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Discrete Spectrum in the Gaps for Perturbations of the Magnetic Schrödinger Operator

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ABSTRACT. We consider the operator $B(\alpha) = A+F-\alpha V$ on $L_2(\mathbb{R}^d)$, where $A = (-i\nabla - \mathfrak{A})^2$, $\mathfrak{A}$ is the vector potential, F, V are electric potentials, $V(x) \geqslant 0$, $\alpha > 0$. We obtain a Weyl type asymptotic formula, as $\alpha \to \infty$, for the number of eigenvalues of $B(\alpha)$ that have crossed a fixed point in a gap of the spectrum of $A + F$.

This paper adjoins [B2], but in fact can be read independently. The main result is an example of the application of the abstract Theorem 1.2 in [B2]. For earlier results on the discrete spectrum asymptotics for the perturbed magnetic Schrödinger operator, see the authors' papers [B1, R].

§1. Formulation of the problem. Statement of the main result

1. Let $\mathfrak{H}$ be a Hilbert space, $\|\cdot\|$ and $(\cdot, \cdot)$ be the norm and the scalar product in $\mathfrak{H}$. If T is a linear operator on $\mathfrak{H}$, then $\mathfrak{D}(T)$ stands for its domain and $\rho(T)$ for its resolvent set. If T is compact and $s_k(T)$ are its singular numbers, then

$$\nu(s, T) := \operatorname{card}\{ k : s_k(T) > s \}, \qquad s > 0,$$

is the corresponding distribution function. If, moreover, T is selfadjoint, then $n_\pm(\cdot, T)$ are the distribution functions for its positive and negative eigenvalues; clearly $\nu = n_+ + n_-$. The class Σ_p, $p > 0$, is distinguished by the condition

$$|T|_p^p := \sup_{s>0} s^p \nu(s, T) < \infty.$$

We denote by Σ_p^0 the separable subclass of Σ_p:

$$\Sigma_p^0 = \{ T \in \Sigma_p : \nu(s, T) = o(s^{-p}), \ s \to 0 \}.$$

In the sequel the role of the principal Hilbert space $\mathfrak{H}$ will be played by $L_2(\mathbb{R}^d)$. *Throughout* $d \geqslant 3$. For $\Omega \subset \mathbb{R}^d$, by $H^1(\Omega)$ we denote the Sobolev

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class with the metric form $\int_\Omega(|\nabla u|^2 + |u|^2)\,dx$. The class $\dot{H}^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. Below

$$D := -i\nabla;$$
$$\omega_d := \operatorname{vol}\{x \in \mathbb{R}^d : |x| < 1\};$$

integration without indication of the domain extends over all of $\mathbb{R}^d$. By C, c (possibly with subscripts) we denote various constants in the estimates. For the operator of multiplication by a function, we use the same notation as for the function itself.

2. Let $\mathfrak{A}(X) = \{\mathfrak{A}_1, \dots, \mathfrak{A}_d\}$ be the magnetic potential (a real vector-valued function). Let

$$\mathfrak{A} \in L_{d,\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d). \tag{1.1}$$

The magnetic Schrödinger operator $A = A(\mathfrak{A})$ is introduced with the help of the sesquilinear form

$$a[u_1, u_2] = \int (D - \mathfrak{A})u_1 \overline{(D - \mathfrak{A})u_2}\,dx$$

whose domain $d[a]$ is given by

$$d[a] = \{u \in L_2(\mathbb{R}^d) : u \in H^1_{\mathrm{loc}}(\mathbb{R}^d),\ a[u, u] < \infty\}.$$

The set $d[a]$ is a complete Hilbert space under the metric form $a[u, u] + \|u\|^2$. The class $C_0^\infty(\mathbb{R}^d)$ is dense in $d[a]$. According to general theory,

$$a[u, v] = (A^{1/2}u, A^{1/2}v), \qquad d[a] = \mathfrak{D}(A^{1/2}).$$

The free particle energy operator $A(0)$ corresponds to the potential $\mathfrak{A} \equiv 0$. The associated quadratic form is

$$a(0)[u, u] = \int |\nabla u|^2\,dx, \qquad d[a(0)] = H^1(\mathbb{R}^d).$$

The operator $A(0)$ turns out to be extremal in many respects. In particular, the following estimate is true (cf. [AHS, Theorems 2.3, 2.5]).

PROPOSITION 1.1. *Let φ be multiplication by a measurable function. Then*

$$\|\varphi(A + \gamma I)^{-1/2}\| \leqslant \|\varphi(A(0) + \gamma I)^{-1/2}\|, \qquad \gamma > 0. \tag{1.2}$$

The following proposition can easily be derived from (1.2) (compare with Proposition 9.1 from [BS3]).

PROPOSITION 1.2. *If for some $\gamma \geqslant 0$*

$$\|\varphi u\|^2 \leqslant C\int(|\nabla u|^2 + \gamma|u|^2)\,dx, \qquad u \in C_0^\infty(\mathbb{R}^d),$$

then with the same C

$$\|\varphi u\|^2 \leqslant C(a[u, u] + \gamma\|u\|^2), \qquad u \in C_0^\infty(\mathbb{R}^d).$$

We emphasize that $\gamma = 0$ is not forbidden in Proposition 1.2. This proposition implies, in particular, the "magnetic" version of the Hardy inequality

$$\int \frac{|u|^2}{|x|^2}\,dx \leqslant \frac{4}{(d-2)^2} a[u, u]. \tag{1.3}$$

Now let Ω be a ball. Then

$$\int_\Omega (|\nabla u|^2 + |u|^2)\,dx \leqslant C(\mathfrak{A}, \Omega) a[u, u], \qquad u \in d[a]. \tag{1.4}$$

Indeed, under condition (1.1), the following inequality can easily be derived from the Sobolev embedding theorem of limit order:

$$\left| \int_\Omega |(\mathfrak{D} - \mathfrak{A})u|^2\,dx - \int_\Omega |\nabla u|^2\,dx \right| \\ \leqslant \varepsilon \int_\Omega |\nabla u|^2\,dx + c(\varepsilon) \int_\Omega |u|^2\,dx, \qquad \forall \varepsilon > 0.$$

This, together with (1.3), implies (1.4).

3. Now let us introduce the form

$$v[u, u] = \int V(x)|u|^2\,dx, \tag{1.5}$$

$$V = \overline{V} \in L_{d/2}(\mathbb{R}^d). \tag{1.6}$$

By $2V_\pm$ we denote the function $|V| \pm V$.

Let $n_\pm(\gamma, s)$ be the distribution function for the positive eigenvalues (i.e., the consecutive maxima) of the quotient of quadratic forms

$$\pm v[u, u]/(a[u, u] + \gamma\|u\|^2), \qquad \gamma \geqslant 0. \tag{1.7$_\pm$}$$

In other words, $n_\pm(\gamma, s) = n_\pm(s, T)$, where T is the operator generated by the form (1.5) on the space with the metric form $a[u, u] + \gamma\|u\|^2$. For $\gamma > 0$ the quotient (1.7) is considered for $u \in d[a]$. For $\gamma = 0$ one should consider (1.7) on the set $\tilde{d}[a]$, the completion of $C_0^\infty(\mathbb{R}^d)$ in the metric corresponding to the form $a[u, u]$. It should be emphasized that $\tilde{d}[a]$ is a function space by (1.3).

The following *Lieb estimate* is crucial for what follows.

PROPOSITION 1.3. *Under conditions* (1.1), (1.6), *the following estimate for* $n_\pm(\gamma, s)$ *holds*:

$$n_\pm(\gamma, s) \leqslant C(d) s^{-d/2} \int V_\pm^{d/2}\,dx. \tag{1.8$_\pm$}$$

The constant $C(d)$ *in* (1.8) *does not depend on* $\mathfrak{A}$ *and on* $\gamma \geqslant 0$.

We make some necessary explanations. Under condition (1.6) the form $a_\pm(\alpha) := a \mp \alpha v$ is bounded from below and closed on $d[a]$, whatever be $\alpha > 0$. Let $A_\pm(\alpha)$ be the operator generated by this form on $L_2(\mathbb{R}^d)$.

Denote by $N_+(\gamma, \mathfrak{A}, V; \alpha)$ the number of eigenvalues of $A_+(\alpha)$ lying to the left of the point $\lambda = -\gamma \leqslant 0$. Then

$$N_+(\gamma, \mathfrak{A}, V; \alpha) = n_+(\gamma, \alpha^{-1}), \tag{1.9}$$

and (1.8_+) turns into an estimate for N_+. If $\mathfrak{A} = 0$, this estimate becomes the well-known inequality of Rozenblyum, Lieb, and Cwikel (cf., e.g., [RS] or [BS2]). In [AHS] it was indicated (Theorem 2.15) that Lieb's method of deriving estimates for $N_+(\gamma, 0, V; \alpha)$ extends directly to the "magnetic" case; moreover, the constant in the estimate does not become larger.

REMARK 1.1. Of course, (1.8_-) follows from (1.8_+) by substituting $-V$ for V.

4. In what follows we suppose that V in (1.5), (1.6) is nonnegative (the only exception is Subsection 5 of §2). In accordance to that, (1.5) can be rewritten as

$$v[u, u] = \|Wu\|^2, \qquad W(x) \geqslant 0, \qquad W \in L_d(\mathbb{R}^d). \tag{1.10}$$

In the case considered, (1.8_+) with $\gamma > 0$ is equivalent to the following statement.

PROPOSITION 1.4. *Let* (1.1) *be satisfied and* $W \in L_d(\mathbb{R}^d)$. *Then for* $\gamma > 0$ *we have*

$$W(A + \gamma I)^{-1/2} \in \Sigma_d \tag{1.11}$$

and

$$|W(A + \gamma I)^{-1/2}|_d \leqslant (C(d))^{1/d} \|W\|_{L_d(\mathbb{R}^d)}, \qquad \gamma > 0. \tag{1.12}$$

5. To pass to the operator $A_\pm(\alpha)$ means to incorporate the electric potential $\mp\alpha W^2$ into the magnetic operator $A(\mathfrak{A})$. Here $\alpha > 0$ is the coupling constant that we shall suppose large in the sequel. Besides that, we incorporate into $A(\mathfrak{A})$ an electric potential F that does not contain α. Let us impose the condition

$$\int |F(x)||u|^2\,dx \leqslant \varepsilon \int |\nabla u|^2\,dx + c(\varepsilon) \int |u|^2\,dx, \quad \forall \varepsilon > 0, \quad u \in C_0^\infty(\mathbb{R}^d). \tag{1.13}$$

PROPOSITION 1.5. *Let the function* F *satisfy* (1.13). *Then, with the same constant* $c(\varepsilon)$, *we have*

$$\int |F(x)||u|^2\,dx \leqslant \varepsilon a[u, u] + c(\varepsilon)\|u\|^2, \quad \forall \varepsilon > 0, \quad u \in C_0^\infty(\mathbb{R}^d). \tag{1.14}$$

The proof reduces to a reference to (1.2). It should only be taken into account that (1.14) is equivalent to the inequality

$$\|\varphi(A + \gamma I)^{-1/2}\| \leqslant \varepsilon^{1/2}, \qquad \varphi = |F|^{1/2}, \qquad \gamma = \varepsilon^{-1} c(\varepsilon).$$

REMARK 1.2. A sufficient condition for (1.13) (and then also for (1.14)) is

$$\sup_{n\in\mathbb{Z}^d}\int_{Q_n}|F|^r\,dx<\infty,\qquad Q_n=[0,1)^d+n,\qquad 2r>d.$$

This condition is a particular case of condition (2.14) in [B2].

6. Let $F=\overline{F}$ and (1.14) hold. Then the form b,

$$b:=a+f,\qquad f[u_1,u_2]:=(Fu_1,u_2),\tag{1.15}$$

is bounded from below and closed on $d[a]$. The same is true for the form

$$b_{\pm}(\alpha):=b\mp\alpha v=a+f\mp\alpha v,\qquad \alpha>0,\tag{1.16}$$

where v is the form given by (1.10). Let us denote by B, $B_{\pm}(\alpha)$ the selfadjoint operators on $L_2(\mathbb{R}^d)$ generated by the forms b, $B_{\pm}(\alpha)$. Furthermore, let $\lambda=\overline{\lambda}\in\rho(B)$. *We denote by* $N_{\pm}(\lambda,B,W;\alpha)$ *the number of eigenvalues of the operator* $B_{\pm}(t)$ *that cross the point* λ *as* t *increases from* 0 *to* $\alpha>0$.

The following theorem is the main result of this paper.

THEOREM 1.1. *Suppose that conditions* (1.1), (1.13) *are satisfied and let the form* v *be defined by* (1.10). *Let the operators* B, $B_{\pm}(\alpha)$ *be generated by the forms* b, $b_{\pm}(\alpha)$ *defined in* (1.15), (1.16). *Then for every* $\lambda=\overline{\lambda}\in\rho(B)$

$$\lim_{\alpha\to\infty}\alpha^{d/2}N_{+}(\lambda,B,W;\alpha)=(2\pi)^{-d}\omega_d\int W^d\,dx,\tag{1.17}$$

$$\lim_{\alpha\to\infty}\alpha^{d/2}N_{-}(\lambda,B,W;\alpha)=0.\tag{1.18}$$

REMARK 1.2. For $\lambda<0$ the function $N_{+}(\lambda,A,W;\cdot)$ coincides with the number of eigenvalues of A lying to the left of λ. In this case considerations of variational nature can be applied to obtain asymptotic formulas (see [R], and also Theorem 2.1 below). The general case can be reduced to the case $B=A$, $\lambda<0$ with the help of the abstract Theorem 1.2 from [B2]. This plan of the proof will be implemented in §2.

REMARK 1.3. Gaps inside the spectrum of B can appear, for example, in the following cases. 1) The potentials $\mathfrak{A}$, F are periodic; 2) $B=A$ and the spectrum of $A=A(\mathfrak{A})$ is discrete ("magnetic bottles"; cf., say, [AHS, CV]); 3) the case of constant magnetic field of maximal rank (this is possible only if d is even).

REMARK 1.4. The asymptotic formula (1.17) was obtained in [B1] under much more restrictive assumptions on $\mathfrak{A}$. In particular, the boundedness of $\mathfrak{A}$ and $\operatorname{div}\mathfrak{A}$ was required.

§2. Proof of Theorem 1.1

1. Under the assumptions of Theorem 1.1, the relations (1.11), (1.14) hold. Suppose also that the following relation has already been established under these assumptions:

$$W(A+I)^{-1}\in\Sigma_d^0.\tag{2.1}$$

Then, by virtue of Theorem 1.2 from [B2], (1.18) holds and it is sufficient to obtain the asymptotic formula (1.17) for $B = A$ (that is, for $F = 0$) and $\lambda < 0$. The last result is contained in Theorem 1.1 of [R]. However, here we shall give an independent argument, differing from that in [R] (see Theorem 2.1 below). Our main objective is to prove inclusion (2.1).

2. We begin with statements of technical nature pertaining to the case of a bounded domain. Let Ω be a ball in $\mathbb{R}^d$; by $\|\cdot\|_\Omega$ we denote the norm in $L_2(\Omega)$. The following lemma can easily be derived from the Sobolev embedding theorem of limit order.

LEMMA 2.1. *Let $h \in L_d(\Omega)$. Then the form $\|hu\|^2_\Omega$, $u \in H^1(\Omega)$, is compact on $H^1(\Omega)$.*

Let us introduce the form a_Ω, closed on $L_2(\Omega)$ and defined on $d[a_\Omega] = H^1(\Omega)$ by the formula:

$$a_\Omega[u, u] := \int_\Omega |(D - \mathfrak{A})u|^2\, dx = \int_\Omega |\nabla u|^2 dx + \int_\Omega |\mathfrak{A}u|^2 dx - 2\operatorname{Re}\int_\Omega u\mathfrak{A}\overline{Du}\, dx, \quad u \in H^1(\Omega). \tag{2.2}$$

LEMMA 2.2. *Let (1.1) be satisfied. Then the form $a_\Omega[u, u] - \|\nabla u\|^2_\Omega$ is compact on $H^1(\Omega)$.*

Indeed, by Lemma 2.1, the second (and hence also the third) summand on the right-hand side of (2.2) is a compact form on $H^1(\Omega)$. □

Below we denote by $\overset{0}{a}_\Omega$ the restriction of a_Ω to $d[\overset{0}{a}_\Omega] := \overset{\circ}{H}{}^1(\Omega)$; let A_Ω and $\overset{0}{A}_\Omega$ be the selfadjoint operators on $L_2(\Omega)$ generated by the forms a_Ω and $\overset{0}{a}_\Omega$ respectively.

LEMMA 2.3. *Let $\varphi = \overline{\varphi} \in L_{d/2}(\Omega)$ and*

$$n_\pm(\gamma, \Omega, \mathfrak{A}, \varphi; \cdot), \qquad \overset{0}{n}_\pm(\gamma, \Omega, \mathfrak{A}, \varphi; \cdot)$$

be the distribution functions for the positive spectrum of the quotient of quadratic forms

$$\pm \int_\Omega \varphi |u|^2\, dx\, (a_\Omega[u, u] + \gamma\|u\|^2_\Omega)^{-1}, \qquad \gamma > 0, \quad u \in H^1(\Omega), \tag{$2.3_\pm$}$$

$$\pm \int_\Omega \varphi |u|^2\, dx\, (a_\Omega[u, u] + \gamma\|u\|^2_\Omega)^{-1}, \qquad \gamma \geqslant 0, \quad u \in \overset{\circ}{H}{}^1(\Omega). \tag{$2.4_\pm$}$$

Under condition (1.1), we have the following asymptotic formula:

$$\lim_{s\to 0} s^{d/2} n_\pm(\gamma, \Omega, \mathfrak{A}, \varphi; s) = \lim_{s\to 0} s^{d/2} \overset{0}{n}_\pm(\gamma, \Omega, \mathfrak{A}, \varphi; s) = (2\pi)^{-d}\omega_d \int_\Omega \varphi_\pm^{d/2}\, dx. \tag{$2.5_\pm$}$$

PROOF. A power type spectral asymptotic formula for the quotient of two quadratic forms will not change if a compact perturbation is incorporated into the denominator (i.e., into the metric form; cf. [BS1, Lemma 1.3]). Consequently, Lemma 2.2 allows one to replace the form $a[u, u]$ in (2.3), (2.4) by the form $\|\nabla u\|_\Omega^2$, that is, to reduce the verification of $(2.5_\pm)$ to the case $\mathfrak{A} \equiv 0$. In this case the desired result is well known (see, e.g., [BS2] or [Rs]). □

REMARK 2.1. If $\varphi \equiv 1$, then (2.5_+) gives a spectral asymptotic formula for compact operators $(A_\Omega + \gamma I)^{-1}$, $\gamma > 0$, and $(A_\Omega^0 + \gamma I)^{-1}$, $\gamma \geqslant 0$. In particular, these operators belong to $\Sigma_{d/2}$.

3. Let us return to the operator $A = A(\mathfrak{A})$ on $L_2(\mathbb{R}^d)$.

LEMMA 2.4. *Let* $\zeta \in C_0^\infty(\mathbb{R}^d)$, $u \in \mathfrak{D}(A)$. *Then* $\zeta u \in \mathfrak{D}(A)$ *and*

$$A(\zeta u) - \zeta A u = -2i(\nabla\zeta)(D - \mathfrak{A})u - (\Delta\zeta)u =: \tilde{h}. \tag{2.6}$$

PROOF. The relation $u \in \mathfrak{D}(A)$ means that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$, $(D - \mathfrak{A})u \in L_2(\mathbb{R}^d)$ and

$$a[u, \psi] = (h, \psi), \qquad h \in L_2(\mathbb{R}^d), \qquad \forall \psi \in C_0^\infty(\mathbb{R}^d). \tag{2.7}$$

Moreover, $h = Au$. Substituting $\zeta\psi$ for ψ in (2.7), we obtain after elementary calculations

$$a[\zeta u, \psi] = (\zeta h + \tilde{h}, \psi), \qquad \forall \psi \in C_0^\infty(\mathbb{R}^d). \tag{2.8}$$

Since evidently $\zeta u \in d[a]$, we get the desired result from (2.8). □

LEMMA 2.5. *Let* $\zeta \in C_0^\infty(\mathbb{R}^d)$. *Then*

$$\|(A + I)(\zeta u)\| \leqslant C(\zeta)\|(A + I)u\|, \qquad u \in \mathfrak{D}(A). \tag{2.9}$$

PROOF. By (2.6) we have

$$\begin{aligned} \|(A + I)(\zeta u)\| &\leqslant \|\zeta(A + I)u\| + \|\tilde{h}\| \\ &\leqslant c(\|(A + I)u\| + \|(D - \mathfrak{A})u\| + \|u\|). \end{aligned}$$

It remains to take into account the inequalities

$$\|(D - \mathfrak{A})u\| = \|A^{1/2}u\| \leqslant \|(A + I)u\|, \qquad \|u\| \leqslant \|(A + I)u\|. \quad □$$

Observe also the following version of Lemma 2.4 involving the operator A_Ω^0 from Subsection 2.

LEMMA 2.6. *Suppose that under the assumptions of Lemma* 2.4 *we have* $\operatorname{supp}\zeta \subset \Omega$. *Then*

$$u_0 := (\zeta u)|_\Omega \in \mathfrak{D}(A_\Omega^0), \qquad A_\Omega^0 u_0 = \zeta h + \tilde{h}|_\Omega = A(\zeta u)|_\Omega. \tag{2.10}$$

PROOF. Since $u_0 \in \dot{H}^1(\Omega)$, the desired result follows from (2.8) with $\psi \in C_0^\infty(\Omega)$. □

4. Now we proceed to the proof of (2.1). Since clearly (1.12) implies the estimate

$$|W(A+I)^{-1}|_d \leqslant C\|W\|_{L_d(\mathbb{R}^d)},$$

it is sufficient to obtain (2.1) for

$$W \in C_0^\infty(\mathbb{R}^d) \tag{2.11}$$

(compare with [B, §2]). In this case a sharper estimate holds.

LEMMA 2.7. *Let* (1.1), (2.11) *be satisfied. Then for* $A(\mathfrak{A})$ *we have the inclusion*

$$W(A+I)^{-1} \in \Sigma_{d/2}. \tag{2.12}$$

PROOF. The squares of the singular numbers of $W(A+I)^{-1}$ coincide with the consecutive maxima (i.e. the spectrum) of the quotient of quadratic forms

$$\|Wu\|^2/\|(A+I)u\|^2, \qquad u \in \mathfrak{D}(A). \tag{2.13}$$

Variational considerations allow one to obtain upper estimates for the spectrum of the quotient (2.13). Let Ω be a ball, supp $W \subset \Omega$, and the function $\zeta \in C_0^\infty(\Omega)$ satisfy $\zeta(x)W(x) = W(x)$. Let us use (2.9), (2.10). We obtain

$$\|Wu\|^2 = \|Wu_0\|_\Omega^2, \qquad \|(A+I)u\|^2 \geqslant c\|(\overset{0}{A}_\Omega + I)u_0\|_\Omega^2 \quad (c>0),$$

and so the quotient (2.13) is estimated from above (we omit the constant factor) by the quotient

$$\|Wu_0\|_\Omega^2/\|(\overset{0}{A}_\Omega + I)u_0\|_\Omega^2, \qquad u_0 \in \mathfrak{D}(\overset{0}{A}_\Omega). \tag{2.14}$$

Moreover, an arbitrary $u_0 \in \mathfrak{D}(\overset{0}{A}_\Omega)$ instead of the u_0 from (2.10) can be allowed in (2.14), and that can only increase the consecutive maxima. Finally, it is possible to estimate W in the numerator in (2.14) by a constant. Again omitting a constant factor, we arrive at the spectrum of the quotient

$$\|u_0\|_\Omega^2/\|(\overset{0}{A}_\Omega + I)u_0\|_\Omega^2, \qquad u_0 \in \mathfrak{D}(\overset{0}{A}_\Omega).$$

The latter coincides with the spectrum of the operator $(\overset{0}{A}_\Omega + I)^{-2}$. In view of Remark 2.1, we have $(\overset{0}{A}_\Omega + I)^{-2} \in \Sigma_{d/4}$. Thus the eigenvalues (the consecutive maxima) σ_n of the quotient (2.13) also satisfy the estimate $\sigma_n = O(n^{-d/4})$. This is equivalent to (2.12).

Along with Lemma 2.7, we have established estimate (2.1) under the assumptions of Theorem 1.1.

5. It remains to establish (1.17) for $A = B$, $\lambda = -\gamma < 0$. Let us prove a somewhat more general statement. As in Subsection 3 of §1, we assume that the form v is defined by (1.5), (1.6) rather than by (1.10). The operator $A_+(\alpha)$ is generated by the form $a - \alpha v$, $\alpha > 0$; $N_+(\gamma, \mathfrak{A}, V; \alpha)$ is the number of eigenvalues of $A_+(\alpha)$ lying to the left of $\lambda = -\gamma \leqslant 0$.

THEOREM 2.1. *Let* (1.1), (1.6) *be satisfied. Then*

$$\lim_{\alpha\to\infty} \alpha^{-d/2} N_+(\gamma, \mathfrak{A}, V; \alpha) = (2\pi)^{-d} \omega_d \int V_+^{d/2}\, dx, \qquad \gamma \geqslant 0. \tag{2.15}$$

PROOF. By virtue of (1.9), the problem is equivalent to that of determining the asymptotics for $n_+(\gamma, s)$ (we recall that this is the spectrum distribution function for the quotient (1.7_+)). Estimate (1.8) enables us to restrict ourselves to calculating the asymptotics under the condition $V \in C_0^\infty(\mathbb{R}^d)$. Let Ω be a ball, $\operatorname{supp} V \subset \Omega$. The lower estimate for $n_+(\gamma, s)$ follows if we pass from (1.7_+) to the quotient (2.4_+) with $\varphi = V$. Similarly for $\gamma > 0$, the upper estimate follows by passing to the quotient (2.3_+) with $\varphi = V$. If $\gamma = 0$, one should first add the form $\|u\|_\Omega^2$ to the denominator of the quotient (1.7_+). By (1.4), this form is compact with respect to the form a, and hence adding this form does not change the asymptotics of $n_+(0, s)$. After that the upper estimate follows by passing to the quotient (2.3_+) with $\varphi = V$, $\gamma = 1$. These two-sided estimates together with (2.5_+), (1.9) lead to (2.15). □

If the form v is defined by (1.10) and $\gamma \geqslant 0$, then evidently

$$N_+(\gamma, \mathfrak{A}, V; \alpha) = N_+(-\gamma, A, W; \alpha).$$

Therefore, once the asymptotic formula (2.15) established, the proof of Theorem 1.1 is complete.

Finally we note that the second-named author has established (cf. [R, Theorem 1.1]) that the asymptotic formula (2.15) holds for every $\gamma = -\lambda$, $\lambda < \lambda_e$; here λ_e is the lower bound of the essential spectrum of $A(\mathfrak{A})$. If $\lambda_e = 0$, then formally (2.15) with $\gamma = 0$ is not implied by this result, but can be obtained by the method of [R]. The proof of Theorem 2.1 presented here is slightly simpler than the argument in [R].

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