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Asymptotic bounds on the number of the eigenvalues in the gaps of the 2D magnetic Schrödinger operator

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1 Introduction

1.1. For $u \in C_0^\infty(\mathbf{R}^2)$ introduce the quadratic form

$$\chi_g[u] := \int_{\mathbf{R}^2} \left\{ |i\nabla u + Au|^2 - gV|u|^2 \right\} \, d\mathbf{x}$$

where $A \in L^2_{\rm loc}(\mathbf{R}^2; \mathbf{R}^2)$ is the vector (magnetic) potential, $-V : \mathbf{R}^2 \to \mathbf{R}$ is the scalar (electric) potential, and $g \geq 0$ is the coupling constant. We suppose that the function V is non-negative, and the multiplier by it is $-\Delta$-form-compact. Then the quadratic form χ_g is lower-bounded and closable in $L^2(\mathbf{R}^2)$ (see [Av.Her.Sim, Sect. 2]). Denote by $H_g \equiv H_g(A, V)$ the unique selfadjoint operator generated by the closure of χ_g.
Note that since the multiplier by V is $-\Delta$-form-compact, it is also H_0-form-compact (see [Av.Her.Sim, Sect. 2]). Hence, the essential spectrum of H_g is independent of g:

$$\sigma_{\rm ess}(H_g) = \sigma_{\rm ess}(H_0), \forall g \geq 0. \tag{1.1}$$

Assume that the real point λ belongs to the resolvent set $\varrho(H_0)$ of the operator H_0. Denote by $\mathcal{N}_g(\lambda)$ the number of the eigenvalues of the operator $H_{g'}$ which cross the point λ as the parameter g' grows from zero to the positive value $g > 0$. More precisely, we set

$$\mathcal{N}_g(\lambda) := \sum_{0<g'<g} \dim \, {\rm Ker} \, (H_{g'} - \lambda).$$

Further, assume that the interval $[\lambda, \mu]$, $\lambda < \mu$, belongs to $\varrho(H_0)$, and denote by $\mathcal{N}_g(\lambda, \mu)$ the number of the eigenvalues of the operator H_g lying on the interval $[\lambda, \mu)$. Then we have

$$\mathcal{N}_g(\lambda, \mu) = \mathcal{N}_g(\mu) - \mathcal{N}_g(\lambda).$$

In the present paper we consider electric potentials V which decay rapidly at infinity (for example, potentials satisfying the estimate $V(\mathbf{x}) \leq C|\mathbf{x}|^{-\alpha}$ with $\alpha > 2$ and $C > 0$, for $|\mathbf{x}|$ large enough).

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1.2. The first result we obtain (see below Theorem 2.1) concerns the asymptotics of $\mathcal{N}_g(\lambda)$ as $g \to \infty$, the number $\lambda \in \varrho(H_0)$ being fixed. We find that under some additional assumptions about A the main asymptotic term of $\mathcal{N}_g(\lambda)$ is Weylian, i.e. we have

$$\lim_{g \to \infty} g^{-1} \mathcal{N}_g(\lambda) = \frac{1}{4\pi} \int_{\mathbf{R}^2} V(\mathbf{x})\, d\mathbf{x}. \tag{1.2}$$

Asymptotic relations similar to (1.2) can be found in [Rai 1] and [Bir.Rai]. However, in these works only the case of dimensions higher or equal to three has been studied. The two-dimensional case is, in a way, more difficult, since the Rozenblum-Lieb-Cwikel estimate and, in particular, its "magnetic" version (see e.g [Sim, Sect. 15]) does not hold in this case. In Lemma 4.1 below we obtain an estimate which partly substitutes the "magnetic" Lieb estimate in the two-dimensional case.

1.3. Note that the right-hand side of (1.2) is independent of λ. Hence, this formula entails just the rough estimate

$$\mathcal{N}_g(\lambda, \mu) = o(g), \; g \to \infty, \; [\lambda, \mu] \subset \varrho(H_0), \tag{1.3}$$

but does not provide any precise information about the asymptotic behaviour of $\mathcal{N}_g(\lambda, \mu)$: the estimate (1.3) does not even imply that $\mathcal{N}_g(\lambda, \mu)$ grows unboundedly as $g \to \infty$.

Our second result (see below Theorem 2.2) concerns an asymptotic lower bound of $\mathcal{N}_g(\lambda, \mu)$ as $g \to \infty$ in the case where the magnetic field

$$b := \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$$

is constant and positive. In this case the spectrum of H_0 is purely essential and coincides with a sequence of eigenvalues of infinite multiplicity. More precisely, we have

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = \bigcup_{q=0}^{\infty} \{\Lambda_q\}, \tag{1.4}$$

where the numbers

$$\Lambda_q := b(2q+1), \; q \in \mathbf{N} := \{0, 1, 2, \ldots\},$$

are known as Landau levels (see e.g. [Av.Her.Sim]). Under some additional assumptions about A and V we get the estimate

$$\liminf_{g \to \infty} g^{-2/\alpha} \mathcal{N}_g(\lambda, \mu) \geq \mathcal{C}, \; [\lambda, \mu] \subset \varrho(H_0), \tag{1.5}$$

where the quantity $\mathcal{C}$ is strictly positive. In particular, the estimate (1.5) implies that $\mathcal{N}_g(\lambda, \mu)$ grows unboundedly as $g \to \infty$.

2 Formulation of the main results

2.1. For $\mathbf{m} \in \mathbf{Z}^2$ set $\mathcal{Q}_{\mathbf{m}} := (0,1)^2 + \mathbf{m}$. Define the space $\mathcal{L}_p$, $p > 1$, as the space of (classes of) functions f defined over $\mathbf{R}^2$ for which the norm

$$\|f\|_{\mathcal{L}_p} := \sum_{\mathbf{m} \in \mathbf{Z}^2} \|f\|_{L^p(\mathcal{Q}_{\mathbf{m}})}$$

is finite (see e.g. [Bir.Sol 2]).
We shall say that the electromagnetic potential (A, V) belongs to the class $\mathcal{K}_1$ if and only if the following conditions are satisfied:
(i) The magnetic potential can be written as $A = A^{(1)} + A^{(2)}$ where $A^{(1)} \in C^1_{\rm loc}(\mathbf{R}^2; \mathbf{R}^2)$ satisfies the estimates

$$\sup_{\mathbf{x} \in \mathbf{R}^2} |\partial A_k^{(1)}(\mathbf{x})/\partial x_j| \leq C, \; j = 1,2, \; k = 1,2, \tag{2.1}$$

and $|A^{(2)}|^2 \in \mathcal{L}_q$ for some $q > 1$;
(ii) $V \in \mathcal{L}_p$ for some $p > 1$, and $V \geq 0$.

Remark. Note that if $(A, V) \in \mathcal{K}_1$, then the multiplier by V is $-\Delta$-form-compact, and the relation (1.1) is valid.

Theorem 2.1 *Suppose that $(A, V) \in \mathcal{K}_1$. Assume $\lambda \in \varrho(H_0)$. Then the asymptotic relation (1.2) holds.*

Remark. We do not seek for the most general class of magnetic potentials A for which Theorem 2.1 is valid. Our aim is to consider a class of magnetic potentials which admits a comparatively brief description and a simple proof of Theorem 2.1, and at the same time contains the important from physics point of view linear potentials A.

2.2. We shall say that a function $f \in C^\infty(\mathbf{R}^2)$ belongs to the class $\mathcal{D}_\alpha$, $\alpha > 0$ if and only if the estimates

$$\left| \frac{\partial^{\beta_1+\beta_2} f(\mathbf{x})}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} \right| \leq C_\beta \langle \mathbf{x} \rangle^{-\alpha-\beta_1-\beta_2}, \langle \mathbf{x} \rangle := (1 + |\mathbf{x}|^2)^{1/2},$$

hold for each $\mathbf{x} \in \mathbf{R}^2$ and each multiindex $\beta = (\beta_1, \beta_2) \in \mathbf{N}^2$;
We shall say the electromagnetic potential (A, V) belongs to the class $\mathcal{K}_{2,\alpha}$, $\alpha > 0$, if and only if the following conditions are satisfied:
(i) The magnetic field b is constant;
(ii) The electric potential V belongs to the class $\mathcal{D}_\alpha$, and $V \geq 0$;
(iii) There exists a positive function $v \in C(\mathbf{S}^1)$ such that we have

$$\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}|^\alpha V(\mathbf{x}) = v(\hat{\mathbf{x}}), \; \hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}|.$$

Remark. If the magnetic field b is constant, we assume without any loss of generality $A = (-\frac{bx_2}{2}, \frac{bx_1}{2})$. Hence, the class $\mathcal{K}_{2,\alpha}$ with $\alpha > 2$ is contained in the class $\mathcal{K}_1$.

It is convenient to remind here the results obtained in [Rai 1] and [Rai 2] (see also [Rai 3]) concerning the asymptotics of $\mathcal{N}_g(\lambda)$ as $g \to \infty$ in the case where $(A, V) \in \mathcal{K}_{2,\alpha}$, $\alpha \in (0, 2]$, and $\lambda \in \varrho(H_0)$, i.e. either $\lambda < \Lambda_0$, or $\Lambda_q < \lambda < \Lambda_{q+1}$ for some $q \in \mathbf{N}$ (see (1.4)). Namely, the asymptotic relations

$$\lim_{g\to\infty} g^{-2/\alpha}\mathcal{N}_g(\lambda) = \frac{b}{4\pi} \sum_{q:\Lambda_q>\lambda} (\Lambda_q - \lambda)^{-2/\alpha} \int_{\mathbf{S}^1} v(\omega)^{2/\alpha}\, dS(\omega),\ \alpha \in (0, 2), \tag{2.2}$$

$$\lim_{g\to\infty} g^{-1}(\ln g)^{-1}\mathcal{N}_g(\lambda) = \frac{b}{8\pi} \int_{\mathbf{S}^1} v(\omega)\, dS(\omega),\ \alpha = 2, \tag{2.3}$$

are valid. Note that the series $\sum_{q:\Lambda_q>\lambda}(\Lambda_q - \lambda)^{-2/\alpha}$ occurring at the right-hand side of (2.2) converges if and only if $\alpha \in (0, 2)$.
If $[\lambda, \mu] \in \varrho(H_0)$, the asymptotics (2.2) entails

$$\lim_{g\to\infty} g^{-2/\alpha}\mathcal{N}_g(\lambda, \mu) = \frac{b}{2\alpha\pi} \int_\lambda^\mu \left\{ \sum_{q:\Lambda_q>\mu} (\Lambda_q - t)^{-1-2/\alpha} \right\} dt \int_{\mathbf{S}^1} v(\omega)^{2/\alpha}\, dS(\omega), \alpha \in (0, 2), \tag{2.4}$$

while the asymptotics (2.3) implies just the rough estimate

$$\mathcal{N}_g(\lambda, \mu) = o(g \ln g),\ g \to \infty.$$

Note that the series $\sum_{q:\Lambda_q>\mu}(\Lambda_q - t)^{-1-2/\alpha}$ occurring at the right-hand side of (2.4) is uniformly convergent with respect to $t \in [\lambda, \mu]$ for every $\alpha > 0$, and not only for $\alpha \in (0, 2)$.

Theorem 2.2 *Suppose that* $(A, V) \in \mathcal{K}_{2,\alpha}$*,* $\alpha \geq 2$*. Assume* $[\lambda, \mu] \subset \varrho(H_0)$*. Then we have*

$$\liminf_{g\to\infty} g^{-2/\alpha}\mathcal{N}_g(\lambda, \mu) \geq \frac{b}{2\alpha\pi} \int_\lambda^\mu \left\{ \sum_{q:\Lambda_q>\mu} (\Lambda_q - t)^{-1-2/\alpha} \right\} dt \int_{\mathbf{S}^1} v(\omega)^{2/\alpha}\, dS(\omega). \tag{2.5}$$

Remark. The quantity at the right-hand side of (2.5) (coinciding with $\mathcal{C}$ in (1.5)) is strictly positive. Hence, the estimate (2.5) implies that $\mathcal{N}_g(\lambda, \mu)$ grows unboundedly as $g \to \infty$ under the assumptions of Theorem 2.2.

Our conjecture is that the asymptotic formula (2.4) is valid for all $\alpha > 0$ and not only for $\alpha \in (0, 2)$. The lower bound (2.5) is the first step of the proof of this conjecture. Unfortunately, at present we are not able to prove the corresponding upper bound.
Asymptotics of the type of (2.4) for $\alpha > 0$ has been obtained in [Sob] for the operator

$$h_g u := -\frac{d^2 u}{dx^2} + pu + gqu$$

acting in $L^2(\mathbf{R})$. Here p is a periodic potential, q is a perturbation decaying at infinity, and g is the coupling constant. In this case however some methods from the theory of the ordinary differential equations have been used. Evidently, those methods are not applicable in the case considered in the present paper.

3 Notations and preliminaries

3.1. Let T be a linear selfadjoint operator in a Hilbert space $\mathbf{H}$. Then $\sigma(T)$ denotes the spectrum of T, and $\sigma_{\rm ess}(T)$ denotes its essential spectrum. Let $\mathcal{P}_{\mathcal{I}}(T)$ be the spectral projection of T corresponding to the interval $\mathcal{I} \subset \mathbf{R}$. Set

$$N(\lambda, \mu; T) = \text{rank } \mathcal{P}_{(\lambda,\mu)}(T), \ \lambda, \mu \in \mathbf{R}, \ \lambda < \mu,$$

$$N(\lambda; T) = \text{rank } \mathcal{P}_{(-\infty,\lambda)}(T), \ \lambda \in \mathbf{R}.$$

Assume that T is a compact linear operator acting in $\mathbf{H}$. For $s > 0$ set

$$\nu(s; T) = \text{rank } \mathcal{P}_{(s^2,\infty)}(T^*T).$$

If, in addition $T = T^*$, put

$$n_\pm(s; T) = \text{rank } \mathcal{P}_{(s,\infty)}(\pm T), \ s > 0.$$

3.2. Set $W = V^{1/2}$. Here we remind an important representation of the quantity $\mathcal{N}_g(\lambda)$, $\lambda \in \varrho(H_0)$.

Lemma 3.1 ([Bir, Proposition 1.6]) *Let $\lambda \in \varrho(H_0)$. Then under the hypotheses of Theorem 2.1 or Theorem 2.2 we have*

$$\mathcal{N}_g(\lambda) = n_+(g^{-1}; W(H_0 - \lambda)^{-1}W), \ g > 0. \tag{3.1}$$

4 Proof of Theorem 2.1

4.1. Let $(A, V) \in \mathcal{K}_1$. Introduce the quadratic-forms ratio

$$\int_{\mathbf{R}^2} V|u|^2 \, d\mathbf{x}/a[u] \tag{4.1}$$

where

$$a[u] := \int_{\mathbf{R}^2} \left\{|i\nabla u + Au|^2 + |u|^2\right\} \, d\mathbf{x},$$

and u belongs to the domain $D[a]$ of the closure of the quadratic form a defined originally on $C_0^\infty(\mathbf{R}^2)$. Denote by $T = T(V)$ the operator generated by the quadratic-forms ratio (4.1).

Lemma 4.1 *Let $(A,V)\in\mathcal{K}_1$. Then the estimate*

$$n_+(s;T(V))\le Cs^{-1}\|V\|_{\mathcal{L}_p} \tag{4.2}$$

holds for every $s>0$ with a constant C which depends on A and p but is independent of V and s.

Proof. Denote by $T^{(1)}=T^{(1)}(V)$ the operator generated by the quadratic-forms ratio

$$\int_{\mathbf{R}^2} V|u|^2\,d\mathbf{x}/a^{(1)}[u],\, u\in D[a],$$

where

$$a^{(1)}[u]:=\int_{\mathbf{R}^2}\left\{|i\nabla u+A^{(1)}u|^2+|u|^2\right\}\,d\mathbf{x}.$$

Note that the quadratic form $a[u]-a^{(1)}[u]$ is compact in $D[a]$. Moreover, for each $u\in D[a]$, $u\neq 0$, we have $a[u]>0$. Hence, there exists a positive constant c such that the estimate

$$a[u]\ge ca^{(1)}[u],\forall u\in D[a].$$

is valid. Therefore, we get

$$n_+(s;T)\le n_+(cs;T^{(1)}),\forall s>0. \tag{4.3}$$

Next, fix $\mathbf{m}\in\mathbf{Z}^2$ and denote by $T_{\mathbf{m}}\equiv T_{\mathbf{m}}(V)$ the selfadjoint compact operator generated by the quadratic-forms ratio

$$\frac{\int_{\mathcal{Q}_{\mathbf{m}}} V|u|^2\,d\mathbf{x}}{\int_{\mathcal{Q}_{\mathbf{m}}}\{|i\nabla u+A^{(1)}u|^2+|u|^2\}\,d\mathbf{x}},\, u\in H^1(\mathcal{Q}_{\mathbf{m}}).$$

Then the minimax principle enatails

$$n_+(s;T^{(1)})\le\sum_{\mathbf{m}\in\mathbf{Z}^2} n(s;T_{\mathbf{m}}),\forall s>0. \tag{4.4}$$

Fix $\mathbf{m}\in\mathbf{Z}^2$ and denote by $T_{\mathbf{m}}^{(1)}\equiv T_{\mathbf{m}}^{(1)}(V)$ the operator generated by the quadratic-forms ratio

$$\frac{\int_{\mathcal{Q}_0} V(\mathbf{x}+\mathbf{m})|w(\mathbf{x})|^2\,d\mathbf{x}}{\int_{\mathcal{Q}_0}\{|i\nabla w(\mathbf{x})+(A^{(1)}(\mathbf{x}+\mathbf{m})-A^{(1)}(\mathbf{m}))\,w(\mathbf{x})|^2+|w(\mathbf{x})|^2\}\,d\mathbf{x}},\, w\in H^1(\mathcal{Q}_0).$$

Note that we have

$$T_{\mathbf{m}}=U_{\mathbf{m}}^*T_{\mathbf{m}}^{(1)}U_{\mathbf{m}}$$

where

$$(U_{\mathbf{m}}u)(\mathbf{x})=\exp\left(-iA^{(1)}(\mathbf{m}).\mathbf{x}\right)u(\mathbf{x}+\mathbf{m}),\mathbf{x}\in\mathcal{Q}_{\mathbf{0}},\mathbf{m}\in\mathbf{Z}^2.$$

Thus we obtain

$$n_+(s;T_{\mathbf{m}})=n_+(s;T_{\mathbf{m}}^{(1)}),\forall s>0,\forall\mathbf{m}\in\mathbf{Z}^2. \tag{4.5}$$

Taking into account the estimate (2.1) combined with the Lagrange formula

$$A_j^{(1)}(\mathbf{x}+\mathbf{m}) - A_j^{(1)}(\mathbf{m}) = \int_0^1 \mathbf{x}.\nabla A_j^{(1)}(\mathbf{m}+\tau\mathbf{x})\, d\tau,\ j=1,2,$$

we find that the quantity

$$\sup_{\mathbf{m}\in\mathbf{Z}^2}\sup_{\mathbf{x}\in\mathcal{Q}_0} |A^{(1)}(\mathbf{x}+\mathbf{m}) - A^{(1)}(\mathbf{m})|$$

is bounded. Hence, there exists a constant c_1 independent of $\mathbf{m}\in\mathbf{Z}^2$ such that we have

$$\int_{\mathcal{Q}_0} \left\{|i\nabla w(\mathbf{x}) + \left(A^{(1)}(\mathbf{x}+\mathbf{m}) - A^{(1)}(\mathbf{m})\right) w(\mathbf{x})|^2 + |w(\mathbf{x})|^2\right\}\, d\mathbf{x} \geq$$

$$c_1 \int_{\mathcal{Q}_0} \left\{|\nabla w(\mathbf{x})|^2 + |w(\mathbf{x})|^2\right\}\, d\mathbf{x}, \forall w \in H^1(\mathcal{Q}_\mathbf{0}). \tag{4.6}$$

Fix $\mathbf{m}\in\mathbf{Z}^2$ and denote by $T_\mathbf{m}^{(2)} \equiv T_\mathbf{m}^{(2)}(V)$ the operator generated by the quadratic-forms ratio

$$\frac{\int_{\mathcal{Q}_0} V(\mathbf{x}+\mathbf{m})|w(\mathbf{x})|^2\, d\mathbf{x}}{\int_{\mathcal{Q}_0}\{|\nabla w(\mathbf{x})|^2 + |w(\mathbf{x})|^2\}\, d\mathbf{x}}, w\in H^1(\mathcal{Q}_\mathbf{0}).$$

Then the estimate (4.6) entails

$$n_+(s; T_\mathbf{m}^{(1)}) \leq n_+(c_1 s; T_\mathbf{m}^{(2)}), \forall s>0, \forall \mathbf{m}\in\mathbf{Z}^2. \tag{4.7}$$

By [Bir.Sol 1, Theorem 4.12] we have

$$n_+(s; T_\mathbf{m}^{(2)}) \leq c_p s^{-1}\|V(.+\mathbf{m})\|_{L^p(\mathcal{Q}_0)}, \forall s>0, \forall \mathbf{m}\in\mathbf{Z}^2, \tag{4.8}$$

where the constant c_p is independent of V and $\mathbf{m}$. Putting together the estimates (4.3), (4.4), (4.5), (4.7) and (4.8), and taking into account the obvious identity

$$\|V(.+\mathbf{m})\|_{L^p(\mathcal{Q}_0)} = \|V\|_{L^p(\mathcal{Q}_\mathbf{m})}, \forall \mathbf{m}\in\mathbf{Z}^2,$$

we get (4.2).

4.2. The following lemma implies that it suffices to prove (1.2) for $\lambda = -1$.

Lemma 4.2 *Under the hypotheses of Theorem 2.1 we have*

$$\liminf_{g\to\infty} g^{-1}\mathcal{N}_g(\lambda) = \liminf_{g\to\infty} g^{-1}\mathcal{N}_g(-1), \tag{4.9}$$

$$\limsup_{g\to\infty} g^{-1}\mathcal{N}_g(\lambda) = \limsup_{g\to\infty} g^{-1}\mathcal{N}_g(-1). \tag{4.10}$$

Remark. The advantage of the point $-1 \in \varrho(H_0)$ occurring at the right-hand side of (4.9)–(4.10) in comparison with the arbitrary point $\lambda = \overline{\lambda} \in \varrho(H_0)$, is that the point -1 is situated below the bottom Λ_0 of $\sigma(H_0)$.

Proof of Lemma 4.2. Writing the resolvent identity

$$W(H_0-\lambda)^{-1}W = W(H_0+1)^{-1}W + (1+\lambda)W(H_0-\lambda)^{-1}(H_0+1)^{-1}W,$$

and taking into account the equality (3.1) and the Weyl inequalities for the eigenvalues of compact operators, we find that it suffices to prove the estimate

$$\nu(s;(H_0+1)^{-1}W) = o(s^{-2}),\, s \downarrow 0, \tag{4.11}$$

in order to deduce (4.9)–(4.10). Note that we have

$$\nu(s;(H_0+1)^{-1}W) \le \nu(s;(H_0+1)^{-1/2}W), \forall s > 0,$$

$$\nu(s;(H_0+1)^{-1/2}W) = n_+(s^2; T(W^2)), \forall s > 0.$$

Since the class $C_0^\infty(\mathbf{R}^2)$ is evidently dense in the space of (classes) of functions f for which the norm $\left(\sum_{\mathbf{m}\in\mathbf{Z}^2} \|f\|^2_{L^{2p}(\mathcal{Q}_\mathbf{m})}\right)^{1/2}$ is finite, the estimate (4.2) implies that it suffices to prove (4.11) for $W \in C_0^\infty(\mathbf{R}^2)$ (cf. [Bir, Sect. 2]).
The proof of the estimate (4.11) for $W \in C_0^\infty(\mathbf{R}^2)$ is completely analogous to the demonstration of Lemma 2.7 in [Bir.Rai].

4.3. The following lemma is a special case of Theorem 2.1 for $\lambda = -1$.

Lemma 4.3 *Let $(A,V) \in \mathcal{K}_1$. Then we have*

$$\lim_{g\to\infty} g^{-1}\mathcal{N}_g(-1) = \frac{1}{4\pi}\int_{\mathbf{R}^2} V(\mathbf{x})\, d\mathbf{x}. \tag{4.12}$$

The proof of the lemma is quite the same as the one of Theorem 2.1 in [Bir.Rai]. However, here we apply again the estimate (4.2) instead of the "magnetic" Lieb estimate.

Now the asymptotic relation (1.2) follows immediately from (4.9), (4.10) and (4.12).

5 Proof of Theorem 2.2

5.1. Introduce the selfadjoint operator

$$h := b\left(-\frac{d^2}{dx^2} + x^2\right),\ x \in \mathbf{R},$$

defined originally on $C_0^\infty(\mathbf{R})$ and then closed in $L^2(\mathbf{R})$. The spectrum of h is purely discrete and consists of simple eigenvalues coinciding with the Landau levels Λ_q, $q \in \mathbf{N}$. The corresponding orthonormal eigenfunctions can be written in the form

$$f_q(x) := \mathcal{P}_q(x) \exp(-x^2/2), q \in \mathbf{N}, x \in \mathbf{R},$$

where $\mathcal{P}_q$, $q \in \mathbf{N}$, are polynomials with real coefficients (normalized Hermite polynomials).

Introduce the operator

$$\mathcal{H}_0 := \int_{\mathbf{R}}^{\oplus} h \; dy$$

which is selfadjoint in $L^2(\mathbf{R}^2_{x,y})$. Further, set

$$V_b(x, y) := V(b^{-1/2}x, b^{-1/2}y), (x, y) \in \mathbf{R}^2,$$

and define the the operator $\mathcal{V}$, selfadjoint and bounded in $L^2(\mathbf{R}^2_{x,y})$, as a pseudodifferential operator (ΨDO) with Weyl symbol

$$V_b(x - \eta, y - \xi), (x, y; \xi, \eta) \in T^*\mathbf{R}^2_{x,y},$$

(see [Shu, Chapter 4]). Thus, for $u \in L^2(\mathbf{R}^2_{x,y})$ we have

$$(\mathcal{V}u)(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^4} \exp\{i[\xi.(x - x') + \eta.(y - y')]\}$$

$$\times V_b \left(\frac{1}{2}(x + x') - \eta, \frac{1}{2}(y + y') - \xi \right) u(x', y') \; dx'dy'd\xi d\eta.$$

For $g \geq 0$ set

$$\mathcal{H}_g := \mathcal{H}_0 - g\mathcal{V}.$$

Lemma 5.1 ([Rai 2, Subsection 2.1]) *The operators H_g and $\mathcal{H}_g$ are unitarily equivalent.*

Since we have $(\lambda, \mu) \subset [\lambda, \mu)$, Lemma 5.1 entails the following elementary inequality

$$\mathcal{N}_g(\lambda, \mu) \geq N(\lambda, \mu; \mathcal{H}_g), \; [\lambda, \mu] \subset \varrho(H_0), g \geq 0. \tag{5.1}$$

5.2. For $\lambda < \mu$ such that $[\lambda, \mu] \subset \varrho(H_0)$ set

$$\gamma = \frac{\lambda + \mu}{2}, \; \tau = \frac{\mu - \lambda}{2}. \tag{5.2}$$

Then the spectral theorem for selfadjoint operators entails

$$N(\lambda, \mu; \mathcal{H}_g) = N(\tau^2; (\mathcal{H}_g - \gamma)^2). \tag{5.3}$$

Denote by m the number of the Landau levels smaller than μ (hence, amongst the Landau levels greater than μ, the level Λ_m is nearest to μ). Fix an arbitrary integer M greater than m. Define the orthogonal projection $P \equiv P_M$ by

$$(P_M u)(x,y) := \sum_{q=m}^{M} f_q(x) \int_{\mathbf{R}^2} u(x',y) f_q(x')\, dx', \;\; u \in L^2(\mathbf{R}^2_{x,y}).$$

Since M is finite, the set

$$\mathcal{L}_M := P_M L^2(\mathbf{R}^2_{x,y})$$

coincides with $P_M D(\mathcal{H}_0)$. Moreover, we have $\mathcal{L}_M \subset D(\mathcal{H}_0)$. Put

$$Q \equiv Q_M := \mathrm{Id} - P_M.$$

Denote by $\mathcal{F}_1 \equiv \mathcal{F}_{1,g}$ and $\mathcal{F}_2 \equiv \mathcal{F}_{2,g}$ respectively the restrictions of the operators $P(\mathcal{H}_g - \gamma)P(\mathcal{H}_g - \gamma)$ and $P(\mathcal{H}_g - \gamma)Q(\mathcal{H}_g - \gamma)$ onto the subspace $\mathcal{L}_M$. Obviously, the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ are selfadjoint and bounded in $\mathcal{L}_M$. Since the projection P commutes with the operator $\mathcal{H}_0 - \gamma$, we have

$$\mathcal{F}_2 = g^2 P\mathcal{V}Q\mathcal{V}_{|\mathcal{L}_M} = g^2 \left(P\mathcal{V}^2_{|\mathcal{L}_M} - P\mathcal{V}P\mathcal{V}_{|\mathcal{L}_M} \right). \tag{5.4}$$

The identity

$$P(\mathcal{H}_g - \gamma)^2_{|\mathcal{L}_M} = \mathcal{F}_1 + \mathcal{F}_2,$$

combined with the minimax principle, entails

$$N(\tau^2; (\mathcal{H}_g - \gamma)^2) \geq N(\tau^2; \mathcal{F}_1 + \mathcal{F}_2). \tag{5.5}$$

On the other hand, the minimax principle implies that the inequality

$$N(\tau^2; \mathcal{F}_1 + \mathcal{F}_2) \geq N(\tau^2 - \varepsilon; \mathcal{F}_1) - n_+(\varepsilon; \mathcal{F}_2) \tag{5.6}$$

holds for each $\varepsilon \in (0, \tau^2)$.

5.3. In this subsection we prove the estimate

$$\lim_{g \to \infty} g^{-2/\alpha} n_+(\varepsilon; \mathcal{F}_{2,g}) = 0, \forall \varepsilon > 0. \tag{5.7}$$

Introduce the operators $\mathcal{O}(V^j)$, $j = 1, 2$, selfadjoint and bounded in $\{L^2(\mathbf{R}_y)\}^{M-m+1}$, as ΨDOs with matrix-valued Weyl symbols

$$s^{(j)}(y,\eta) = \left\{ s^{(j)}_{pr}(y,\eta) \right\}_{p,r=m}^{M}, \;\; (y,\eta) \in T^*\mathbf{R}_y,$$

where

$$s^{(j)}_{pr}(y,\eta) := \frac{1}{2\pi} \int_{\mathbf{R}^3} V_b^j \left(\frac{x_1 + x_2}{2} - \eta, y - \xi \right) e^{i(x_1 - x_2)\cdot\xi} f_p(x_1) f_r(x_2)\, d\xi dx_1 dx_2,$$

$$p,r\in\{m,\ldots,M\},\ j=1,2.$$

Then, according to (5.4), the operator $\mathcal{F}_{2,g}$ is unitarily equivalent to the operator $g^2\left(\mathcal{O}(V^2)-\mathcal{O}(V)^2\right)$, and, therefore, we have

$$n_+(\varepsilon;\mathcal{F}_{2,g})=n_+(\varepsilon g^{-2};\mathcal{O}(V^2)-\mathcal{O}(V)^2),\ \varepsilon>0,\ g>0. \tag{5.8}$$

The functions $s_{pr}^{(j)}(y,\eta)-\delta_{pr}V_b^j(-\eta,y)$, $p,r\in\{m,\ldots,M\}$, belong to the class $\mathcal{D}_{j\alpha+1}$, $j=1,2$. Therefore, the entries $\tilde{s}_{pr}(y,\eta)$ of the symbol of the operator $\mathcal{O}(V^2)-\mathcal{O}(V)^2$ satisfy $\tilde{s}_{pr}\in\mathcal{D}_{2\alpha+1}$, $p,r\in\{m,\ldots,M\}$, (see [Shu, Theorem 23.6]). Hence, we have

$$n_+(s;\mathcal{O}(V^2)-\mathcal{O}(V)^2)=O(s^{-\frac{2}{2\alpha+1}})=o(s^{-\frac{1}{\alpha}}),\ s\downarrow 0. \tag{5.9}$$

The combination of (5.8) and (5.9) yields (5.7).

5.4. Set

$$\tilde{\mathcal{H}}_g:=P\mathcal{H}_{g|\mathcal{L}_M},$$

and

$$\lambda_1:=\gamma-\sqrt{\tau^2-\varepsilon},\ \mu_1:=\gamma+\sqrt{\tau^2-\varepsilon},\ \varepsilon\in(0,\tau^2),$$

the numbers γ and τ being defined in (5.2). Obviously, we have

$$N(\tau^2-\varepsilon;\mathcal{F}_{1,g})=N(\lambda_1,\mu_1;\tilde{\mathcal{H}}_g). \tag{5.10}$$

Moreover, the inequality

$$N(\lambda_1,\mu_1;\tilde{\mathcal{H}}_g)\geq N(\mu_1;\tilde{\mathcal{H}}_g)-N(\lambda_2;\tilde{\mathcal{H}}_g) \tag{5.11}$$

holds for each $\lambda_2\in(\lambda_1,\mu_1)$.

Let $u=\{u_q\}_{q=m}^M\in\{L^2(\mathbf{R})\}^{M-m+1}$. Define the operator $\chi(t):\{L^2(\mathbf{R})\}^{M-m+1}\to\{L^2(\mathbf{R})\}^{M-m+1}$ by

$$\left(\chi(t)u\right)_q:=(\Lambda_q-t)^{-1/2}u_q,\ q\in\{m,\ldots,M\},\ t\in[\lambda,\mu].$$

Set

$$\mathcal{T}(t):=\chi(t)\mathcal{O}(V)\chi(t),\ t\in[\lambda,\mu].$$

Then the Birman–Schwinger principle entails

$$N(t;\tilde{\mathcal{H}}_g)=n_+(g^{-1};\mathcal{T}(t)),\ t\in[\lambda,\mu]. \tag{5.12}$$

The operator $\mathcal{T}(t)$ can be written as the sum

$$\mathcal{T}(t)=\mathcal{T}^{(1)}(t)+\mathcal{T}^{(2)}(t),\ t\in[\lambda,\mu], \tag{5.13}$$

where $\mathcal{T}^{(1)}(t)$ is a ΨDO with matrix-valued symbol

$$\left\{\delta_{pr}(\Lambda_p-t)^{-1}V_b(-\eta,y)\right\}_{p,r=m}^M,\ (y,\eta)\in T^*\mathbf{R}_y,$$

while the entries of the symbol of the operator $\mathcal{T}^{(2)}(t)$ belong to the class $\mathcal{D}_{\alpha+1}$. Utilizing the general results on the spectral asymptotics for ΨDOs of negative order (see e.g. [Dau.Rob]), we get

$$n_+(s;\mathcal{T}^{(1)}(t)) =$$

$$\frac{1}{2\pi}\sum_{q=m}^{M} \text{vol } \{(y,\eta)\in T^*\mathbf{R}_y : V_b(-\eta,y) > s(\Lambda_q - t)\}\,(1+o(1)),\ s\downarrow 0,\ t\in[\lambda,\mu], \tag{5.14}$$

$$n_+(s;\mathcal{T}^{(2)}(t)) = O(s^{-\frac{2}{\alpha+1}}) = o(s^{-2/\alpha}),\ s\downarrow 0,\ t\in[\lambda,\mu]. \tag{5.15}$$

Note that for $t\in[\lambda,\mu]$ and $q\geq m$ we have

$$\frac{1}{2\pi}\text{vol } \{(y,\eta)\in T^*\mathbf{R}_y : V_b(-\eta,y) > s(\Lambda_q - t)\} =$$

$$s^{-2/\alpha}(\Lambda_q - t)^{-2/\alpha}\frac{b}{4\pi}\int_{\mathbf{S}^1} v(\omega)^{2/\alpha}\,dS(\omega)(1+o(1)),\ s\downarrow 0. \tag{5.16}$$

Putting together (5.1), (5.3), (5.5)–(5.7) and (5.10)-(5.16), and writing

$$(\Lambda_q-\mu_1)^{-2/\alpha} - (\Lambda_q-\lambda_2)^{-2/\alpha} = \frac{2}{\alpha}\int_{\lambda_2}^{\mu_1}(\Lambda_q - t)^{-1-2/\alpha}\,dt,\ q\in\{m,\ldots,M\},$$

we obtain the estimate

$$\liminf_{g\to\infty} g^{-2/\alpha}\mathcal{N}_g(\lambda,\mu)\geq$$

$$\frac{b}{2\alpha\pi}\sum_{q=m}^{M}\int_{\lambda_2}^{\mu_1}(\Lambda_q - t)^{-1-2/\alpha}\,dt\int_{\mathbf{S}^1} v(\omega)^{2/\alpha}\,dS(\omega),\ \forall M>m,\ \forall\mu_1\in(\lambda,\mu),\ \forall\lambda_2\in(\lambda,\mu_1). \tag{5.17}$$

Since the series

$$\sum_{q=m}^{\infty}(\Lambda_q - t)^{-1-2/\alpha} = \sum_{q:\Lambda_q>\mu}(\Lambda_q - t)^{-1-2/\alpha}$$

converges uniformly with respect to $t\in[\lambda,\mu]$, and the number $\varepsilon>0$ in (5.10) as well as the difference $\lambda_2-\lambda_1$ in (5.11) can be chosen arbitrarily small, we may let $M\to\infty$, $\lambda_2\downarrow\lambda$ and $\mu_1\uparrow\mu$ in (5.17), thus obtaining (2.5).

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