), Q.E.D.

For the next theorems we need two well-known lemmata about \( \varepsilon \)-entailment and \( \varepsilon \)-consistency and one lemma about material counterparts. Some terminology: A truth valuation \( u \) verifies [falsifies] \( A \Rightarrow B \) iff \( u(A \land B) = 1 \) \[ u(A \land \neg B) = 1 \], respectively; \( u \) falsifies \( L \) iff \( u \) falsifies some \( L \in L \); \( u \) confirms \( L \) iff \( u \) verifies some \( L \in L \) and does not falsify \( L \); \( L \) is confirmable iff there exists \( u \) which confirms \( L \); \( L \) is nontrivial iff \( L \)'s antecedent is consis-
tent; finally \( L \) is nontrivial iff \( L \) is nonempty and every \( L \in L \) is nontrivial.

For lemma 1 see [2, p.61]; for lemma 2 see [2, p.52], [23, p.488].

**Lemma 1** \( L \) \( \epsilon \)-entails \( A \Rightarrow B \) iff either (a) \( L \) is \( \epsilon \)-inconsistent or (b) some subset \( L^* := \{ A_i \Rightarrow B_i \mid 1 \leq i \leq n \} \subseteq L \) yields \( A \Rightarrow B \), which means that the following holds: (i) each truth valuation confirming \( L^* \) verifies \( A \Rightarrow B \), and (ii) each truth valuation falsifying \( A \Rightarrow B \) falsifies \( L^* \).

**Lemma 2** \( L \) is \( \epsilon \)-inconsistent iff some nontrivial subset of \( L \) is not confirmable iff for some consistent \( A \in BLang \), \( L \vdash \epsilon A \Rightarrow \bot \).

**Lemma 3** If \( L \) is \( \epsilon \)-consistent, then \( L \vdash \epsilon L \) implies \( L \vdash L \).

**Proof of lemma 3:** If \( L \) is \( \epsilon \)-consistent, then every \( L \) derivable from \( L \) by the rules of \( P_\epsilon \) is derivable without use of the rule (\( \epsilon EFQ \)), because this rule applies only to laws \( A \Rightarrow \bot \) with consistent antecedent, and whenever \( L \) implies such a law it is \( \epsilon \)-inconsistent by lemma 2. All rules of \( P_\epsilon \) distinct from (\( \epsilon EFQ \)) are propositionally valid for the material counterparts of the default laws. Hence \( L \) is derivable from \( L \) by rules of propositional logic. Q.E.D.

**Proof of theorem 3:**

For th.3.1: Let \( L = \{ A_i \Rightarrow B_i \mid 1 \leq i \leq n \} \). Given \( L \vdash F \vdash Aa \), then (a): \( Fx \land \land_{1 \leq i \leq n}(A_i \rightarrow B_i) \vdash Fx \land A \) by prop. logic and universal generalization. This implies that (b): \( Fx \land \land_{1 \leq i \leq n}(Fx \land A_i \rightarrow B_i) \vdash Fx \land A \), because by prop. logic, (b)'s antecedent implies (a)'s antecedent. Moreover, (b) implies (c): \( \land_{1 \leq i \leq n}(Fx \land A_i \rightarrow B_i) \vdash Fx \rightarrow A \) as well as (d): \( (Fx \lor \lor_{1 \leq i \leq n} A_i) \land \land_{1 \leq i \leq n}(Fx \land A_i \rightarrow B_i) \vdash Fx \land A \) by prop. logic. But (c)+(d) imply that \( D_F(L) \) yields \( F \Rightarrow A \) in the sense of lemma 1(b) (by prop. logic). This implies by lemma 1 that \( D_F(L) \vdash Fx \Rightarrow A \).

To prove th. 3.2, we note that by lemma 3, \( D_F(L) \vdash Fx \Rightarrow A \) implies (e): \( (D_F(L))^- \vdash Fx \rightarrow A \), since \( D_F(L) \) is \( \epsilon \)-consistent by assumption. From (e) it follows that (f): \( L^- \vdash Fx \rightarrow A \) (since \( L^- \vdash (D_F(L))^- \)), and (f) implies \( L^- \cup F \vdash Aa \) by universal instantiation. Q.E.D.

**Proof of theorem 4:**

Put \( L_\epsilon := T(L, C) \). For th. 4.(ii): We prove (1) \( I \cup L \cup C \vdash \epsilon L \) iff (2)
\[ I \cup C \cup L_c \vdash \epsilon \ L. \] The right-to-left-direction holds because \( I \cup C \cup L_c \vdash \epsilon \) by applications of the rule (Cont). The left-to-right direction holds because (Cont) is the only rule applying to elements \( C \in C \), and the conclusions of its applications are elements of \( L_c \). If we replace every \( L := (A \Rightarrow B) \) in the proof of (1) to which (Cont) has been applied by \( L_c := (\neg B \Rightarrow \neg A) \), and remove all \( C \)-premises from this proof, we obtain a proof of (2).

**Concerning 4(i):** We prove that if (3) \( I \cup \mathcal{L} \cup \mathcal{C} \vdash \epsilon \), then (4) \( I \cup \mathcal{L} \cup \mathcal{C} \vdash \epsilon \)-entails \( I \cup \mathcal{L} \cup \mathcal{C} \). Assume \( \not\vdash \epsilon \mathcal{L} \cap \mathcal{C} \) is consistent. Then by lemma 2 and prop. logic, there is a nontrivial subset \( \{ \mathcal{F} \mathcal{L} \cap \mathcal{A}^i \vdash \mathcal{B}^i \mid 1 \leq i \leq n \} \) of \( \mathcal{D}_F(L'_F) \) such that (a) \( \mathcal{F} \mathcal{L} \cap \mathcal{A}^i \vdash \mathcal{B}^i \mid 1 \leq i \leq n \) \vdash A_{1 \leq i \leq n} -(\mathcal{F} \mathcal{L} \cap \mathcal{A}^i). Each \( A^i \vdash \mathcal{B}^i \) is in \( L'_F \), and following from def.3.1-2 and prop. logic, \( \mathcal{L} \vdash (L'_F)^- \). Hence \( \{ A^i \vdash \mathcal{B}^i \mid 1 \leq i \leq n \} \) and thus also (b) \( \{ \mathcal{F} \mathcal{L} \cap \mathcal{A}^i \vdash \mathcal{B}^i \mid 1 \leq i \leq n \} \)

**Proof of theorem 5:**

We start with 2(a): If \( (A \Rightarrow B) \in \mathcal{C} \), then \( \mathcal{M} \cup \mathcal{F} \not\vdash \neg \mathcal{A} \mathcal{A} \) by def.3.1, and \( \mathcal{M} \cup \mathcal{F} \not\vdash \neg \mathcal{A} \mathcal{A} \) for any disjunct \( \mathcal{B} \) because by assumption, \( \mathcal{B} \) is a single literal, and \( \mathcal{M} \cup \mathcal{F} \vdash \neg \mathcal{A} \mathcal{A} \) cannot hold, for else \( \mathcal{M} \cup \mathcal{F} \vdash \neg \mathcal{A} \mathcal{A} \) would hold, contradiction. If \( \neg \mathcal{B} \Rightarrow \neg \mathcal{A} \) is \( \mathcal{L} \), then \( \mathcal{M} \cup \mathcal{F} \vdash \neg \mathcal{A} \mathcal{A} \) by def.3.1, whence (P*a) is satisfied, and \( \mathcal{M} \cup \mathcal{F} \not\vdash \mathcal{B} \mathcal{A} \) by def.3.1, whence (P*b) is satisfied. Finally, if \( \mathcal{T} \Rightarrow \neg \mathcal{A} \mathcal{V} \mathcal{B} \) is \( \mathcal{L} \), then (P*b) is satisfied since \( \mathcal{M} \cup \mathcal{F} \not\vdash \neg \mathcal{T} \) and (P*a) is satisfied since \( \mathcal{M} \cup \mathcal{F} \vdash \neg \mathcal{A} \mathcal{A} \) (as well as \( \mathcal{M} \cup \mathcal{F} \vdash \mathcal{B} \mathcal{A} \) by def.3.1.

**For 1:** Here we must show that \( \mathcal{M} \cup \mathcal{F} \) is consistent iff \( \mathcal{D}_F(L'_F) \) is \( \epsilon \)-consistent. **Left-to-right:** Assume \( \mathcal{D}_F(L'_F) \) is \( \epsilon \)-inconsistent. Then by lemma 2 and prop. logic, there is a nontrivial subset \( \{ \mathcal{F} \mathcal{L} \cap \mathcal{A}^i \vdash \mathcal{B}^i \mid 1 \leq i \leq n \} \) of \( \mathcal{D}_F(L'_F) \) such that (a) \( \mathcal{F} \mathcal{L} \cap \mathcal{A}^i \vdash \mathcal{B}^i \mid 1 \leq i \leq n \) \vdash A_{1 \leq i \leq n} -(\mathcal{F} \mathcal{L} \cap \mathcal{A}^i). Each \( A^i \vdash \mathcal{B}^i \) is in \( L'_F \), and following from def.3.1-2 and prop. logic, \( \mathcal{L} \vdash (L'_F)^- \). Hence \( \{ A^i \vdash \mathcal{B}^i \mid 1 \leq i \leq n \} \) and thus also (b) \( \{ \mathcal{F} \mathcal{L} \cap \mathcal{A}^i \vdash \mathcal{B}^i \mid 1 \leq i \leq n \} \)

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is implied by $\mathcal{M}$. (a)+(b) imply that $\mathcal{M} \cup \mathcal{F} \vdash \bigwedge_{1 \leq i \leq n} \neg A_i^t a$, by prop. logic. But this would mean that $\mathcal{L}_F^i$ does not satisfy principle $(P^*b)$, whence by th. 5.2(a), $\mathcal{M} \cup \mathcal{F}$ cannot be consistent. – Right-to-left: Assume $\mathcal{M} \cup \mathcal{F}$ is inconsistent. Since $(\mathcal{L}_F^i)^{-} \not\models \mathcal{M}$ (by prop. logic), it follows that $(\mathcal{L}_F^i)^{-} \cup \mathcal{F}$ is inconsistent. Hence $\mathcal{D}_F(\mathcal{L}_F^i) \vdash_{e} \mathcal{F} x \models \bot$ by th. 3.1. Since $\mathcal{F}$ is consistent, lemma 2 implies that $\mathcal{D}_F(\mathcal{L}_F^i)$ is $\epsilon$-inconsistent.

For 2(b): By th. 5.2(a), $(P^*a)$ holds, and by th. 5.1, $\mathcal{D}_F(\mathcal{L}_F^i)$ is $\epsilon$-consistent. This implies by th. 3.2 that also $(P)$ is satisfied.

For 3: $\mathcal{M} \not\models (\mathcal{L}_F^i)^{-}$ by prop. logic, and $\mathcal{D}_F(\mathcal{L}_F^i)$ is $\epsilon$-consistent by th. 5.1; so th.5.3 follows by th.3.

For 4: $\mathcal{E} \in POOLE((\mathcal{L}^*,\mathcal{F}))$ iff $\mathcal{E} = Cn(\mathcal{L}^{-} \cup \mathcal{F})$ for some maximal materialization $\mathcal{L}^{-}$ of $(\mathcal{L},\mathcal{F})$ iff (i): $\mathcal{E} = \mathcal{E}[\mathcal{D}_F(\mathcal{L}_F^i)]$ by th. 3, since $(\mathcal{L}_F^i)^{-} \models \mathcal{L}^{-}$ and $\mathcal{D}_F(\mathcal{L}_F^i)$ is $\epsilon$-consistent (i.e. is a Poolean update) by th. 5.1. It remains to show that $\mathcal{D}_F(\mathcal{L}_F^i)$ is maximal. Assume for reductio that for some $L \in (\mathcal{L}^* \setminus \mathcal{L})$, $\mathcal{D}_F((\mathcal{L} \cup \{L\})^i_F)$ would be $\epsilon$-consistent. Then $(\mathcal{L} \cup \{L\})^{-} \cup \mathcal{F}$ must be consistent by th. 5.1. Because $(\mathcal{L} \cup \{L\})^{-}$ properly extends $\mathcal{M}$, $\mathcal{M}$ would then not be a maximal materialization; a contradiction. – So $\mathcal{D}_F(\mathcal{L}_F^i)$ is a maximal Poolean update, whence (i) holds iff $\mathcal{E} \in P,DP((\mathcal{L}^*,\mathcal{F}))$ (def. 3.3-4).

For 5(a): By th. 4, $\mathcal{L} \cup \mathcal{L}_F(\mathcal{L}) \cup \mathcal{I}_F(\mathcal{L}_F^i) \epsilon$-implies the same default laws as $\mathcal{L} \cup \mathcal{L}_F^i \cup \mathcal{I}_F(\mathcal{L}_F^i)$, which by th.2 $\epsilon$-implies the same default laws as $\mathcal{L} \cup \mathcal{L}_F^i \cup \mathcal{D}_F(\mathcal{L}_F^i)$. To prove 5(a) we first have to prove:

(A): $\mathcal{L} \cup \mathcal{L}_F^i \cup \mathcal{D}_F(\mathcal{L}_F^i)$ is $\epsilon$-consistent.

For reductio, assume that the negation of (A) is true. Then by lemma 2, there is a nonempty subset $X \subseteq \mathcal{L} \cup \mathcal{L}_F^i \cup \mathcal{D}_F(\mathcal{L}_F^i)$ such that (B): $X^{-} \vdash \bigwedge \mathcal{Y}$, where $\mathcal{Y} = \{ \neg P \mid (P \Rightarrow Q) \in X \}$. Now note that (C): $\mathcal{L}^{-} \vdash [\mathcal{L} \cup \mathcal{L}_F^i \cup \mathcal{D}_F(\mathcal{L}_F^i)][-]$ must hold, because $(\mathcal{L}_F^i)^{-} \vdash (\mathcal{D}_F(\mathcal{L}_F^i))^-$ and $(\mathcal{L}_F^i)^- \vdash \mathcal{L}^{-}$ (by prop. logic). (C) and (B) imply (D): $\mathcal{L}^{-} \vdash \bigwedge \mathcal{Y}$. For each element of $\mathcal{Y} \in \mathcal{Y}$, one of the following cases must apply: (i) $Y = \neg A$ for $(A \Rightarrow B) \in \mathcal{L}$, or (ii) $Y = \neg A$ or $Y = \neg (\mathcal{F} x \land A)$ for $(A \Rightarrow B) \in \mathcal{L}_F$. (D) + case (ii) imply $\mathcal{M} \cup \mathcal{F} \vdash \neg A$ for $(A \Rightarrow B) \in \mathcal{L}_F$, which is impossible since $(P^*)$ is satisfied by th. 5.2. Thus only case (i) can apply. But this means that $X \subseteq \mathcal{L}$, and hence by lemma 2 that $\mathcal{L}$ would be $\epsilon$-inconsistent. But by assumption and th. 5.1, $\mathcal{L}$ is $\epsilon$-consistent – a contradiction.

Right-to-left of th.5(a) is easy, since $\mathcal{L} \cup \mathcal{L}_F(\mathcal{L}) \cup \mathcal{I}_F(\mathcal{L}_F^i) \vdash_{e} \mathcal{F} x \models \bot$. By the arguments above, (1) holds iff
(2): $L \cup L_x^f \cup D_x(L_x^f) \vdash \mathcal{F}x \Rightarrow A$. The premises of (2) are $\varepsilon$-consistent by (A) above. Hence by lemma 3, (3): $L^- \cup (L_x^f)^- \cup (D_x(L_x^f))^- \vdash \mathcal{F}x \Rightarrow A$. (3) and (C) above imply that $L^- \vdash \mathcal{F}x \Rightarrow A$ and hence (4): $\mathcal{M} \cup \mathcal{F} \vdash \mathcal{A}$. (4) implies by th.3.1 that $D_x(L_x^f) \vdash \mathcal{F}x \Rightarrow A$.

For 5(b): The antecedent of 5(b) implies $D_x(L_x^f) \vdash \mathcal{F}x \Rightarrow A$ by th.5.5(a). Every $D \in D_x(L_x^f)$ results from a unique $L \in L$ (by transforming and updating) such that for all $p$ satisfying all contraposition- and irrelevance assumptions in $C_x(L)$ and $I_x(L)$, $p(D) \geq p(L)$. By th.1.2, our claim follows for all $p$ proper for $L \cup I_x(L_x^f)$; but this restriction is not necessary because $D_x(L_x^f)$ is $\varepsilon$-consistent. So (by the argument for lemma 3), $\mathcal{F}x \Rightarrow A$ follows from $D_x(L_x^f)$ in the weaker calculus $P$ (recall §2.2), and here the inequality of th. 1.2 holds without the restriction to proper $p$’s (cf. [1], th.2). Q.E.D.

References


