OPTIMAL SEQUENTIAL CHANGE-DETECTION FOR FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. The sequential detection of an abrupt and persistent change in the dynamics of an arbitrary continuous-path stochastic process is considered; the optimality of the cumulative sums (CUSUM) test is established with respect to a modified Lorden’s criterion. As a corollary, sufficient conditions are obtained for the optimality of the CUSUM test when the observed process is described by a fractional stochastic differential equation. Moreover, a novel family of model-free, Lorden-like criteria is introduced and it is shown that these criteria are optimized by the CUSUM test when a fractional Brownian motion adopts a polynomial drift. Finally, a modification of the continuous-time CUSUM test is proposed for the case that only discrete-time observations are available.

1. INTRODUCTION

1.1. Problem formulation and literature review. The fast and accurate detection of an abrupt change in the behavior of a stochastic system is an important problem in many application areas, such as quality control, target tracking, navigation, seismology, bio-surveillance, intrusion detection in computer networks and finance (see for example [1], [18], [23], [37]). These changes usually represent anomalies that must be detected in an on-line mode so that any appropriate action be taken immediately.

In order to be more precise, we assume that we observe sequentially the path of a stochastic process \( \{\xi_t\}_{t \geq 0} \) which is defined on some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). We denote by \( \{\mathcal{F}_t\} \) the filtration generated by \( \{\xi_t\} \), i.e.

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\( \mathcal{F}_t := \sigma(\xi_s : 0 \leq s \leq t), \ t > 0, \) whereas we set \( \xi_0 := 0, \) thus \( \mathcal{F}_0 \) is the trivial \( \sigma \)-algebra. The process \( \{\xi_t\} \) is initially in control, but at some unknown, deterministic time \( \tau \) a disorder occurs and the process gets out of control.

The unknown time of the change \( \tau \) takes values in \([0, \infty]\) and parametrizes the true underlying probability measure \( P \), which we denote by \( P_\tau \). Thus, under \( P_\infty \) the change never occurs, whereas under \( P_0 \) the change has occurred from the beginning, before any observation has been collected. We assume that the probability measures \( P_0 \) and \( P_\infty \) are equivalent when restricted to the \( \sigma \)-algebra \( \mathcal{F}_t \), i.e. \( P_0|_{\mathcal{F}_t} \sim P_\infty|_{\mathcal{F}_t} \) for every \( t \in [0, \infty) \).

Therefore, we can define the following likelihood ratio process:

\[
e^u_t := \frac{dP_0}{dP_\infty}|_{\mathcal{F}_t} = \left( \frac{dP_\infty}{dP_0}|_{\mathcal{F}_t} \right)^{-1}, \quad 0 \leq t < \infty,
\]

where \( u_0 := 1 \). This assumption guarantees that the change-detection problem is well-defined, since it implies that the measures \( P_\tau \) and \( P_\infty \) are equivalent when restricted to \( \mathcal{F}_t \) with

\[
\frac{dP_\tau}{dP_\infty}|_{\mathcal{F}_t} = \left( \frac{dP_\infty}{dP_\tau}|_{\mathcal{F}_t} \right)^{-1} = e^{u_t - u_\tau},
\]

for any \( \tau \in [0, \infty) \) and \( t \in [\tau, \infty) \).

The main goal in the change-detection problem is to find a detection rule \( T \) that raises an alarm as soon as possible after the change has occurred. Since the path of \( \{\xi_t\} \) is sequentially observed, the detection rule cannot anticipate the future and is naturally defined as an \( \{\mathcal{F}_t\} \)-stopping time.

A good detection rule \( T \) should minimize –in some sense– the detection delay \( (T - \tau)^+ \), but it should also have a small frequency of false alarms (when applied repeatedly). Indeed, raising an alarm when the change has not occurred may also have severe undesirable consequences; consider for example the social cost of a false epidemic detection. Since the minimization of the detection delay and of the frequency of false alarms are two antithetic goals (which cannot be attained simultaneously), it is crucial to formulate the change-detection problem properly in order to balance this trade-off.

Lorden [16] quantified the performance of a detection rule \( T \) using the worst (with respect to \( \tau \)) conditional expected detection delay given the
worst possible scenario up to time $\tau$, i.e.

$$\mathcal{J}_L[\mathcal{T}] := \sup_{\tau \geq 0} \text{esssup} E_\tau[(\mathcal{T} - \tau)^+] | \mathcal{F}_\tau]$$

and suggested the minimization of $\mathcal{J}_L$ over $\{\mathcal{F}_t\}$-stopping times $\mathcal{T}$ whose period of false alarms $E_\infty[\mathcal{T}]$ is bounded below by a fixed constant $\gamma$, i.e.

$$\inf_{\mathcal{T}} \mathcal{J}_L[\mathcal{T}] \quad \text{where} \quad E_\infty[\mathcal{T}] \geq \gamma.$$  \hfill (4)

Although this (strongly minimax) approach seems to be rather conservative, it was shown by Moustakides in [22] that it is appropriate when the change can be triggered by the observations. For more details on this point, as well as a comparison with the minimax approach due to Pollack [29] and the Bayesian approach due to Shiryaev [35], we refer to [22].

Lorden’s criterion has a deep connection with the cumulative sums (CUSUM) test, which was proposed by Page [28] and is defined as follows

$$S_c := \inf\{t \geq 0 : u_t - \inf_{0 \leq s \leq t} u_s \geq c\}. \hfill (5)$$

Thus, $S_c$ is the first time the process $\{u_t\}$ exceeds its running infimum by $c$, where $c$ is a fixed positive constant. Moustakides [20]–and later Ritov [32]–proved that $S_c$ is $\mathcal{J}_L$-optimal, i.e. solves problem (4), in the case of independent and identically distributed observations before and after the change. The threshold $c$ that guarantees this optimality is the one that makes the false alarm constraint be satisfied with equality, i.e. $E_\infty[S_c] = \gamma$.

The $\mathcal{J}_L$-optimality of $S_c$ was extended in continuous-time, independently by Shiryaev [36] and Beibel [2], when the observed process is a standard Brownian motion that adopts a linear drift after the change.

Moustakides [21] proposed the following modification of Lorden’s optimization problem:

$$\inf_{\mathcal{T}} \mathcal{J}_M[\mathcal{T}] \quad \text{where} \quad E_\infty[-u_\mathcal{T}] \geq \gamma,$$  \hfill (6)

where

$$\mathcal{J}_M[\mathcal{T}] := \sup_{\tau \geq 0} \text{esssup} E_\tau[(u_\mathcal{T} - u_\tau)1_{\{\mathcal{T} \geq \tau\}}] | \mathcal{F}_\tau]. \hfill (7)$$

The main difference between $\mathcal{J}_L$ and $\mathcal{J}_M$ is that the latter incorporates the underlying dynamics. Indeed, $\mathcal{J}_M$ quantifies detection delay with the
information (in a Kullback-Leibler sense) that is required until an alarm is raised. On the other hand, $J_L$ considers the actual time that has elapsed until an alarm is raised, independently of the underlying dynamics.

Moustakides proved in [21] that the cusum test $S_c$ is $J_M$-optimal, i.e. solves problem (6), among stopping times $T$ such that $E_0[u_T], E_\infty[-u_T] < \infty$, when $\{\xi_t\}$ is a diffusion-type process and the change is persistent. Despite its generality, this optimality result does not include many important continuous-path processes. For example, suppose that $\xi_t = B_t + (t - \tau)^+$, where $\{B_t\}$ is an (unobserved) fractional Brownian motion with Hurst index $H \in (0, 1)$. When $H = 1/2$, this problem reduces to the detection of a linear drift in a standard Brownian motion and, as we previously discussed, in this case $S_c$ solves both (4) and (6), i.e. it is both $J_L$-optimal and $J_M$-optimal. However, analogous optimality results are not known when $H \neq 1/2$. It is one of the main goals of this work to understand the optimality properties of $S_c$ in this case and more generally when the post-change distribution is described by a fractional stochastic differential equation.

Fractional Brownian motion and fractional stochastic differential equations have been applied extensively in areas such as hydrology, traffic networks, finance and economics (see for example [3], [7], [10]). These diverse applications have triggered a great interest in the statistical inference of such processes. In particular, Kleptsyna and Le Breton in [12] and [13] studied the maximum likelihood estimator (mle) for the drift of the fractional Ornstein-Uhlenbeck process. Tudor and Viens [38] used Malliavin calculus techniques to study the mle in the case of a general fractional diffusion whose drift is linear with respect to the unknown parameter. For the same class of processes, Rao [31] studied a sequential version of the mle. The problem of sequential testing for fractional diffusions was considered by Rao in [30]. All these results were based on the pioneering work of Norros et al. in [24], where an integral transformation of the observed process was used in order to compute and analyze the mle for the linear drift of a fractional Brownian motion.
1.2. Main contributions. First of all, we extend the optimality result of Moustakides in [21] and show that the \textit{cusum} test $S_c$ is $J_M$-optimal when \{u_t\} has continuous paths and the change is persistent (in the sense of (9)). We then assume that the observed process is described by a fractional stochastic differential equation and obtain sufficient conditions for the $J_M$-optimality of $S_c$ in this framework. We focus in particular on the case that a fractional Brownian motion adopts a polynomial drift or turns into a fractional diffusion after the change.

Moreover, we introduce a novel family of model-free optimization problems where detection delay and period of false alarms are measured in non-linear time-scales. We then generalize the $J_L$-optimality of $S_c$ for the detection of a linear drift in a standard Brownian motion by showing an analogous result for the detection of a polynomial drift in a fractional Brownian motion.

Finally, we propose a modification of the continuous-time \textit{cusum} $S_c$ in the case that the underlying process can be observed only at discrete times.

The paper is organized as follows: in section 2, we prove the $J_M$-optimality of $S_c$ for arbitrary stochastic processes with continuous paths. In Section 3, we review stochastic differential equations driven by fractional Brownian motion. In Section 4, we consider the sequential change detection problem for fractional stochastic differential equations. In Section 5, we propose a novel class of Lorden-like, model-free criteria and show that they are optimized by the \textit{cusum} test under certain dynamics. In Section 6, we discuss the implementation of the \textit{cusum} test in the case of discrete-time observations and we conclude in Section 7.

2. Cusum optimality for continuous-path processes

In this section we assume that the likelihood ratio process \{e^{u_t}\} – defined in (1) – is a continuous-time process with continuous paths. Then, there exists (see [11], Ch. 6.1) a $(P_\infty, \mathcal{F}_t)$-local martingale \{X_t\} with continuous paths and a $(P_\infty, \mathcal{F}_t)$-predictable and increasing process \{A_t\} so that

$$u_t = X_t - \frac{1}{2} A_t, \quad t \in [0, \infty).$$

(8)
We also assume that the probability measures $P_0$ and $P_\infty$ are singular on the $\sigma$-algebra $\mathcal{F}_\infty := \sigma(\mathcal{V}_{t \geq 0})$, i.e.

$$P_0|_{\mathcal{F}_\infty} \perp P_\infty|_{\mathcal{F}_\infty},$$

which implies (see [11], Ch.8.1) that

$$P_0(A_\infty = \infty) = P_\infty(A_\infty = \infty) = 1.$$  

This assumption implies that the change is persistent, or – in other words – that it has sufficient “energy”. For example, this requirement excludes the case of a transient change, where the process returns to its old regime after some finite time.

Based on representation (8) and property (10), we can derive the operating characteristics of $S_c$ and prove its $J_M$-optimality following the method of Moustakides in [21]. Before we do so, we introduce the notation $y_t := u_t - \inf_{0 \leq s \leq t} u_s$, that is, $\{y_t\}$ is the so-called CUSUM process.

**Proposition 2.1.** The CUSUM test $S_c$ terminates almost surely, i.e. $P_t(S_c < \infty) = 1$ for any $t \in [0, \infty]$. Moreover:

$$J_M[S_c] = E_0[u_{S_c}] = g(c), \quad -E_\infty[u_{S_c}] = g(-c),$$

where $g(x) := e^{-x} + x - 1$.

**Proof.** Suppose that $t \in [0, \infty)$. With an application of Itô’s rule, optional sampling theorem and a localization argument, it can be shown – as in [21] – that $P_t$-a.s. on $\{S_c > t\}$ we have:

$$E_t[(A_{S_c} - A_t) \mathbb{1}_{\{S_c = \infty\}}|\mathcal{F}_t] \leq E_t[A_{S_c} - A_t|\mathcal{F}_t] = g(c) - g(y_t) \leq g(c) < \infty.$$  

(12)

It is now clear that $P_t(S_c = \infty) = 0$, otherwise we are led to a contradiction due to condition (10). Similarly, $P_\infty$-a.s. on $\{S_c > t\}$ we have

$$E_\infty[(A_{S_c} - A_t) \mathbb{1}_{\{S_c = \infty\}}|\mathcal{F}_t] \leq E_\infty[A_{S_c} - A_t|\mathcal{F}_t] = g(-c) - g(-y_t) \leq g(-c) < \infty,$$

(13)

which implies – again due to (10) – that $P_\infty(S_c = \infty) = 0$.

From (12) we also have $P_T$-a.s. on $\{S_c > t\}$ that

$$E_t[u_{S_c} - u_t|\mathcal{F}_t] = E_t[A_{S_c} - A_t|\mathcal{F}_t] = g(c) - g(y_t).$$

(14)
Consequently, since $g$ is an increasing function and a $\{y_t\}$ is a continuous, non-negative, increasing process, we obtain:

$$J_M[S_c] = \sup_{\tau \geq 0} \text{esssup}_{\tau} E_\tau[(u_{S_c} - u_\tau)1_{\{S_c \geq \tau\}} | \mathcal{F}_\tau]$$

$$= \sup_{\tau \geq 0} \text{esssup}_{\tau} E_\tau[(g(c) - g(y_\tau))1_{\{S_c \geq \tau\}} | \mathcal{F}_\tau]$$

$$= E_0[u_{S_c}] = g(c) = e^{-c} + c - 1, \quad (15)$$

which proves the first part of (11).

Similarly, from (13) we have $P_\infty$-a.s. on $\{S_c > t\}$:

$$-E_\infty[u_{S_c} - u_t|\mathcal{F}_t] = E_\infty[A_{S_c} - A_t|\mathcal{F}_t] = g(-c) - g(-y_t). \quad (16)$$

The second part in (11) now follows by setting $t = 0$ in (16). \qed

**Proposition 2.2.** For any stopping time $\mathcal{T}$ such that $E_0[A_T], E_\infty[A_T] < \infty$, we have:

$$J_M(\mathcal{T}) = \sup_{\tau \geq 0} \text{esssup}_{\tau} E_\tau[(A_T - A_\tau)^+ | \mathcal{F}_\tau], \quad -E_\infty[u_T] = E_\infty[A_T] \quad (17)$$

and

$$J_M(\mathcal{T}) \geq \frac{E_\infty[e^{\gamma \mathcal{T}_c} g(y_{\mathcal{T}_c})]}{E_\infty[e^{\gamma \mathcal{T}_c}]}, \quad \text{where } \mathcal{T}_c := \mathcal{T} \wedge S_c := \min(\mathcal{T}, S_c) \quad (18)$$

**Proof.** In order to prove (17), it suffices to show that $P_\tau$-a.s. on $\{S_c \geq \tau\}$

$$E_\tau[(u_T - u_\tau)|\mathcal{F}_\tau] = E_\tau[A_T - A_\tau|\mathcal{F}_\tau]$$

for any fixed $\tau \in [0, \infty]$. This follows from an application of optional sampling theorem and a localization argument, as long as the right-hand side is finite, which is indeed the case if $E_0[A_T] < \infty$ and $E_\infty[A_T] < \infty$. The lower bound (18) can now be shown as in [21], Theorem 2. \qed

We now present the main result of this section.

**Theorem 2.3.** Suppose that the likelihood-ratio process $\{e^{u_t}\}$ has continuous paths and that condition (2) is satisfied. Then, the CUSUM test $S_c$ solves problem (6) among stopping times $\mathcal{T}$ such that $E_0[A_T], E_\infty[A_T] < \infty$, as long as the threshold $c$ is chosen so that $\gamma = e^c - c - 1$. 


Proof. Let $\mathcal{T}$ be a stopping time so that $\mathbb{E}_0[A_{\mathcal{T}}], \mathbb{E}_\infty[A_{\mathcal{T}}] < \infty$ and $\mathbb{E}_\infty[A_{\mathcal{T}}] > \gamma$. Consider the function $\psi(c) := \mathbb{E}_\infty[-u_{\mathcal{T}_c}] = \mathbb{E}_\infty[A_{\mathcal{T}_c}]$, where $\mathcal{T}_c = \mathcal{T} \wedge \mathcal{S}_c$. Then, from (17) follows that $\psi(\infty) := \mathbb{E}_\infty[A_{\mathcal{T}_c}] > \gamma$ and it can be shown as in [21] that $\psi$ is continuous and increasing. Therefore, since $\psi(0) = 0$, there is some threshold $c$ so that $\psi(c) = \gamma$. Consequently, the corresponding stopping time $\mathcal{T}_c$ satisfies the false alarm constraint in (6) with equality, while having a better $\mathcal{J}_M$-performance than $\mathcal{T}$, since $\mathcal{T}_c \leq \mathcal{T}$.

Therefore, if we restrict ourselves to stopping times $\mathcal{T}$ such that $\mathbb{E}_0[A_{\mathcal{T}}] < \infty$ and $\mathbb{E}_\infty[A_{\mathcal{T}}] < \infty$, it suffices to take the infimum in $\mathcal{J}_M$ over stopping times that satisfy the false alarm constraint in (6) with equality.

From the previous discussion it becomes clear that for the CUSUM test $\mathcal{S}_c$ to be $\mathcal{J}_M$-optimal, its threshold $c$ must be chosen so that $\mathbb{E}_\infty[-u_{\mathcal{S}_c}] = \gamma$, which from (11) implies that $e^c - c - 1 = \gamma$. Moreover, since both sides in (18) are equal for $\mathcal{S}_c$, it suffices to show that $\mathcal{S}_c$ is the solution to the following optimization problem

$$
\inf_{\mathcal{T}} \frac{\mathbb{E}_\infty[e^{y_{\mathcal{T}_c}}g(y_{\mathcal{T}_c})]}{\mathbb{E}_\infty[e^{y_{\mathcal{T}_c}}]} \quad \text{when} \quad \mathbb{E}_\infty[A_{\mathcal{T}}] = \gamma, \quad (19)
$$

where $\mathcal{T}_c = \mathcal{T} \wedge \mathcal{S}_c$. But this – due to (17) – can be shown following exactly the same steps as in [21], Theorem 3.

We will use this general theorem in order to understand the optimality properties of $\mathcal{S}_c$ when the distribution of $\{\xi_t\}$ is induced by a fractional stochastic differential equation. Before we do so, we present a brief review on stochastic differential equations driven by fractional Brownian motion.

3. A QUICK REVIEW ON FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a centered, Gaussian process with continuous paths, $\{B_t\}_{t \geq 0}$, is defined. We say that $\{B_t\}$ is fractional Brownian motion with Hurst parameter (or index) $H \in (0, 1)$, if it has the following covariance structure

$$
\mathbb{E}(B_tB_s) = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H}\right), \quad t, s \geq 0. \quad (20)
$$
Clearly, when $H = 1/2$, $\{B_t\}$ reduces to the standard Brownian motion. Fractional Brownian motion was initially introduced by Kolmogorov [15] and was later studied by Mandelbrot and Van Ness, [17]. For an exhaustive treatment on this process we refer to [25], Chapter 5.

Fractional Brownian motion is $H$-self-similar, that is, $\{B_t\}$ has the same finite-dimensional distribution as $\{c^{-H} B_{ct}\}$ for every $c > 0$. Moreover, it has stationary increments, which are independent only when $H = \frac{1}{2}$.

More specifically, when $H > 1/2$ ($H < 1/2$), the increments of $\{B_t\}$ over disjoint intervals are positively (negatively) correlated and we say that fractional Brownian motion exhibits long-range (short-range) dependence, in the sense that $\sum_{n=1}^{+\infty} \mathbb{E}[(B_n - B_{n-1})B_1] = +\infty (< +\infty)$.

The Hurst parameter $H$ does not determine only the distributional properties of $\{B_t\}$, but also the regularity of its trajectories. Thus, $\{B_t\}$ is $\gamma$-Hölder continuous for every $0 \leq \gamma < H$, that is, for all $T > 0$, there exists a nonnegative random variable $C_{\gamma,T}$, which is $L^p$-integrable for all $p \geq 1$, so that:

$$|B_t - B_s| \leq C_{\gamma,T}|t - s|^{H - \epsilon}, \quad \forall s, t \in [0, T] \quad \mathbb{P} - a.s.$$ 

It is well-known (see for example [33]) that –when $H \neq 1/2$– fractional Brownian motion is not a semimartingale. Therefore, it is not possible to define an Itô stochastic integral with respect to it. Nevertheless, using the results of Young [40], it is possible to define an integral with respect to fractional Brownian motion pathwise. More specifically, if $\{u_t\}$ is a stochastic process with $\gamma$-Hölder continuous paths for some $\gamma > 1 - H$, then the integral $\int_0^t u_s dB_s$ is well-defined $\mathbb{P}$-a.s. for any $t \geq 0$ as a Riemann-Stieljes integral [40].

Finally, consider the following stochastic differential equation driven by a fractional Brownian motion $\{B_t\}$ with Hurst index $H$:

$$X_t = \int_0^t f(s, X_s) \, ds + B_t, \quad t \geq 0$$

(21)

where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a Borel function. In the case that $H = 1/2$, Zvonkin [41] and Veretennikov [39] obtained sufficient conditions for the existence and pathwise uniqueness of a strong solution with continuous paths. Nualart
and Ouksnine (in [26] and [27]) obtained the corresponding conditions when $H \neq 1/2$. More specifically, they showed that when $H < 1/2$, $f(t, x)$ must satisfy the following integrability condition:

$$f^2(t, x) \leq K + F(t, x), \quad t \geq 0, \quad x \in \mathbb{R},$$

where $K$ is a positive constant and $F$ a nonnegative Borel function for which there exist real numbers $p > 1$ and $\beta > \frac{p}{p-\frac{p}{2}}$, so that for every $T > 0$:

$$\int_0^T \left( \int_{\mathbb{R}} |F(t, x)|^p \, dx \right)^{\frac{\beta}{p}} \, dt < \infty.$$

On the other hand, when $H > \frac{1}{2}$, $f(t, x)$ must satisfy the following Hölder-continuity condition:

$$|f(t, x) - f(s, y)| \leq K(|x-y|^\alpha + |t-s|^\gamma), \quad t \geq 0, \quad x, y \in \mathbb{R},$$

where $1 > \alpha > 1 - \frac{1}{2H}$ and $\gamma > H - \frac{1}{2}$.

In the special case that $\sigma(t, x) = 1$ and $f(t, x) = ax$, where $a$ is a real constant, the solution to (21) is the fractional analogue of the Ornstein-Uhlenbeck process and is called fractional Ornstein-Uhlenbeck process. This process has been studied by Cheridito et al. in [8] and has been used extensively in financial modeling (see for example [4], [5], [6] and [7]).

4. Change-detection for fractional stochastic differential equations

In this section we assume that the observed process $\{\xi_t\}$ is governed by the following fractional stochastic differential equation

$$\xi_t = B_t + \mathbb{1}_{\{t \geq \tau\}} \int_{\tau}^t f(s, \xi_s) \, ds, \quad t \geq 0, \quad (22)$$

where $f(t, x)$ is such that there exists a solution to this equation, as described in the previous section. Then, from Theorem 2.3 it is clear that the CUSUM test $S_c$ will be $J_M$-optimal if the log-likelihood ratio $\{u_t\}$ admits representation (8) and condition (10) is satisfied. However, since $\{\xi_t\}$ is not a semimartingale, we cannot apply Girsanov’s theorem and compute $\{u_t\}$. Thus, it is not straightforward to compute $\{u_t\}$ and verify whether this condition is satisfied or not.
In order to get around this problem, we work in a similar way as Norros et. al [24] (see also Molchan et. al. [19]) and consider the following integral transformation of \( \{ \xi_t \} \):

\[
\zeta_t := \int_0^t k_H(t,s) \, d\xi_s, \quad t \geq 0,
\]

where

\[
k_H(t,s) := c_H^{-1} \Phi^{-H} (t-s)^{\frac{1}{2}-H}, \quad c_H := 2H \frac{\Gamma \left( \frac{3}{2} \right)}{\Gamma \left( \frac{3}{2} - H \right) \Gamma \left( H + \frac{1}{2} \right)}.
\]

Then, setting \( \theta_t := f(t,\xi_t) \) and assuming that the sample paths of \( \{ \theta_t \} \) are sufficiently smooth (see Samko et al. [34]), from (22) and (23) we obtain:

\[
\zeta_t = \int_0^t k_H(t,s) \, dB_s + \int_0^t Q_s \, dm_s, \quad t \geq 0.
\]

In the following proposition, based on the ideas of Kleptsyn et al. [12, 13, 14], we describe the main properties of the transformed process \( \{ \zeta_t \} \) and express the log-likelihood ratio \( \{ u_t \} \) in terms of \( \{ \zeta_t \} \).

**Proposition 4.1.** Suppose that the observed process \( \{ \xi_t \} \) is given by (22) and consider the transformed process \( \{ \zeta_t \} \), defined in (23). Then:

(i) Under \( P_\infty \), \( \{ \zeta_t \} \) is a centered, Gaussian \( \{ \mathcal{F}_t \} \)-martingale with quadratic variation \( \{ m_t \} \), where

\[
m_t := \lambda_H^{-1} \beta^{-2H}, \quad \lambda_H := \frac{\Gamma \left( \frac{3}{2} - H \right)}{2H \Gamma(3-2H) \Gamma \left( H + \frac{1}{2} \right)}.
\]

Under \( P_0 \), \( \{ \zeta_t \} \) is an \( \{ \mathcal{F}_t \} \)-semimartingale with the following decomposition

\[
\zeta_t = \int_0^t k_H(t,s) \, dB_s + \int_0^t Q_s \, dm_s
\]
(ii) The filtrations generated by \( \{ \xi_t \} \) and \( \{ \zeta_t \} \) coincide, thus \( F_t = \sigma(\zeta_s : 0 \leq s \leq t) \). In particular, \( \{ \xi_t \} \) admits the following representation in terms of \( \{ \zeta_t \} \)

\[
\xi_t = \int_0^t K_H(t, s) \, d\zeta_s, \tag{29}
\]

where \( K_H(t, s) := -2H \frac{d}{ds} \int_s^t r^{H-1/2}(r-s)^{H-1/2} dr, 0 \leq s \leq t \) is a deterministic kernel.

(iii) The log-likelihood ratio process \( \{ u_t \} \) has the following form:

\[
u_t = \int_0^t Q_s \, d\zeta_s - \frac{1}{2} \int_0^t Q_s^2 \, dm_s, \quad 0 \leq t < \infty. \tag{30}
\]

**Proof.** Under \( P_\infty \), from (25) we obtain \( \zeta_t = \int_0^t k_H(t, s) \, dB_s \) for every \( t \geq 0 \). Thus, under \( P_\infty \), \( \{ \zeta_t \} \) is a Gaussian process, since it is a stochastic integral with a non-random kernel with respect to a Gaussian process. Moreover, it is a martingale, as it was shown by Molchan [19] (see also Norros et al. [24]). Based on this fact, it is clear from (28) that \( \{ \zeta_t \} \) is a semi-martingale under \( P_0 \).

The fact that the filtrations of \( \{ \xi_t \} \) and \( \{ \zeta_t \} \) coincide follows directly from definition (23) and representation (29). The latter was proved by Kleptsyna et al. in [12] and [13] and the proof is based on techniques of fractional calculus (see for example [34]).

The final claim of the proposition follows from the previous parts and an application of Girsanov’s theorem.

\[\square\]

We can now state a sufficient condition for the optimality of the CUSUM test \( S_c \) under (22).

**Proposition 4.2.** Suppose that \( \{ \xi_t \} \) is described by (22) and the following condition is satisfied:

\[
\int_0^\infty Q_s^2 \, dm_s = \infty, \quad P_0, P_\infty - a.s. \tag{31}
\]

Then, the CUSUM test \( S_c \) is \( J_M \)-optimal, in the sense that it solves (6) among stopping times \( T \) such that \( E_\infty \left[ \int_0^T Q_s^2 \, dm_s \right] < \infty \), \( E_0 \left[ \int_0^T Q_s^2 \, dm_s \right] < \infty \).
Proof. From (30), the log-likelihood ratio process $\{u_t\}$ is written in the form (8), i.e. $u_t = X_t - \frac{1}{2}A_t$, with $X_t = \int_0^t Q_s d\zeta_s$ and $A_t = \int_0^t Q_s^2 dm_s$. The proposition then follows from Theorem 2.3. □

Remark 4.3. The previous analysis can be repeated in a straightforward way if (22) is replaced by

$$\xi_t = \int_0^t \sigma_s dB_s + \text{1}_{\{t \geq \tau\}} \int_0^t \theta_s ds, \quad t \geq 0$$  \hspace{1cm} (32)

where $\{\theta_t\}$ is an $\{F_t\}$-adapted process and $\{\sigma_t\}$ a non-vanishing, deterministic function such that the paths of $\{\theta_t/\sigma_t\}$ are sufficiently smooth (see [34] for details on this point). Indeed, we can still apply the integral transformation (23) and obtain the process $\{\zeta_t\}$, which now admits the following decomposition

$$\zeta_t = \int_0^t k_H(t,s) \sigma_s dB_s + \text{1}_{\{t \geq \tau\}} \int_0^t Q_s dm_s,$$  \hspace{1cm} (33)

with

$$Q_t = \frac{d}{dm} \int_0^t k_H(t,s) \frac{\theta_s}{\sigma_s} ds, \quad t \geq 0.$$  \hspace{1cm} (34)

After these modifications, the remaining of the proof is straightforward.

We now consider two interesting families of stochastic processes that are included in (22).

Proposition 4.4. Suppose that $\{\xi_t\}$ is a fractional Brownian motion with Hurst parameter $H$ that adopts a polynomial drift after the change, i.e.

$$\xi_t = B_t + \text{1}_{\{t \geq \tau\}} \int_\tau^t s^\alpha ds, \quad t \geq 0,$$  \hspace{1cm} (35)

where $\alpha > H - 1$. Then, the cusum test $S_c$ is $J_M$-optimal.

Remark 4.5. A direct corollary of this proposition is that $S_c$ is $J_M$-optimal for detecting a linear drift ($\alpha = 0$) in a fractional Brownian motion for any value of the Hurst parameter $H$. 


Proof. Recalling the definition of \( \{Q_t\} \) in (26) and of \( k_H(t, s) \) in (24), we have:

\[
Q_t = \frac{d}{dm_t} \int_0^t s^\alpha k_H(t, s) \, ds = c_H^{-1} \frac{d}{dm_t} \int_0^t s^\frac{1}{2}-H+\alpha (t - s)^\frac{1}{2}-H \, ds.
\]

Then, from the definition of the Beta function we obtain:

\[
Q_t = d_{H, \alpha} t^\alpha, \quad \int_0^t Q_s^2 \, ds = v_{H, \alpha} t^{2-2H+2\alpha}, \tag{36}
\]

where

\[
d_{H, \alpha} := \frac{B\left(\frac{3}{2} - H + \alpha, \frac{3}{2} - H\right)}{4H^2 \Gamma(3 - 2H) \Gamma^2(H + 1/2)} \frac{1 - H + \alpha}{1 - H}, \tag{37}
\]

and

\[
v_{H, \alpha} := d_{H, \alpha}^2 \frac{1 - H}{1 - H - \alpha}. \tag{38}
\]

It is now clear that condition (31) is satisfied if and only if \( \alpha > H - 1 \), which finishes the proof. \( \square \)

**Proposition 4.6.** Suppose that the evolution of \( \{\xi_t\} \) is described by the following dynamics:

\[
\xi_t = B_t + \mathbb{1}_{\{t \geq \tau\}} \int_\tau^t b(\xi_s) \, ds, \quad t \geq 0, \tag{39}
\]

where \( b(x) = c_0 + c_1 x + (|x| \wedge 1)^\alpha \), with \( c_0, c_1 \in \mathbb{R}_+ \) and \( \alpha \in [0, 1) \). Then, the CUSUM test \( \mathcal{S}_c \) is \( \mathcal{J}_M \)-optimal.

Proof. In order to prove this proposition, it suffices to show that condition (31) holds, which can be done as in Tudor and Viens [38], Lemma 3. \( \square \)

**Remark 4.7.** A direct consequence of this proposition is that \( \mathcal{S}_c \) is \( \mathcal{J}_M \)-optimal when \( \{\xi_t\} \) becomes a fractional Ornstein-Uhlenbeck process after the change \( (c_0 = \alpha = 0) \).

**Remark 4.8.** If we follow the approach of Tudor and Viens in [38], it is actually possible to establish the \( \mathcal{J}_M \)-optimality of \( \mathcal{S}_c \) for the more general class of functions \( b(\cdot) \) for which the following conditions are satisfied:

(i) \( xb(x) \) has a constant sign for all \( x \geq 0 \) and a constant sign for all \( x \leq 0 \).
(ii) there exists a function $h(x)$ with $\lim_{x \to \pm\infty} h(x) = 0$ so that $\left| \frac{b(x) - c_0}{x} \right| = c_1 + h(x)$ as $x \to \pm\infty$, where the constants $c_0$, $c_1$ are as in Proposition 4.6.

5. A class of model-free Lorden-like criteria

For the detection of a linear drift in a standard Brownian motion, the cusum test $S_c$ is not only $J_M$-optimal, but also $J_L$-optimal, since in this case the optimization problems (11) and (6) are equivalent. Our goal in this section is to obtain an analogue of this result for the detection a polynomial drift in a fractional Brownian motion.

In order to do so, we introduce a family of Lorden-like, model-free criteria. In particular, for each $k > 0$ we define

$$J^k_L[T] := \sup_{\tau \geq 0} \text{esssup} E_\tau [(T^k - \tau^k)^+ | \mathcal{F}_\tau]$$

and consider the constrained optimization problem

$$\inf_T J^k_L[T], \quad \text{where } E_\infty[T^k] \geq \gamma. \quad (41)$$

Problem (41) is a modification of Lorden’s optimization problem (11) (note that the latter is recovered when $k = 1$). The difference in (41) is that detection delay and period of false alarms are measured, not in terms of the actual time, but in terms of an expanded or contracted time, depending on whether $k > 1$ or $k < 1$, respectively.

**Proposition 5.1.** Suppose that the observed process $\{\xi_t\}$ is a fractional Brownian motion with Hurst parameter $H$ that adopts a polynomial drift after the change, i.e.

$$\xi_t = B_t + 1_{\{t \geq \tau\}} \int_\tau^t s^\alpha \, ds, \quad t \geq 0. \quad (42)$$

If $\alpha > H - 1$, then the cusum test $S_c$ is $J^k_L$-optimal, i.e. solves (41), for $k = 2 + 2\alpha - 2H$ and the corresponding performance characteristics are

$$J^k_L[S_c] = \frac{2}{v_{H,\alpha}} (e^{-c} + c - 1), \quad E_\infty[S^k_c] = \frac{2}{v_{H,\alpha}} (e^c - c - 1) \quad (43)$$

where $v_{H,\alpha}$ is defined in (68).
Proof. We start by setting $k = 2 + 2\alpha - 2H$ and recalling expressions (30) and (36) for the processes $\{u_t\}$ and $\{Q_t\}$, respectively. Then, for an arbitrary $\{\mathcal{F}_t\}$-stopping time $T$ such that $\mathbb{E}_0[T^k], \mathbb{E}_\infty[T^k] < \infty$ we have

$$- \mathbb{E}_\infty[u_T] = \frac{d_{H,\alpha}}{2} \mathbb{E}_\infty\left[ \int_0^T s^{2\alpha} \, dm_s \right] = \frac{v_{H,\alpha}}{2} \mathbb{E}_\infty[T^k]$$

and for any $t \in [0, \infty)$ we obtain $\mathbb{P}_t$-a.s. on $\{T > t\}$ that

$$\mathbb{E}_t[u_T - u_t | \mathcal{F}_t] = \frac{d_{H,\alpha}}{2} \mathbb{E}_t\left[ \int_t^T s^{2\alpha} \, dm_s | \mathcal{F}_t \right] = \frac{v_{H,\alpha}}{2} \mathbb{E}_t[T^k - t^k | \mathcal{F}_t].$$

Thus, $J_M[T]$ is proportional to $J^k_L[T]$ and Theorem 2.3 implies that $S_c$ solves (41) among such stopping times $T$. However, these integrability conditions can be removed, since they are needed only in order to establish the optimality of $S_c$ with respect to $J_M$.

Finally, if we set $T = S_c$ in (44) and (45), we have:

$$J_M[S_c] = \frac{v_{H,\alpha}}{2} \mathbb{E}_0[S_c^k] = \frac{v_{H,\alpha}}{2} J^k_L[S_c], \quad \mathbb{E}_\infty[-u_{S_c}] = \frac{v_{H,\alpha}}{2} \mathbb{E}_\infty[S_c^k].$$

(46)

Then, (43) follows directly from (41). \qed

We finish this section with two corollaries that generalize in two different directions the $J^k_L$-optimality of $S_c$ for the detection of a linear drift in a standard Brownian motion.

**Corollary 5.2.** Suppose that $\{\xi_t\}$ is a fractional Brownian motion which adopts a linear drift after the change, i.e. we have $\alpha = 0$ in (42). Then, $S_c$ is $J^k_L$-optimal with $k = 2(1 - H)$.

**Corollary 5.3.** Suppose that $\{\xi_t\}$ is a standard Brownian motion which adopts a polynomial drift after the change, i.e. we have $H = 1/2$ in (42). Then, $S_c$ is $J^k_L$-optimal with $k = 2(1 + \alpha)$ as long as $\alpha > -1$.

6. THE CASE OF DISCRETE-TIME OBSERVATIONS

In this section we assume that the process $\{\xi_t\}$ is not observed in continuous-time, but only at a sequence of discrete times $\{s_n\}$. Thus, the available information at time $s_n$ is given by the $\sigma$-algebra $\mathcal{G}_n := \sigma(\xi_{s_1}, \ldots, \xi_{s_n})$, instead of $\mathcal{F}_{s_n} = \sigma(\xi_u : 0 \leq u \leq s_n)$. For simplicity, we also assume that the observation times are equidistant with step $h$, i.e. $s_n = nh$. 
We assume in particular that the underlying continuous-time process is as follows: \( \xi_t = B_t + \mu (t - \tau)^+ \), where \( \mu \) is a known real number and \( \{B_t\} \) a fractional Brownian motion with Hurst index \( H \). When \( H = 1/2 \), we have \( u_{s_n} = \mu \xi_{s_n} - \frac{\mu^2}{2} s_n \) and a straightforward discretization of the continuous-time CUSUM \( S_c \) leads to the following detection rule:

\[
\hat{S}_c := \inf \{ s_n : u_{s_n} - \inf_{k=0,\ldots,n} u_{s_k} \geq c \}.
\]

(47)

On the other hand, when \( H \neq 1/2 \), recalling (26) and (30) we obtain

\[
u_{s_n} = \mu d_H \zeta_{s_n} - \frac{\mu^2}{2} (d_H s_n^{1-H})^2, \quad \zeta_{s_n} = \int_0^{s_n} k_H(s_u, u) d\xi_u,
\]

(48)

where \( d_H \) is a constant obtained by setting \( \alpha = 0 \) in (37). Thus, in order to implement the rule (47), we need to additionally approximate the integral in (48), which depends on the (unavailable) continuous path of the process \( \xi \).

In order to do so, we approximate the process \( \zeta \) (and consequently \( u \)) at the sequence of time instants \( \{t_n\} \) with \( t_n := s_{nd} = ndh \) for any \( n \in \mathbb{N} \), where \( d \) is a (fixed) integer. In other words, the period with which we update our detection rule is \( dh \). We then propose the following discretization of the continuous-time CUSUM

\[
\hat{S}_c := \inf \{ t_n : \tilde{u}_{t_n} - \inf_{0 \leq k \leq n} \tilde{u}_{t_k} \geq c \},
\]

(49)

where

\[
\tilde{u}_{t_n} := \mu d_H \tilde{\zeta}_{t_n} - \frac{\mu^2}{2} (d_H t_n^{1-H})^2, \quad \tilde{\zeta}_{t_n} := \sum_{j=1}^{nd} k_H(t_n, s_j) (\xi_{s_{j+1}} - \xi_{s_j}).
\]

(50)

We illustrate this method in Figs. 1, 2 and 3. Fig. 1 shows the observed path \( \{\xi_{s_n}\} \), which is a fractional Brownian motion with Hurst index \( H = 0.65 \) that adopts a drift \( \mu = 0.5 \) at the (unknown to the statistician) time \( \tau = 10 \). In Fig. 2 we plot the discretized transformed process \( \{\tilde{\zeta}_{t_n}\} \) which is computed using (50) with \( d = 40 \). In Fig. 3 we plot the discretized log-likelihood ratio \( \{\tilde{u}_{t_n}\} \) and the corresponding CUSUM process \( \{\tilde{y}_{t_n} := \tilde{u}_{t_n} - \inf_{0 \leq k \leq n} \tilde{u}_{t_k}\} \) up to the stopping rule \( \hat{S}_c \), where \( c = 1.5 \).
Remark 6.1. (i) It is clear that the smaller the sampling period $h$, the smaller the performance loss due to the discrete nature of the observations. On the other hand, for a given sampling period $h$, there is a trade-off in the choice of the period $d$ with which the proposed detection rule is updated. Indeed, a large $d$ leads to a better approximation of the integral $\zeta_{t_n}$ by the Riemann sum $\tilde{\zeta}_{t_n}$, but at the same time deteriorates the resolution of the detection rule, thus leading to a larger detection delay.

(ii) We can work similarly for the discretization of the continuous-time CUSUM test $S_c$ when $\{\xi_t\}$ is described by a fractional stochastic differential equation. In this case, there is an additional discretization bias, because of the need to approximate the process $Q$, defined in \[26]. Of course, the discretization of $S_c$ is also challenging when the underlying process is a diffusion under both $P_0$ and $P_\infty$. Our goal
in this section was to emphasize the difficulties that arise in the implementation of $S_c$ in practice when trying to detect of a linear drift in a fractional – in comparison to a standard – Brownian motion.

7. Conclusions

In this work, we established the optimality of the CUSUM test –with respect to the criterion suggested by Moustakides [21] – for the detection of abrupt and persistent changes in arbitrary continuous-path processes. Moreover, we proved that the CUSUM test is optimal –in this sense– when the underlying process is governed by a fractional stochastic differential equation. For the problem of detecting a polynomial drift in a fractional Brownian motion, we also showed that CUSUM test optimizes certain Lorden-like, model-free criteria; in that way, we generalized its optimality – for the detection of a linear drift in a standard Brownian motion– with respect to the
original Lorden criterion. Finally, we discussed issues regarding the practical implementation of the continuous-time CUSUM test when only discrete observations are available.

References


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