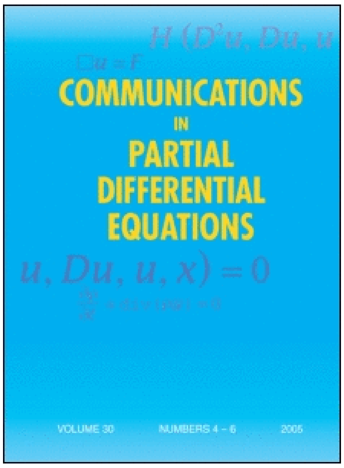

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THE ESSENTIAL SPECTRUM OF A LINEAR MAGNETOHYDRODYNAMIC MODEL CONTAINING A VACUUM REGION

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1 Introduction

1.1. The essential spectrum of the force operator of the ideal linear magnetohydrodynamics has been extensively studied in the mathematical literature (see e.g. [A.L.M.S, Section 5], [Des.Gey], [Kako 1], [Kako 2], [Kako 3], [Lan.Möl], [Rai 1], [Rai 2, Section 4]). However, all the rigorous from mathematical point of view works on this topic concern the magnetohydrodynamic (MHD) model of a plasma confined in a bounded domain $\mathcal{O}_p \subset \mathbf{R}^3$ with perfectly conducting boundary. On the other hand, there exists another MHD model which is more realistic from physical point of view. According to this model, the plasma region $\mathcal{O}_p$ is surrounded by a vacuum region $\mathcal{O}_v$ whose boundary consists of two disjoint surfaces $\mathcal{S}_p$ and $\mathcal{S}_v$. The surface $\mathcal{S}_p$ coincides with the plasma-vacuum interface, while the outer surface $\mathcal{S}_v$ is perfectly conducting. The interaction between the plasma filling $\mathcal{O}_p$ and the exterior magnetic field is described by the MHD equations, while the dynamics of the electromagnetic field in the vacuum region $\mathcal{O}_v$ is governed by the Maxwell equations. Usually the domains $\mathcal{O}_p$ and $\mathcal{O}_v$ are assumed to be axisymmetric. In other words, they are obtained by the rotation of two plane domains Ω_p and Ω_v (Ω_p being surrounded by Ω_v), around an axis situated at a positive distance from the closure of $\Omega_p \cup \Omega_v$. In the case where the ratio of the small and the large characteristic radii of the toroidal domain $\mathcal{O}_v \cup \mathcal{O}_p$ is sufficiently small, one may consider $\mathcal{O}_p$ and, respectively, $\mathcal{O}_v$ as the cylindrical manifolds $\Omega_p \times \mathbf{S}^1$ and, respectively, $\Omega_v \times \mathbf{S}^1$ where $\mathbf{S}^1 := \mathbf{R}/2\pi\mathbf{Z}$. This is the exact meaning of the notations $\mathcal{O}_p$ and $\mathcal{O}_v$ adopted in the present paper. We denote by Γ_p and Γ_v respectively the boundaries of Ω_p and Ω_v. Thus we have $\mathcal{S}_p = \Gamma_p \times \mathbf{S}^1$ and $\mathcal{S}_v = \Gamma_v \times \mathbf{S}^1$.

1.2. The stationary (i.e. independent of time) equilibrium of the plasma occupying $\mathcal{O}_p$ is given by the macroscopic velocity $\mathbf{V} : \overline{\mathcal{O}_p} \to \mathbf{R}^3$, the exterior magnetic field $\mathbf{B}_p : \overline{\mathcal{O}_p} \to \mathbf{R}^3$, the pressure $P : \overline{\mathcal{O}_p} \to [0, \infty)$, and the mass density $\varrho : \overline{\mathcal{O}_p} \to (0, \infty)$. Moreover, we denote by $\gamma > 1$ the constant adiabaticity index.

Throughout the paper we assume that the plasma equilibrium is static, i.e. we have

$$\mathbf{V} \equiv 0 \quad \text{in} \quad \mathcal{O}_p.$$

The plasma equilibrium quantities P and $\mathbf{B}_p$ satisfy the equations

$$\nabla P = [\text{rot}\, \mathbf{B}_p,\ \mathbf{B}_p] \quad \text{in} \quad \mathcal{O}_p, \tag{1.1}$$

$$\text{div}\, \mathbf{B}_p = 0 \quad \text{in} \quad \mathcal{O}_p. \tag{1.2}$$

The equilibrium mass density ϱ is an arbitrary sufficiently smooth strictly positive function over $\overline{\mathcal{O}_p}$.

The stationary equilibrium vacuum magnetic field $\mathbf{B}_v$ satisfies the equations

$$\left\{ \begin{array}{l} \text{rot}\, \mathbf{B}_v = 0, \\ \text{div}\, \mathbf{B}_v = 0, \end{array} \right. \quad \text{in} \quad \mathcal{O}_v. \tag{1.3}$$

Moreover, the equilibrium quantities P, $\mathbf{B}_p$ and $\mathbf{B}_v$ satisfy the boundary conditions

$$(\mathbf{n}, \mathbf{B}_p) = 0 \quad \text{on} \quad \mathcal{S}_p, \tag{1.4}$$

$$(\mathbf{n}, \mathbf{B}_v) = 0 \quad \text{on} \quad \mathcal{S}_p, \tag{1.5}$$

$$\frac{1}{2}\mathbf{B}_v^2 = P + \frac{1}{2}\mathbf{B}_p^2 \quad \text{on} \quad \mathcal{S}_p, \tag{1.6}$$

$$(\tilde{\mathbf{n}}, \mathbf{B}_v) = 0 \quad \text{on} \quad \mathcal{S}_v, \tag{1.7}$$

where $\mathbf{n}$ (respectively, $\tilde{\mathbf{n}}$) denotes the unit normal to $\mathcal{S}_p$ (respectively, to $\mathcal{S}_v$) vector exterior with respect to $\mathcal{O}_p$ (respectively, to $\mathcal{O}_v$). More details concerning the physical approach to the plasma equilibrium could be found in [Frei], and a rigorous mathematical approach to this problem is contained in [Tem].

1.3. The linear perturbations $\mathbf{V}_1$, $\mathbf{B}_{p,1}$, P_1, ϱ_1 and $\mathbf{B}_{v,1}$ of the equilibrium quantities $\mathbf{V}$, $\mathbf{B}_p$, P, ϱ and $\mathbf{B}_v$ can be written in the terms of the displacement vector

$$\xi(t,x) := \int_0^t \mathbf{V}_1(\tau,x)\,d\tau,$$

and the equilibrium quantities themselves. Here t denotes the time, and $x \in \mathcal{O}_p$ denotes the spatial variable. The vector $\xi(t,x)$ satisfies the mixed problem

$$\begin{cases} \frac{\partial^2 \xi}{\partial t^2} = -\mathcal{F}\xi \quad \text{in} \quad (0,\infty)\times\mathcal{O}_p, \\ L\xi = 0 \quad \text{on} \quad (0,\infty)\times\mathcal{S}_p, \\ \xi_{|t=0} = 0, \quad \frac{\partial\xi}{\partial t}_{|t=0} = \mathbf{V}_{1|t=0}, \quad \text{in} \quad \mathcal{O}_p. \end{cases} \tag{1.8}$$

Here we have used the notations

$$\mathcal{F}\xi := -\frac{1}{\varrho}\left\{\nabla(\gamma P \operatorname{div}\xi + (\nabla P,\xi)) + [\operatorname{rot}\operatorname{rot}[\xi,\mathbf{B}_p],\mathbf{B}_p] + [\operatorname{rot}\mathbf{B}_p,\operatorname{rot}[\xi,\mathbf{B}_p]]\right\},$$

and

$$L\xi := -\left\{\gamma P \operatorname{div}\xi - (\mathbf{B}_p,(\operatorname{rot}[\xi,\mathbf{B}_p] + (\xi.\nabla)\mathbf{B}_p)) + (\mathbf{B}_v,(R\xi + (\xi.\nabla)\mathbf{B}_v))\right\}_{|\mathcal{S}_p},$$

where

$$R\xi = \operatorname{rot}\mathbf{A}, \tag{1.9}$$

and $\mathbf{A}$ is the solution of the following auxiliary boundary-value problem:

$$\begin{cases} \operatorname{rot}\operatorname{rot}\mathbf{A} = 0 \quad \text{in} \quad \mathcal{O}_v, \\ [\mathrm{n},\mathbf{A}] = -(\mathrm{n},\xi)\,\mathbf{B}_v \quad \text{on} \quad \mathcal{S}_p, \\ [\tilde{\mathrm{n}},\mathbf{A}] = 0 \quad \text{on} \quad \mathcal{S}_v. \end{cases} \tag{1.10}$$

The existence of solutions of the boundary-value problem (1.10) follows from [Mor, Theorem 7.8.2]. Obviously, the mapping R is linear and continuous as a mapping from the Sobolev space of vector-valued functions $\{H^1(\mathcal{O}_p)\}^3$ into $\{L^2(\mathcal{O}_v)\}^3$ as well as from $\{H^2(\mathcal{O}_p)\}^3$ into $\{H^1(\mathcal{O}_v)\}^3$ (see below Lemma 3.3). Moreover, we have $\operatorname{Ker} R = \{0\}$.

Remark. Note that we do not claim that (1.10) has a unique solution $\mathbf{A}$, but that this boundary-value problem determines rot $\mathbf{A}$ uniquely.

The linear mixed problem (1.8) was derived heuristically from the original non-linear-problem in the pioneer work [B.F.K.K] (see also the survey article [Frei]). Some rigorous results concerning the derivation of this mixed problem can be found in [Lau.Shen] and [Lau].

1.4. It has been shown in [Ush] and [Lau.Shen] that under some additional hypotheses concerning the equilibrium, the operator $\mathcal{F}$ with domain

$$D_0(F) := \left\{\xi \in \left\{H^2(\mathcal{O}_p)\right\}^3 : L\xi = 0 \text{ on } \mathcal{S}_p\right\}$$

is symmetric and lower-bounded in the Hilbert space $\{L^2(\mathcal{O}_p; \varrho dx)\}^3$. Note that the validity of the inequality

$$G := \left(\mathbf{n}, \nabla\left(\frac{1}{2}\mathbf{B}_v^2 - \frac{1}{2}\mathbf{B}_p^2 - P\right)\right) \geq 0 \quad \text{on} \quad \mathcal{S}_p \tag{1.11}$$

is one of the important assumptions imposed in [Ush] and [Lau.Shen]; together with some hypotheses about the smoothness and the regularity of the equilibrium quantities it guarantees the lower-boundedness of the force operator F (see (1.14) and (1.16) below).

Denote by F the Friedrichs extension of the operator $\mathcal{F}$. The operator F is known in the physics literature as the linear MHD force operator. The present paper is devoted to the localization of the essential spectrum of the operator F related to a particular MHD equilibrium described explicitly below in Subsection 2.1.

The reason of the existence of non-empty essential spectrum of the operator F could be explained heuristically in the following manner. The principal matrix-valued symbol $\mathbf{F}(x,p)$, $(x,p) \in T^*\mathcal{O}_p$, of the operator F is equal to

$$\mathbf{F}(x,p) =$$

$$\varrho(x)^{-1}\left\{\delta_{jk}(\mathbf{B}_p(x),p)^2 + \varrho(x)v(x)^2 p_j p_k - \right.$$

$$\left.(\mathbf{B}_p(x),p)(\mathbf{B}_{p,j}(x)p_k + \mathbf{B}_{p,k}(x)p_j)\right\}_{j,k=1}^3,$$

where

$$v^2 := v_S^2 + v_A^2,\ v_S^2 := \gamma P/\varrho,\ v_A^2 := |\mathbf{B}_p|^2/\varrho. \tag{1.12}$$

The quantities v_A^2, v_S^2 and v^2 have respectively the physical meaning of the squares of the Alfvén velocity, the sound velocity and the magnetosonic velocity.

The eigenvalues $\mu_A(x,p)$, $\mu_+(x,p)$ and $\mu_-(x,p)$ of the matrix $\mathbf{F}(x,p)$ can be easily calculated explicitly:

$$\mu_A(x,p) = \varrho(x)^{-1}(\mathbf{B}_p(x),p)^2,$$

$$\mu_\pm(x,p) = \frac{1}{2}\left\{v(x)^2|p|^2 \pm \sqrt{v(x)^4|p|^4 - 4v_S(x)^2\mu_A(x,p)|p|^2}\right\}.$$

In the physics literature the eigenvalue μ_A is associated with the Alfvén polarization, the eigenvalue μ_+ - with the fast magnetosonic polarization, and the eigenvalue μ_- - with the slow magnetosonic polarization. Evidently, the eigenvalue $\mu_+(x,p)$ is elliptic under the natural assumption $v^2(x) > 0$, $x \in \overline{\mathcal{O}_p}$, while $\mu_A(x,p)$ and $\mu_-(x,p)$ are not elliptic on the set $\{(x,p) \in T^*\mathcal{O}_p : (\mathbf{B}_p(x),p) = 0\}$. Moreover, we have $\mu_-(x,p) = 0$ for all $x \in \overline{\mathcal{O}_p}$ such that $P(x) = 0$.

The operator F admits an equivalent description. For

$$\xi \in D_0[a] := \left\{H^1(\mathcal{O}_p)\right\}^3$$

set

$$a_p[\xi] := \int_{\mathcal{O}_p} \left\{\gamma P|\text{div}\xi|^2 + |\text{rot }[\xi, \mathbf{B}_p]|^2 + \right.$$

$$\left. \text{Re }\left((\nabla P, \xi)\,\overline{\text{div}\xi} - \left([\text{rot }\mathbf{B}_p, \text{rot }[\xi, \mathbf{B}_p]],\ \bar{\xi}\right)\right)\right\}\, dx, \tag{1.13}$$

$$a_s[\xi] := \int_{S_p} G|(\mathbf{n}, \xi)|^2\, dS, \tag{1.14}$$

$$a_v[\xi] := \int_{\mathcal{O}_v} |R\xi|^2\, dx, \tag{1.15}$$

$$a[\xi] := a_p[\xi] + a_s[\xi] + a_v[\xi]. \tag{1.16}$$

If $\xi \in D_0(F)$, then we have

$$(\mathcal{F}\xi, \xi)_{\{L^2(\mathcal{O}_p;\varrho dx)\}^3} = a[\xi]$$

(see e.g. [Lau.Shen] or [Ush, Lemma 2.1]). Moreover, the quadratic form a defined on $D_0[a]$ is lower-bounded and closable in $\{L^2(\mathcal{O}_p; \varrho dx)\}^3$. The operator generated by the closed quadratic form a coincides with the force operator F (see [Ush, Proposition 2.4]).

It will be useful to compare here the force operator F with the force operator F_p occurring in the MHD model related to a domain $\mathcal{O}_p$ with perfectly conducting boundary. In order to introduce the operator F_p, one defines the quadratic form a_p on the domain

$$D_0[a_p] := \left\{\xi \in \left\{H^1(\mathcal{O}_p)\right\}^3 : (\mathbf{n}, \xi)_{|S_p} = 0\right\},$$

and then closes it in $\{L^2(\mathcal{O}_p; \varrho dx)\}^3$. The operator F_p is defined as the operator generated by the closure of a_p. Note that if we come back to the MHD model containing a vacuum region studied in the present paper, and restrict the quadratic form a on the domain $D_0[a_p]$, we would get

$$a[\xi] = a_p[\xi]$$

(see (1.14), (1.15), (1.9), (1.10) and (1.16)).

Hence, the operators F and F_p could be considered as two different self-adjoint realizations of one and the same formal differential operation, corresponding to two different boundary conditions. Since the operators F and F_p are not elliptic, one of the most natural and interesting questions in the spectral theory of these two operators is whether their essential spectra coincide or not.

1.5. The paper is organized as follows. In Section 2 we describe the particular equilibrium we study, and state the main result of the article. The equilibrium is supposed to possess a translational symmetry, so that in Section 3 we employ the Fourier decomposition of $\xi \in D_0(F)$ with respect to the negligible variable in order to show that F is unitarily equivalent to the orthogonal sum $\sum_{k\in\mathbf{Z}} \oplus F(k)$ where $F(k)$, $k \in \mathbf{Z}$, are operators selfadjoint in $\{L^2(\Omega_p; \varrho\, dy)\}^3$. Moreover, we show that the equality $\sigma_{ess}(F) = \bigcup_{k\in\mathbf{Z}} \sigma_{ess}(F(k))$ holds under certain hypotheses. In Section 4 we introduce the auxiliary Neumann–to–Dirichlet maps and Dirichlet–to–Neumann maps (see e.g. [Hör], [Syl.Uhl]) which play an important role in the proof of the main result. Further, in Section 5, we use the ideas of the Weyl-Friedrichs decomposition of a vector-valued function into an orthogonal sum of a gradient and a divergence-free vector in order to show that $\sigma_{ess}(F(k))$, $k \in \mathbf{Z}$, coincides with the essential spectrum of the orthogonal sum $\sum_{j=1}^{3} \oplus F_j(k)$ where $F_j(k)$, $j = 1, 2, 3$, $k \in \mathbf{Z}$, are scalar operators. In Section 6 we show that $\sigma_{ess}(F_1(k)) = \emptyset$ and localize $\sigma_{ess}(F_3(k))$, $k \in \mathbf{Z}$. Finally, in Section 7 we localize $\sigma_{ess}(F_2(k))$, $k \in \mathbf{Z}$.

2 Formulation of the main result

2.1. In thus subsection we describe the particular MHD equlibrium we investigate in this article.

As stated above, we assume $\mathcal{O}_p = \Omega_p \times \mathbf{S}^1$ and $\mathcal{O}_v = \Omega_v \times \mathbf{S}^1$ where $\Omega_p \subset \mathbf{R}^2$ and $\Omega_v \subset \mathbf{R}^2$. Moreover, $\Gamma_p = \partial\Omega_p$ and $\Gamma_v = \partial\Omega_v \setminus \Gamma_p$ are supposed to be disjoint C^∞-smooth closed simple curves.

For $x \in \mathcal{O}_p$ (respectively, $x \in \mathcal{O}_v$), we set $x = (y, z)$ where $y \in \Omega_p$ (respectively, $y \in \Omega_v$) and $z \in \mathbf{S}^1$. We assume that the equilibrium quantities P, $\mathbf{B}_p$, ϱ and $\mathbf{B}_v$ are independent of the variable z.

Further, we assume that

$$\mathbf{B}_p = (0, 0, b_p) \tag{2.1}$$

where $b_p \in C^\infty(\overline{\Omega_p})$. Thus, the equation (1.2) and the boundary condition (1.4) are satisfied. In the case where (2.1) holds, the equation (1.1) reads

$$P + \frac{1}{2} b_p^2 = \text{const} \quad \text{in} \quad \Omega_p. \tag{2.2}$$

Moreover, we suppose $P \not\equiv 0$, $b_p \not\equiv 0$ in Ω_p.

Finally, we assume that

$$\mathbf{B}_v = (0, 0, b_v) \tag{2.3}$$

where b_v is a non-zero constant. Thus the equations (1.3) and the boundary conditions (1.5) and (1.7) are satisfied.

Note that (2.3) and (2.2) imply that both sides in (1.6) are (equal) constants. Moreover, the relations (2.3) and (2.2) entail $G \equiv 0$ (see (1.11)) and, hence, $a_s[\xi] \equiv 0$ (see (1.14)).

The motivation for the choice of the particular equilibrium described above is explained in Subsection 2.3.

2.2. In this subsection we state the main result of the article.

Set

$$\beta_0(y) := v_S^2(y)/v^2(y), \; v_B^2(y) := \beta_0(y)v_A^2(y), \; y \in \overline{\Omega_p},$$

(see (1.12); in the sequel the notation v_A should be understood as $v_A \equiv b_p/\sqrt{\varrho}$), and for $k \in \mathbf{Z}$ introduce the closed sets

$$I_A = I_A(k) := \bigcup_{y \in \overline{\Omega_p}} \left\{k^2 v_A^2(y)\right\}, \; ; I_B = I_B(k) := \bigcup_{y \in \overline{\Omega_p}} \left\{k^2 v_B^2(y)\right\},$$

$$I_V = I_V(k) := \bigcup_{s \in \Gamma_p} \left\{k^2 \left(b_p^2(s) + b_v^2\right) / \varrho(s)\right\}.$$

Here and below we parametrize Γ_p by its arc length s, and denote by $f(s)$ the restriction onto Γ_p of any quantity $f(y)$ defined over $\overline{\Omega_p}$, or over $\overline{\Omega_v}$.

Theorem 2.1 *Under the hypotheses concerning the equilibrium described in the Introduction and in Subsection 2.1, we have*

$$\sigma_{\text{ess}}(F) = \bigcup_{k \in \mathbf{Z}} \left\{I_A(k) \cup I_B(k) \cup I_V(k)\right\}. \tag{2.4}$$

In order to compare $\sigma_{\text{ess}}(F)$ with the essential spectrum of the force operator F_p arising in the MHD model with perfectly conducting boundary, we recall here the result of Theorem 2.2 and Corollary 2.4 in [Rai 1]:

$$\sigma_{\text{ess}}(F_p) = \bigcup_{k \in \mathbf{Z}} \left\{I_A(k) \cup I_B(k)\right\}. \tag{2.5}$$

Comparing (2.4) and (2.5), one finds easily that there exist MHD equilibria such that the corresponding set $\sigma_{\text{ess}}(F) \setminus \sigma_{\text{ess}}(F_p)$ is not empty. To our knowledge, this effect is described here for the first time not only at rigorous mathematical level, but even at heuristic one.

2.3. In this subsection we discuss the reasons for which we consider the particular MHD equilibrium described in Subsection 2.1.

First of all, we would like to underline that our aim is not the analysis of a general MHD equilibrium. Our result should be considered rather as an explicit comparison of $\sigma_{\text{ess}}(F)$ and $\sigma_{\text{ess}}(F_p)$ for a particular MHD model; as far as we are informed such a result has not been achieved for any MHD equilibrium.

The plasma equilibrium we consider is a generalization of the so-called θ-pinch model (see e.g. [Frei, Section IV.B.1]) where Ω_p is a disk, and b_p depends only on the distance to the axis of the circular cylinder $\mathcal{O}_p$. We restrict our attention to this simple plasma equilibrium since it is among the rare MHD models for which the essential spectrum of the operator F_p has been localized (see (2.5)). The intrinsic and not quite evident reason for the availability of this result is the fact that the force lines of the magnetic field B_v are straight and parallel to the generatrix of the cylinder $\mathcal{O}_p$.

It should be noted that the essential spectrum of the ideal linear MHD equations for quite general equilibria has been studied in various works (see e.g. [Ham], [Lif]). In these works, however, the authors prove only that a given set Σ is included in the essential spectrum, but not that Σ coincides with the essential spectrum.

Moreover, there exist rigorous results on the localization of $\sigma_{\text{ess}}(F_p)$ for some symmetric equilibria (see [Des.Gey], [Kako 1], [Kako 2], [Kako 3], [Rai 2, Section 4]), but they involve only the localization of the essential spectrum of the force operator $F_p(\mathbf{k})$ with fixed wavenumber(s) $\mathbf{k}$ corresponding to the negligible variable(s).

In order to justify the choice of the vacuum equilibrium (see (2.3)), we need several auxiliary assertions.

Set

$$\Xi := \left\{\omega \in \left\{H^1(\mathcal{O}_v)\right\}^3 : \operatorname{rot}\omega = 0, \ \operatorname{div}\omega = 0, \ (\mathrm{n},\omega)_{|\mathcal{S}_p} = 0, \ (\tilde{\mathrm{n}},\omega)_{|\mathcal{S}_v} = 0\right\}. \tag{2.6}$$

Some well-known facts from the Hodge theory (see e.g. [Mor, Chapter 7]) entail the following lemma.

Lemma 2.1 *We have* $\dim \Xi = 2$, *and the orthogonal (not necessarily normalized) basis in* Ξ *can be written in the form* $\{\omega_1, \omega_2\}$ *where* $\omega_1 := (0,0,1)$ *and* $\omega_2 := (\partial_2\psi, -\partial_1\psi, 0)$, $\psi = \psi(y)$ *being the unique solution of the boundary-value problem*

$$\begin{cases} \Delta\psi = 0 & \text{in} \quad \Omega_v, \\ \psi = 1 & \text{on} \quad \Gamma_p, \\ \psi = 0 & \text{on} \quad \Gamma_v. \end{cases} \tag{2.7}$$

Since $\mathbf{B}_v$ satisfies (1.3), (1.5), (1.7), we have $\mathbf{B}_v \in \Xi$, i.e.

$$\mathbf{B}_v = c_1\omega_1 + c_2\omega_2, \tag{2.8}$$

where c_1 and c_2 are real constants. The relation (2.3) is equivalent to $c_2 = 0$.

Note that if $c_2 \neq 0$, then (1.11) is violated already for some quite simple domains $\mathcal{O}_v$ (e.g. we have $G(s) < 0$ for all $s \in \Gamma_p$ in the case where $\mathcal{O}_v$ is a circular annulus). Even if we had demonstrated that the operator F was lower-bounded despite the violation of (1.11), the picture in the case $c_2 \neq 0$ would change dramatically in comparison with the case $c_2 = 0$. The intrinsic reason for this phenomenon would be again the fact that if $c_2 = 0$, the force lines of B_v are straight and parallel to the generatrix of $\mathcal{O}_v$, and if $c_2 \neq 0$ they are not.

Finally, there is yet another methodological reason for the choice of the equilibrium studied here. We would like to display clearly enough the relation between the non-local boundary condition arising in the MHD model containing a vacuum region, and the Neumann–to–Dirichlet and Dirichlet–to–Neumann maps described below in Section 4. In particular, we would like to show that the pseudo-differential methods enter the analysis of the MHD model containing a vacuum region in a fairly natural way. If we had considered a more general MHD equilibrium, this methodological novelty could be completely hidden by the tedious technicalities typical for the MHD theory.

We hope that we shall be able to extend our analysis to more general MHD equilibria in a future work.

3 Fourier decomposition

3.1. Expanding $\xi \in D_0(F)$ into a Fourier series

$$\xi(y,z) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbf{Z}} \xi^{(k)}(y) e^{ikz}, \tag{3.1}$$

we get

$$a_p[\xi] = \sum_{k\in\mathbf{Z}} a_p^{(1)}[\xi^{(k)}; k]$$

(see (1.13)), where

$$a_p^{(1)}[\eta; k] := \int_{\Omega_p} \varrho \left\{ v^2 \left| \partial_1 \eta_1 + \partial_2 \eta_2 + ik\beta_0 \eta_3 \right|^2 + k^2 \left(v_A^2 |\eta_1|^2 + v_A^2 |\eta_2|^2 + v_B^2 |\eta_3|^2 \right) \right\} dy, \tag{3.2}$$

with $\partial_1 := \partial/\partial y_1$, $\partial_2 := \partial/\partial y_2$, $\eta = (\eta_1, \eta_2, \eta_3) \in D_0[a^{(1)}]$, and

$$D_0[a^{(1)}] := \left\{ \eta \in \left\{ L^2(\Omega_p) \right\}^3 : \eta_j \in H^1(\Omega_p),\ j = 1,2 \right\}. \tag{3.3}$$

3.2. Our next goal is to introduce the quadratic forms $a_v^{(1)}[\eta; k]$, $k \in \mathbf{Z}$, $\eta \in D_0[a^{(1)}]$, such that the equality

$$a_v[\xi] = \sum_{k\in\mathbf{Z}} a_v^{(1)}[\xi^{(k)}; k]$$

holds (see (1.15)).

Consider the boundary-value problem

$$\begin{cases} \Delta\Phi = 0 \quad \text{in} \quad \mathcal{O}_v, \\ \frac{\partial\Phi}{\partial \mathbf{n}} = b_v \frac{\partial(\mathbf{n},\xi)}{\partial z} \quad \text{on} \quad \mathcal{S}_p, \\ \frac{\partial\Phi}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \mathcal{S}_v, \\ \int_{\mathcal{S}_p} \Phi \, dS = 0, \end{cases} \tag{3.4}$$

where $\xi \in D_0[a] \equiv \{H^1(\mathcal{O}_p)\}^3$.

Lemma 3.1 *The boundary-value problem (3.4) has a unique solution $\Phi = \Phi(\xi) \in H^1(\mathcal{O}_v)$.*

Proof. The lemma concerns a classical result on the solvability of the Neumann boundary-value problem for the Laplace equation in a three-dimensional bounded cylindrical manifold (see e.g. [Vla, Theorem 23.1] where a slightly different version of the lemma is proved). We shall just note that if we put

$$\varphi = \begin{cases} b_v \frac{\partial(\mathbf{n},\xi)}{\partial z} \quad \text{on} \quad \mathcal{S}_p, \\ 0 \quad \text{on} \quad \mathcal{S}_v, \end{cases}$$

then the condition $\int_{\partial\mathcal{O}_v} \varphi \, dS = 0$ guarantees the solvability of the problem (3.4); here the quantity $\int_{\partial\mathcal{O}_v} \varphi \, dS$ should be understood as the duality pair $< \varphi, 1 >$ between $H^{-1/2}(\partial\mathcal{O}_v)$ and $H^{1/2}(\partial\mathcal{O}_v)$.

Lemma 3.2 *The operator R introduced in (1.9) is uniquely defined by the relation*

$$R\xi = \nabla\Phi(\xi) + \frac{\mathbf{B}_v}{2\pi|\Gamma_p|} \int_{\mathcal{S}_p} (\mathbf{n}, \xi) \, dS, \tag{3.5}$$

where $\xi \in D_0[a]$, $\Phi = \Phi(\xi)$ is the solution of the boundary-value problem (3.4), and $|\Gamma_p|$ denotes the length of Γ_p.

Proof. Set

$$\omega = R\xi - \nabla\Phi \equiv \operatorname{rot} \mathbf{A} - \nabla\Phi, \tag{3.6}$$

where $\mathbf{A}$ is a solution of the boundary-value problem (1.10). Then we have

$$\operatorname{rot}\omega = \operatorname{rot}\operatorname{rot}\mathbf{A} - \operatorname{rot}\nabla\Phi = 0,$$

$$\operatorname{div}\omega = \operatorname{div}\operatorname{rot}\mathbf{A} - \Delta\Phi = 0.$$

Further, on $\mathcal{S}_p$ we have

$$(\mathbf{n}, \omega) = (\mathbf{n}, \operatorname{rot}\mathbf{A}) - (\mathbf{n}, \nabla\Phi) = -\operatorname{Div}\,[\mathbf{n}, \mathbf{A}] - \frac{\partial\Phi}{\partial\mathbf{n}} =$$

$$\operatorname{Div}\,((\mathbf{n},\xi)\mathbf{B}_v) - \frac{\partial\Phi}{\partial\mathbf{n}} = b_v \frac{\partial(\mathbf{n},\xi)}{\partial z} - \frac{\partial\Phi}{\partial\mathbf{n}} = 0,$$

where Div denotes the divergence operator acting on the Riemannian manifold $\mathcal{S}_p$. Analogously, on $\mathcal{S}_v$ we have

$$(\tilde{\mathbf{n}}, \omega) = 0.$$

Therefore, we have $\omega \in \Xi$ (see (2.6)), and it can be written in the form

$$\omega = \sum_{j=1,2} \sigma_j \omega_j,$$

(see Lemma 2.1) with

$$\sigma_j \int_{\mathcal{O}_v} |\omega_j|^2 \, dx = \int_{\mathcal{O}_v} (\omega, \omega_j) \, dx, \quad j = 1, 2.$$

It is easy to check the validity of the equalities

$$\int_{\mathcal{O}_v} (\omega, \omega_j) \, dx = \int_{\mathcal{S}_p} (\mathbf{n}, \xi)(\mathbf{B}_v, \omega_j) \, dS, \quad j = 1, 2.$$

Since $(\mathbf{B}_v, \omega_2) \equiv 0$ in $\overline{\mathcal{O}_v}$, we get $\sigma_2 = 0$. Hence, we obtain

$$\omega = \sigma_1 \omega_1 = \frac{\omega_1 \int_{\mathcal{S}_p} (\mathbf{n}, \xi)(\mathbf{B}_v, \omega_1) \, dS}{\int_{\mathcal{S}_p} |\omega_1|^2 \, dS} = \frac{\mathbf{B}_v \int_{\mathcal{S}_p} (\mathbf{n}, \xi) \, dS}{2\pi |\Gamma_p|}. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain (3.5).

Remark. Lemma 3.2 has been proved in a slightly different form in [Ush, Proposition 4.7]. Heuristically, this result has been known long ago (see [Lüst.Mar]). We include the proof of Lemma 3.2 for reader's convenience.

Corollary 3.1 *Let $\xi \in D_0(F)$ Then we have*

$$a_v[\xi] := \int_{\mathcal{O}_v} |\nabla \Phi(\xi)|^2 \, dx + \mathcal{C} \left| \int_{\mathcal{S}_p} (\mathbf{n}, \xi) \, dS \right|^2$$

where $\Phi(\xi)$ is the solution of the boundary-value problem (3.4), and

$$\mathcal{C} := (b_v / 2\pi |\Gamma_p|)^2 \text{ vol } \mathcal{O}_v. \tag{3.8}$$

For $k \in \mathbf{Z}$, $k \neq 0$, consider the boundary-value problem

$$\begin{cases} -\Delta \phi_k + k^2 \phi_k = 0 \quad \text{in} \quad \Omega_v, \\ \frac{\partial \phi_k}{\partial \nu} = ikb_v(\nu_1 \eta_1 + \nu_2 \eta_2) \quad \text{on} \quad \Gamma_p, \\ \frac{\partial \phi_k}{\partial \tilde{\nu}} = 0 \quad \text{on} \quad \Gamma_v, \end{cases} \tag{3.9}$$

where $\eta \in D_0[a^{(1)}]$ (see (3.3)), and ν (respectively, $\tilde{\nu}$) denotes the unit normal to Γ_p (respectively, Γ_v) vector, exterior with respect to Ω_p (respectively, Ω_v). Hence, we have $(\nu_1 \eta_1 + \nu_2 \eta_2)_{|\Gamma_p} \in H^{1/2}(\Gamma_p)$.

Lemma 3.3 *The boundary-value problem (3.9) has a unique solution* $\phi_k = \phi_k(\eta) \in H^1(\Omega_v)$, $k \in \mathbf{Z}$, $k \neq 0$. *Moreover, if* $\xi \in D_0[a]$ *is written in the form (3.1), we have*

$$\Phi(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbf{Z}\setminus\{0\}} \phi_k\left(\xi^{(k)}\right) e^{ikz}, \tag{3.10}$$

where $\Phi(\xi)$ *is the unique solution of the boundary-value problem (3.4).*

Proof. The first assertion of the lemma concerns a classical result on the solvability of the Helmholtz equation with purely imaginary non-zero frequency (see e.g. [Vla]). The second assertion is implied immediately by the symmetry of $\mathcal{O}_v$. Note that the Fourier series (3.10) does not contain a term corresponding to $k = 0$ since if we search for a solution Φ of (3.4) in the form $\Phi(y, z) = \phi_0(y)$, then ϕ_0 should satisfy the boundary-value problem

$$\begin{cases} \Delta\phi_0 = 0 & \text{in} \quad \Omega_v, \\ \frac{\partial \phi_0}{\partial n} = 0 & \text{on} \quad \Gamma_p, \\ \frac{\partial \phi_0}{\partial n} = 0 & \text{on} \quad \Gamma_v, \\ \int_{\Gamma_p} \phi_0 \, ds = 0, \end{cases}$$

and, therefore, $\phi_0 \equiv 0$.

Let $\eta \in D_0[a^{(1)}]$. Put

$$a_v^{(1)}[\eta; k] := \int_{\Omega_v} \left\{ |\nabla\phi_k(\eta)|^2 + k^2|\eta|^2 \right\} dy, \ k \in \mathbf{Z}, \ k \neq 0,$$

where $\phi_k(\eta)$ is the unique solution of the boundary-value problem (3.9). Further, put

$$a_v^{(1)}[\eta; 0] := C \left| \int_{\Gamma_p} (\nu_1\eta_1 + \nu_2\eta_2) \, ds \right|^2,$$

where C is defined in (3.8). Finally, set

$$a^{(1)}[\eta; k] := a_p^{(1)}[\eta; k] + a_v^{(1)}[\eta; k], k \in \mathbf{Z}.$$

Corollary 3.2 *Let* $\xi \in D_0(F)$ *be decomposed into the Fourier series (3.1). Then we have*

$$a[\xi] = \sum_{k \in \mathbf{Z}} a^{(1)}[\xi^{(k)}; k],$$

$$\int_{\mathcal{O}_v} \varrho |\xi|^2 \, dx = \sum_{k \in \mathbf{Z}} \int_{\Omega_v} \varrho |\xi^{(k)}|^2 \, dy.$$

3.3. It is easy to check that the non-negative quadratic form $a^{(1)}[\eta; k]$, $k \in \mathbf{Z}$, defined on $D_0[a^{(1)}]$, is closable in $\{L^2(\Omega_p; \varrho\, dy)\}^3$. Denote by $F(k)$ the selfadjoint operator generated in $\{L^2(\Omega_p; \varrho\, dy)\}^3$ by the closed quadratic form $a^{(1)}(k)$. Obviously, the force operator F is unitarily equivalent to the orthogonal sum $\sum_{k\in\mathbf{Z}} \oplus F(k)$ and, hence, we have

$$\bigcup_{k\in\mathbf{Z}} \sigma_{\rm ess}(F(k)) \subseteq \sigma_{\rm ess}(F). \tag{3.11}$$

Note that one cannot exclude *a priori* the possibility that the set at the left-hand side of (3.11) is just *contained* in $\sigma_{\rm ess}(F(k))$ since certain sequences of discrete eigenvalues of the operators $F(k)$ with different $k \in \mathbf{Z}$ might converge to points which are in $\sigma_{\rm ess}(F)$ but do not belong to any $\sigma_{\rm ess}(F(k))$, $k \in \mathbf{Z}$.

Corollary 3.3 *Assume that the equality*

$$\sigma_{\rm ess}(F(k)) = I_A(k) \cup I_B(k) \cup I_V(k), \forall k \in \mathbf{Z}, \tag{3.12}$$

holds. Then we have

$$\bigcup_{k\in\mathbf{Z}} \sigma_{\rm ess}(F(k)) = \sigma_{\rm ess}(F). \tag{3.13}$$

Proof. Set

$$e_1^- := \min_{\overline{y\in\Omega_p}} v_B^2(y), \quad e_1^+ := \max_{\overline{y\in\Omega_p}} v_B^2(y).$$

Evidently, we have

$$k^2 e_1^- = \inf\left\{I_A(k) \cup I_B(k) \cup I_V(k)\right\}, \forall k \in \mathbf{Z},$$

and, by (3.12), we get

$$\inf \sigma_{\rm ess}(F(k)) = k^2 e_1^-, \forall k \in \mathbf{Z}.$$

On the other hand, the inequality $v_B^2(y) \le v_A^2(y)$, $y \in \overline{\Omega_p}$, implies

$$a^{(1)}[\eta; k] \ge k^2 e_1^- \int_{\Omega_p} \varrho|\eta|^2\, dy, \quad \forall \eta \in D_0[a^{(1)}], \quad \forall k \in \mathbf{Z}.$$

Therefore, we obtain

$$\inf \sigma(F(k)) \ge k^2 e_1^-, \forall k \in \mathbf{Z}.$$

Hence, we have

$$\inf \sigma(F(k)) = \inf \sigma_{\rm ess}(F(k)) = k^2 e_1^-, \forall k \in \mathbf{Z}. \tag{3.14}$$

Assume at first that $e_1^- = 0$. Note that $e_1^+ > 0$ since we have assumed $P \not\equiv 0$, $b_p \not\equiv 0$. Therefore, $k^2 e_1^+$ tends to $+\infty$ as $k^2 \to \infty$. Thus, we get

$$\sigma(F) \subseteq [0, \infty) = \bigcup_{k\in\mathbf{Z}} [0, k^2 e_1^+] \equiv \bigcup_{k\in\mathbf{Z}} I_1(k) \subseteq \bigcup_{k\in\mathbf{Z}} \sigma_{\rm ess}(F(k)) \subseteq \sigma_{\rm ess}(F).$$

The trivial inclusion $\sigma_{ess}(F) \subseteq \sigma(F)$ then entails

$$\sigma(F) = \sigma_{ess}(F) = \bigcup_{k\in\mathbf{Z}} \sigma_{ess}(F(k)) = [0,\infty).$$

Assume now that e_1^- is strictly positive. Then (3.14) implies that for each $\lambda > 0$ the interval $[0,\lambda]$ may contain points of the spectra of a *finite* number of operators $F(k)$, $k \in \mathbf{Z}$. Hence, we have

$$[0,\lambda] \cap \left\{ \bigcup_{k\in\mathbf{Z}} \sigma_{ess}(F(k)) \right\} = [0,\lambda] \cap \sigma_{ess}(F), \ \forall\lambda > 0.$$

which entails (3.13).

The rest of the paper is devoted to the proof of the equalities (3.12).

4 Neumann – to – Dirichlet and Dirichlet – to – Neumann maps

4.1. Throughout this subsection the parameter $k \in \mathbf{Z}$, $k \neq 0$, is fixed. For a given $f \in H^{-1/2}(\Gamma_p)$ denote by $\tilde{\phi}_k = \tilde{\phi}_k(f) \in H^1(\Omega_v)$ the unique solution of the boundary-value problem

$$\begin{cases} -\Delta\tilde{\phi}_k + k^2\tilde{\phi}_k = 0 & \text{in} \quad \Omega_v, \\ \frac{\partial\tilde{\phi}_k}{\partial\nu} = -f & \text{on} \quad \Gamma_p, \\ \frac{\partial\tilde{\phi}_k}{\partial\tilde{\nu}} = 0 & \text{on} \quad \Gamma_v. \end{cases} \tag{4.1}$$

Note that we have $\phi_k(\eta) = -ikb_v\tilde{\phi}_k(\nu_1\eta_1 + \nu_2\eta_2)$ where $\phi_k(\eta)$ is the solution of the boundary-value problem (3.9).

Define the operator $\mathcal{N}_k : H^{-1/2}(\Gamma_p) \to H^{1/2}(\Gamma_p)$ by

$$\mathcal{N}_k f := \tilde{\phi}_k(f)_{|\Gamma_p}.$$

We denote the restriction of $\mathcal{N}_k$ onto $L^2(\Gamma_p)$ in the same way.

In the sequel we shall use classical pseudo-differential operators (ΨDOs) defined and described briefly for example in [Shu, Subsection 3.7].

Lemma 4.1 *The operator $\mathcal{N}_k$ is a selfadjoint positive compact operator in $L^2(\Gamma_p)$. Moreover, it is a classical ΨDO of order -1 whose principal symbol can be written in the local coordinates $(s,\varsigma) \in T^*\Gamma_p$ as $|\varsigma|^{-1}$ for $|\varsigma| \geq 1$.*

Proof. Let $f_1 \in H^{1/2}(\Gamma_p)$, $f_2 \in H^{1/2}(\Gamma_v)$. Set $\mathbf{f} := (f_1, f_2)$. Let $U_k = U_k(\mathbf{f}) \in H^1(\Omega_v)$ be the unique solution of the boundary-value problem

$$\begin{cases} -\Delta U_k + k^2 U_k = 0 & \text{in} \quad \Omega_v, \\ U_k = f_1 & \text{on} \quad \Gamma_p, \\ U_k = f_2 & \text{on} \quad \Gamma_v. \end{cases}$$

Define the operator $\mathcal{D}_{v,k} : H^{1/2}(\partial\Omega_v) \to H^{-1/2}(\partial\Omega_v)$ by

$$\mathcal{D}_{v,k}\mathbf{f} = \mathbf{g}$$

where $\mathbf{g} = (g_1, g_2)$ and

$$g_1 := -\frac{\partial U_k}{\partial \nu}(\mathbf{f})_{|\Gamma_p}, \quad g_2 := \frac{\partial U_k}{\partial \tilde{\nu}}(\mathbf{f})_{|\Gamma_v}.$$

It is well-known that the operator $\mathcal{D}_{v,k}$ is an elliptic classical ΨDO of order 1 whose principal symbol can be written as $|\varsigma|$ for $|\varsigma| \geq 1$. Moreover, the restriction of $\mathcal{D}_{v,k}$ onto $H^1(\partial\Omega_v)$ is a positive definite selfadjoint operator in $L^2(\partial\Omega_v)$ (see [Hör, Chapter II], [Syl.Uhl, Sections 1-2]). Since we have $L^2(\partial\Omega_v) = L^2(\Gamma_p) \oplus L^2(\Gamma_v)$, the operator $\mathcal{D}_{v,k}$ can be considered as a classical matrix ΨDO which can be written as $\mathcal{D}_{v,k} = \mathcal{D}^{(1)}_{v,k} + \mathcal{D}^{(2)}_{v,k}$ where $\mathcal{D}^{(1)}_{v,k}$ is a classical matrix selfadjoint ΨDO of order 1 whose symbol coincides with

$$\begin{pmatrix} |\varsigma| & 0 \\ 0 & |\varsigma| \end{pmatrix}, \quad |\varsigma| \geq 1,$$

and $\mathcal{D}^{(2)}_{v,k}$ is a classical matrix ΨDO of order at most 0.

Now, let $g_1 \in H^{-1/2}(\Gamma_p)$, $g_2 \in H^{-1/2}(\Gamma_v)$. Set $\mathbf{g} := (g_1, g_2)$. Let $W_k = W_k(\mathbf{g})$ be the solution of the boundary-value problem

$$\begin{cases} -\Delta W_k + k^2 W_k = 0 & \text{in} \quad \Omega_v, \\ \frac{\partial W_k}{\partial \nu} = -g_1 & \text{on} \quad \Gamma_p, \\ \frac{\partial W_k}{\partial \tilde{\nu}} = g_2 & \text{on} \quad \Gamma_v. \end{cases}$$

Define the operator $\mathcal{N}_{v,k} : H^{-1/2}(\partial\Omega_v) \to H^{1/2}(\partial\Omega_v)$ by

$$\mathcal{N}_{v,k}\mathbf{g} = \mathbf{f}$$

where $\mathbf{f} = (f_1, f_2)$ and

$$f_1 := W_k(\mathbf{g})_{|\Gamma_p}, \quad f_2 := W_k(\mathbf{g})_{|\Gamma_v}.$$

Obviously we have

$$\mathcal{N}_{v,k} = \mathcal{D}^{-1}_{v,k}, \ \forall k \in \mathbf{Z}, \ k \neq 0.$$

Therefore, the operator $\mathcal{N}_{v,k}$ is a classical matrix ΨDO which can be written as $\mathcal{N}_{v,k} = \mathcal{N}^{(1)}_{v,k} + \mathcal{N}^{(2)}_{v,k}$ where $\mathcal{N}^{(1)}_{v,k}$ is a classical matrix selfadjoint ΨDO of order -1 whose symbol coincides with

$$\begin{pmatrix} |\varsigma|^{-1} & 0 \\ 0 & |\varsigma|^{-1} \end{pmatrix},$$

for $|\varsigma| \geq 1$, and $\mathcal{N}_{v,k}^{(2)}$ is a classical matrix ΨDO of order at most -2 (see [Shu, Subsection 5.5]).

Let $f_1 \in L^2(\Gamma_p)$, $f_2 \in L^2(\Gamma_v)$ and $\mathbf{f} = (f_1, f_2) \in L^2(\partial\Omega_v)$. Define the orthogonal projection P acting in $L^2(\partial\Omega_v)$ by $P\mathbf{f} = (f_1, 0)$. If $\mathbf{f} = (f_1, 0) \in PL^2(\partial\Omega_v)$ define the isometric operator $\mathcal{I} : PL^2(\partial\Omega_v) \to L^2(\Gamma_p)$ by $\mathcal{I}\mathbf{f} = f_1$. Then we have

$$\mathcal{N}_k = \mathcal{I}P\mathcal{N}_{v,k}P\mathcal{I}^*.$$

Hence, the operator $\mathcal{N}_k$ is a positive selfadjoint classical ΨDO of order -1, and it can be written as $\mathcal{N}_k = \mathcal{N}_k^{(1)} + \mathcal{N}_k^{(2)}$ where $\mathcal{N}_k^{(1)}$ is a selfadjoint classical ΨDO of order -1 whose symbol coincides with $|\varsigma|^{-1}$ for $|\varsigma| \geq 1$, and $\mathcal{N}_k^{(2)}$ is a classical ΨDO of order at most -2. Finally, we note that since Γ_p is a compact manifold, any classical ΨDO of negative order acting in $L^2(\Gamma_p)$ (in particular, $\mathcal{N}_k$) is compact.

Corollary 4.1 *Let $f \in H^{-1/2}(\Gamma_p)$, and let $\tilde{\phi}_k(f) \in H^1(\Omega_v)$ be the solution of the boundary-value problem (4.1). Then we have*

$$\int_{\Omega_v} \left\{ |\nabla\tilde{\phi}_k|^2 + k^2|\tilde{\phi}_k|^2 \right\} dy = \int_{\Gamma_p} \left| \mathcal{N}_k^{1/2} f \right|^2 ds.$$

Proof. The identities

$$0 = \int_{\Omega_v} \left(-\Delta\tilde{\phi}_k + k^2\tilde{\phi}_k \right) \bar{\tilde{\phi}}\, dy = \int_{\Omega_v} \left\{ |\nabla\tilde{\phi}_k|^2 + k^2|\tilde{\phi}_k|^2 \right\} dy +$$

$$\int_{\Gamma_p} \frac{\partial\tilde{\phi}}{\partial\nu} \bar{\tilde{\phi}}\, ds - \int_{\Gamma_v} \frac{\partial\tilde{\phi}}{\partial\tilde{\nu}} \bar{\tilde{\phi}}\, ds$$

imply

$$\int_{\Omega_v} \left\{ |\nabla\tilde{\phi}_k|^2 + k^2|\tilde{\phi}_k|^2 \right\} dy = \int_{\Gamma_p} f\overline{\mathcal{N}_k f}\, ds = \int_{\Gamma_p} \left| \mathcal{N}_k^{1/2} f \right|^2 ds.$$

Corollary 4.2 *For each $k \in \mathbf{Z}$, $k \neq 0$, we have*

$$a_v^{(1)}[\eta; k] = k^2 b_v^2 \int_{\Gamma_p} \left| \mathcal{N}_k^{1/2}(\nu_1\eta_1 + \nu_2\eta_2) \right|^2 ds, \quad \forall \eta \in D_0[a^{(1)}].$$

4.2. It is convenient to introduce here yet another ΨDO acting in $L^2(\Gamma_p)$. For $f \in H^{1/2}(\Gamma_p)$ consider the boundary-value problem

$$\begin{cases} -\mathrm{div}\varrho\nabla\chi + \chi = 0 & \text{in} \quad \Omega_p, \\ \chi = f & \text{on} \quad \Gamma_p. \end{cases} \tag{4.2}$$

Define the operator $\mathcal{D}_p : H^{1/2}(\Gamma_p) \to H^{-1/2}(\Gamma_p)$ by

$$\mathcal{D}_p f = \varrho \frac{\partial\chi}{\partial\nu}(f)_{|\Gamma_p}.$$

We shall denote the restriction of the operator $\mathcal{D}_p$ onto $H^1(\Gamma_p)$ in the same way.

Lemma 4.2 *The operator $\mathcal{D}_p$ is selfadjoint and positive-definite in $L^2(\Gamma_p)$. Moreover, it is an elliptic classical ΨDO of order 1 whose principal symbol could be written in the local coordinates $(s,\varsigma) \in T^*\Gamma_p$ as $\varrho(s)|\varsigma|$ for $|\varsigma| \geq 1$.*

The lemma follows from the general properties of the Dirichlet–to– Neumann maps (see [Hör, Chapter II], [Syl.Uhl, Sections 1-2]).

Corollary 4.3 *Let $f \in H^{1/2}(\Gamma_p)$, and let $\chi = \chi(f)$ be the solution of the boundary-value problem (4.2). Then we have*

$$\int_{\Omega_p} \left\{ \varrho|\nabla\chi|^2 + |\chi|^2 \right\} dy = \int_{\Gamma_p} \left|\mathcal{D}_p^{1/2} f\right|^2 ds. \tag{4.3}$$

5 Weyl-Friedrichs decomposition

5.1. Our next purpose is to introduce for each $k \in \mathbf{Z}$ the scalar selfadjoint operators $F_j(k)$, $j = 1,2,3$, such that we have

$$\sigma_{\text{ess}}(F(k)) = \bigcup_{j=1,2,3} \oplus\, \sigma_{\text{ess}}\left(F_j(k)\right), \quad \forall k \in \mathbf{Z}.$$

Note that there is an approximate correspondence respectively between $F_1(k)$ and the fast magnetosonic polarization, $F_2(k)$ and the Alfvén polarization, and $F_3(k)$ and the slow magnetosonic polarization.

The argument in this subsection follows quite closely the analysis in [Rai 1, Section 3]. Since some differences caused by the change of the boundary conditions arise, we do not omit the details just for reader's convenience.

Introduce the auxiliary differential operators

$$\mathcal{M}u := -\operatorname{div} \varrho^{-1} \nabla u, \quad \mathcal{R}_k u := \mathcal{M} + k^2 \varrho^{-1}\beta_0^2, k \in \mathbf{Z},$$

on the domain

$$D(\mathcal{M}) = D(\mathcal{R}_k) = D_0[a_1^{(2)}] := \left\{ u \in H^2(\Omega_p) : u_{|\Gamma_p} = 0 \right\}.$$

Further, set

$$D_0[a_2^{(2)}] := \left\{ u \in H^2(\Omega_p) : \int_{\Omega_p} u\, dy = 0 \right\}, \quad D_0[a_2^{(3)}] := L^2(\Omega_p).$$

Finally, put

$$D_0[a^{(2)}] := \left\{ \mathrm{u} = (u_1, u_2, u_3) : u_j \in D_0[a_j^{(2)}],\ j = 1,2,3 \right\}.$$

On $D_0[a^{(2)}]$ introduce the operator

$$\mathcal{U}_k = \frac{1}{\varrho}\begin{pmatrix} \partial_1 & -\varrho\partial_2 & ik\partial_1\mathcal{M}^{-1}\beta_0 \\ \partial_2 & \varrho\partial_1 & ik\partial_2\mathcal{M}^{-1}\beta_0 \\ ik\beta_0 & 0 & \varrho \end{pmatrix}, k \in \mathbf{Z}.$$

Evidently, $\mathbf{u} \in D_0[a^{(2)}]$ entails $\mathcal{U}_k\mathbf{u} \in D_0[a^{(1)}]$.

Lemma 5.1 *The operator* $\mathcal{U}_k : D_0[a^{(2)}] \to D_0[a^{(1)}]$, $k \in \mathbf{Z}$, *is bijective.*

Proof. Fix $\eta \in D_0[a^{(1)}]$. Define u_1 as the unique solution of the operator equation

$$\mathcal{R}_k u_1 = -\left(\partial_1\eta_1 + \partial_2\eta_2 + ik\beta_0\eta_3\right).$$

Hence, $u_1 \in D(\mathcal{R}_k) = D_0[a_1^{(2)}]$. Set

$$u_3 = \eta_3 - ik\varrho^{-1}\beta_0 u_1.$$

Obviously, $u_3 \in L^2(\Omega_p) = D[a_3^{(2)}]$. Put

$$\chi_j := \eta_j - \frac{1}{\varrho}\partial_j\left(u_1 + ik\mathcal{M}^{-1}\left(\beta_0 u_3\right)\right),\ j = 1, 2.$$

Then we have

$$\partial_1\chi_1 + \partial_2\chi_2 = 0 \quad \text{in} \quad \Omega_p. \tag{5.1}$$

Fix $y_0 \in \overline{\Omega_p}$ and set

$$\tilde{u}_2(y) := \int_{y_0}^{y} \left(-\chi_1\, dy_2 + \chi_2\, dy_1\right)$$

where the integration is taken along any piece-wise smooth contour lying in $\overline{\Omega_p}$ and connecting y_0 with y. Since (5.1) holds, and Ω_p is simply connected, the function $\tilde{u}_2(y)$ is well-defined, i.e. independent of the integration contour. Put

$$u_2 = \tilde{u}_2 - \frac{1}{\operatorname{vol}\Omega_p}\int_{\Omega_p} \tilde{u}_2\, dy.$$

Then we have $\partial_1 u_2 = \chi_2$, $\partial_2 u_2 = -\chi_1$ (hence, in particular, $u_2 \in H^2(\Omega_p)$), and, moreover, $\int_{\Omega_p} u_2\, dy = 0$. Therefore, $u_2 \in D_0[a_2^{(2)}]$.

Finally, it is obvious that $\mathcal{U}_k\mathbf{u} = \eta$.

Denote by $\mathcal{H}$ the Hilbert space defined as the closure of $D_0[a^{(2)}]$ in the norm generated by the quadratic form

$$b^{(1)}[\mathbf{u}] := \int_{\Omega_p} \varrho\, |\mathcal{U}_k\mathbf{u}|^2\, dy, \quad \mathbf{u} \in D_0[a^{(2)}].$$

Note that we have

$$b^{(1)}[\mathbf{u}] = \sum_{j=1}^{3} b_j^{(1)}[u_j; k], \quad \mathbf{u} = (u_1, u_2, u_3) \in D_0[a^{(2)}],$$

where

$$b_1^{(1)}[u_1;k] = \int_{\Omega_p} \left|\mathcal{R}_k^{1/2} u_1\right|^2 dy,$$

$$b_2^{(1)}[u_2] = \int_{\Omega_p} \varrho \left|\nabla u_2\right|^2 dy,$$

$$b_3^{(1)}[u_3;k] = \int_{\Omega_p} \left\{ \varrho \left|u_3\right|^2 + k^2 \left|\mathcal{M}^{-1/2}(\beta_0 u_3)\right|^2 \right\} dy.$$

Hence, we have $\mathcal{H} = \sum_{j=1}^{3} \oplus \mathcal{H}_j$ where the Hilbert space $\mathcal{H}_1$ coincides with the set $\left\{u_1 \in H^1(\Omega_p) : u_{1|\Gamma_p} = 0\right\}$ equipped with a scalar product generated by the quadratic form $b_1^{(1)}$, the Hilbert space $\mathcal{H}_2$ coincides with the set

$$\left\{ u_2 \in H^1(\Omega_p) : \int_{\Omega_p} u_2 \, dy = 0 \right\}$$

equipped with a scalar product generated by the quadratic form $b_2^{(1)}$, and the Hilbert space $\mathcal{H}_3$ coincides with the set $L^2(\Omega_p)$ equipped with a scalar product generated by the quadratic form $b_3^{(1)}$.

Set

$$a^{(2)}[\mathrm{u}] := a^{(1)}[\mathcal{U}_k \mathrm{u}], \quad \mathrm{u} \in D_0[a^{(2)}].$$

The non-negative quadratic form $a^{(2)}[\mathrm{u};k]$ is, evidently, closable in $\mathcal{H}$, and the selfadjoint operator $\tilde{F}(k)$ generated by the closed quadratic form $a^{(2)}(k)$ in $\mathcal{H}$ is unitarily equivalent to $F(k)$, $k \in \mathbf{Z}$. Hence, in particular, we have

$$\sigma_{\mathrm{ess}}(F(k)) = \sigma_{\mathrm{ess}}(\tilde{F}(k)), \ \forall k \in \mathbf{Z}. \tag{5.2}$$

5.2. Let the scalar product in some Hilbert space $\mathbf{H}$ be generated by the quadratic form $q_0[u]$, $u \in \mathbf{H}$. Let $q[u]$ be a closed lower-bounded quadratic form in $\mathbf{H}$. We shall discuss the spectral properties of the quadratic-forms ratio q/q_0 meaning the corresponding properties of the selfadjoint operator generated by the quadratic form q in the Hilbert space $\mathbf{H}$. In particular, the equality (5.2) could be re-written as

$$\sigma_{\mathrm{ess}}(F(k)) = \sigma_{\mathrm{ess}}(a^{(2)}(k)/b^{(1)}(k)), \ \forall k \in \mathbf{Z}. \tag{5.3}$$

Lemma 5.2 *Let the scalar product in some Hilbert space* $\mathbf{H}$ *be generated by the quadratic form* q_0. *Let* q *be a closed non-negative quadratic form in* $\mathbf{H}$. *Further, let* q_1 *be a real-valued quadratic form compact in* $\mathbf{H}$ *such that the quadratic form* $q_0 + q_1$ *is positive-definite. Finally, let* q_2 *be a real-valued quadratic form compact in the Hilbert space with a scalar product generated by the quadratic form* $q[u] + q_0[u]$, $u \in D[q]$. *Then we have*

$$\sigma_{\mathrm{ess}}(q/q_0) = \sigma_{\mathrm{ess}}((q+q_2)/(q_0+q_1)). \tag{5.4}$$

Lemma 5.3 *Let the scalar product in some Hilbert space* $\mathbf{H}$ *be generated by the quadratic form* q_0. *Let* q *be a bounded real-valued quadratic form in* $\mathbf{H}$.

Further, let $\mathbf{H}_1$ *be a subspace of* $\mathbf{H}$ *such that* $\dim \mathbf{H} \ominus \mathbf{H}_1 < \infty$. *Denote by* $\tilde{q}_0$ *(respectively, by* $\tilde{q}$*) the restriction of* q_0 *(respectively, of* q*) onto* $\mathbf{H}_1$. *Then we have*

$$\sigma_{\mathrm{ess}}(\tilde{q}/\tilde{q}_0) = \sigma_{\mathrm{ess}}(q/q_0).$$

Lemmas 5.2–5.3 follow easily from the well-known Weyl theorem about the invariance of the essential spectrum of selfadjoint operators under relatively compact perturbations (see [Re.Sim, Section XIII.4]).

5.3. Set

$$a_1^{(3)}[u_1;k] := \int_{\Omega_p} \varrho v^2 \left|\mathcal{R}_k u_1\right|^2 dy, \ u_1 \in D[a_1^{(2)}(k)] =: D[a_1^{(3)}(k)], \ k \in \mathbf{Z},$$

$$b_1^{(2)}[u_1;k] := b_1^{(1)}[u_1;k], \ u_1 \in D[b_1^{(1)}(k)] =: D[b_1^{(2)}(k)], \ k \in \mathbf{Z}.$$

Further, put

$$a_2^{(3)}[u_2;k] :=$$

$$\begin{cases} k^2 \left\{ \int_{\Omega_p} \varrho v_A^2 \left|\nabla u_2\right|^2 dy + b_v^2 \int_{\Gamma_p} \left|\mathcal{N}_k^{1/2}\left(\frac{\partial u_2}{\partial s}\right)\right|^2 ds \right\}, & \text{if} \quad k \in \mathbf{Z}, \ k \neq 0, \\ 0, \ \text{if} \quad k = 0, \end{cases}$$

$$u_2 \in H^1(\Omega_p) =: D[a_2^{(3)}(k)], \quad \forall k \in \mathbf{Z},$$

$$b_2^{(2)}[u_2] := \int_{\Omega_p} \left\{ \varrho \left|\nabla u_2\right|^2 + \left|u_2\right|^2 \right\} dy, \ u_2 \in H^1(\Omega_p) =: D[b_2^{(2)}].$$

Note that we have

$$\dim D[b_2^{(2)}] \ominus D[b_2^{(1)}] = 1. \tag{5.5}$$

Next, set

$$a_3^{(3)}[u_3;k] := k^2 \int_{\Omega_p} \varrho v_B^2 \left|u_3\right|^2 dy, \ u_3 \in D[b_3^{(1)}] =: D[a_3^{(3)}(k)], \ k \in \mathbf{Z},$$

$$b_3^{(2)}[u_3] := \int_{\Omega_p} \varrho \left|u_3\right|^2 dy, \ u_3 \in D[b_3^{(1)}] =: D[b_3^{(2)}].$$

Finally, put

$$a^{(3)}[\mathbf{u}] := \sum_{j=1}^{3} a_j^{(3)}[u_j;k], \ \mathbf{u} = (u_1,u_2,u_3), u_j \in D[a_j^{(3)};k], \ j = 1,2,3, \ k \in \mathbf{Z},$$

$$b^{(2)}[\mathbf{u};k] := \sum_{j=1}^{3} b_j^{(2)}[u_j;k], \ \mathbf{u} = (u_1,u_2,u_3), u_j \in D[b_j^{(2)}], \ j = 1,2,3, \ k \in \mathbf{Z}.$$

Note that we have

$$b^{(2)}[\mathbf{u};k] - b^{(1)}[\mathbf{u};k] = \int_{\Omega_p} \left\{ \left|u_2\right|^2 - k^2 \left|\mathcal{M}^{-1/2}(\beta_0 u_3)\right|^2 \right\} dy.$$

Using the compactness of the embedding $H^1(\Omega_p) \to L^2(\Omega_p)$ and the compactness of the operator $\mathcal{M}^{-1/2}$, we get the following result.

Proposition 5.1 *For each $k \in \mathbf{Z}$ the quadratic form $b^{(2)}[\mathbf{u}; k] - b^{(1)}[\mathbf{u}; k]$ is compact in the Hilbert space with the scalar product generated by the quadratic form $b^{(2)}(k)$.*

Now, note that for $k \in \mathbf{Z}$, $k \neq 0$, we have

$$a^{(3)}[\mathbf{u}; k] - a^{(2)}[\mathbf{u}; k] =$$

$$-k^2 \left\{ \int_{\Omega_p} \varrho^{-1} \left\{ v_A^2 \left| \nabla \left(u_1 + ik\mathcal{M}^{-1}(\beta_0 u_3) \right) \right|^2 + k^2 v_B^2 \beta_0^2 |u_1|^2 \right\} dy - \right.$$

$$2\mathrm{Re} \int_{\Omega_p} \left\{ v_A^2 \left(\operatorname{curl} \overline{u}_2, \nabla \left(u_1 + ik\mathcal{M}^{-1}(\beta_0 u_3) \right) \right) + ikv_B^2 \beta_0 u_1 \overline{u}_3 \right\} dy +$$

$$b_v^2 \left\{ \int_{\Gamma_p} \left| \mathcal{N}_k^{1/2} \left(\varrho^{-1} \frac{\partial \left(u_1 + ik\mathcal{M}^{-1}(\beta_0 u_3) \right)}{\partial \nu} \right) \right|^2 ds - \right.$$

$$\left. \left. 2\mathrm{Re} \int_{\Gamma_p} \mathcal{N}_k^{1/2} \left(\varrho^{-1} \frac{\partial \left(u_1 + ik\mathcal{M}^{-1}(\beta_0 u_3) \right)}{\partial \nu} \right) \overline{\mathcal{N}_k^{1/2} \left(\frac{\partial u_2}{\partial s} \right)} ds \right\} \right\},$$

where the vector-valued function curl u_2 is defined as $(\partial_2 u_2, -\partial_1 u_2)$. Moreover, for $k = 0$ we have

$$a^{(3)}[\mathbf{u}; 0] - a^{(2)}[\mathbf{u}; 0] = \mathcal{C} \left| \int_{\Gamma_p} \varrho^{-1} \frac{\partial u_1}{\partial \nu} ds \right|^2,$$

where $\mathcal{C}$ is defined in (3.8).

Using the compactness of the embeddings $H^2(\Omega_p) \to H^1(\Omega_p)$, $H^1(\Omega_p) \to L^2(\Omega_p)$ and $H^1(\Omega_p) \to L^2(\Gamma_p)$, as well as the boundedness of the operator $\mathcal{M}^{-1}$ from $L^2(\Omega_p)$ into $H^2(\Omega_p)$ and the compactness of the operator $\mathcal{N}_k$ in $L^2(\Gamma_p)$, we obtain the following result.

Proposition 5.2 *For each $k \in \mathbf{Z}$ the quadratic form $a^{(3)}[\mathbf{u}; k] - a^{(2)}[\mathbf{u}; k]$ is compact in the Hilbert space with the scalar product generated by the closed positive-definite quadratic form $a^{(3)}[\mathbf{u}; k] + b^{(2)}[\mathbf{u}; k]$, $\mathbf{u} \in D[a^{(3)}(k)]$.*

Applying at first (5.5) combined with Lemma 5.3, and then Propositions 5.1–5.2 combined with Lemma 5.2, we obtain the following result.

Corollary 5.1 *For each $k \in \mathbf{Z}$ we have*

$$\sigma_{\mathrm{ess}}(a^{(2)}(k)/b^{(1)}(k)) = \sigma_{\mathrm{ess}}(a^{(3)}(k)/b^{(2)}(k)) = \bigcup_{j=1,2,3} \sigma_{\mathrm{ess}}(a_j^{(3)}(k)/b_j^{(2)}(k)). \quad (5.6)$$

If we denote by $F_j(k)$ the selfadjoint operator generated by the quadratic-forms ratio $a_j^{(3)}(k)/b_j^{(2)}(k)$, $j = 1,2,3$, and combine (5.3) with (5.6), we get

$$\sigma_{ess}(F(k)) = \bigcup_{j=1,2,3} \sigma_{ess}(F_j(k)), \ \forall k \in \mathbf{Z}. \tag{5.7}$$

6 The essential spectrum due to the fast and slow magnetosonic polarizations

6.1. In this subsection we investigate $\sigma_{ess}(F_1(k)) = \sigma_{ess}(a_1^{(3)}(k)/b_1^{(2)}(k))$, $k \in \mathbf{Z}$. We recall that we have

$$\frac{a_1^{(3)}[u_1;k]}{b_1^{(2)}[u_1;k]} = \frac{\int_{\Omega_p} \varrho v^2 \left|\mathcal{R}_k u_1\right|^2 dy}{\int_{\Omega_p} \left|\mathcal{R}_k^{1/2} u_1\right|^2 dy}, \quad u_1 \in D(\mathcal{M}) \equiv D(\mathcal{R}_k), \quad k \in \mathbf{Z}.$$

Hence, the operator $F_1(k)$ is unitarily equivalent to the operator $\varrho v^2 \mathcal{R}_k$ defined on $D(\mathcal{M})$, and selfadjoint in $L^2(\Omega_p; \varrho^{-1} v^{-2}\, dy)$. Obviously, this operator is elliptic, and since Ω_p is bounded, we obtain

$$\sigma_{ess}(F_1(k)) = \sigma_{ess}(\varrho v^2 \mathcal{R}_k) = \emptyset, \ \forall k \in \mathbf{Z}. \tag{6.1}$$

6.2. In this subsection we localize $\sigma_{ess}(F_3(k)) = \sigma_{ess}(a_3^{(3)}(k)/b_3^{(2)}(k))$, $k \in \mathbf{Z}$. We recall that we have

$$\frac{a_3^{(3)}[u_3;k]}{b_3^{(2)}[u_3]} = \frac{k^2 \int_{\Omega_p} \varrho v_B^2 \left|u_3\right|^2 dy}{\int_{\Omega_p} \varrho \left|u_3\right|^2 dy}, \quad u_3 \in L^2(\Omega_p), \quad k \in \mathbf{Z}.$$

Evidently, $F_3(k)$ is unitarily equivalent to the multiplier by the function $k^2 v_B^2$ in $L^2(\Omega_p)$. Hence, we get

$$\sigma_{ess}(F_2(k)) = \bigcup_{y \in \overline{\Omega_p}} \left\{k^2 v_B^2(y)\right\} \equiv I_B(k), \ \forall k \in \mathbf{Z}. \tag{6.2}$$

7 The essential spectrum due to the Alfvén polarization

7.1. At first we assume $k = 0$. Since $a_2^{(3)}[u_2; 0] \equiv 0$, we have

$$\sigma_{ess}(F_2(0)) = \sigma_{ess}(a_2^{(3)}(0)/b_2^{(2)}(0)) = \{0\}. \tag{7.1}$$

In the sequel we assume $k \neq 0$. Set

$$D[b_{2,1}^{(3)}] := \left\{w_1 \in H^1(\Omega_p) : \ w_{1|\Gamma_p} = 0\right\},$$

$$D[b_{2,2}^{(3)}] := \left\{ w_2 \in H^1(\Omega_p) : \ -\mathrm{div}\varrho\nabla w_2 + w_2 = 0 \right\}.$$

Put

$$b_{2,j}^{(3)}[w_j] := b_2^{(2)}[w_j], \ w_j \in D[b_{2,j}^{(3)}], \ j = 1,2,$$

and

$$b_2^{(3)}[\mathbf{w}] := \sum_{j=1,2} b_{2,j}^{(3)}[w_j], \ \mathbf{w} = (w_1, w_2), \ w_j \in D[b_{2,j}^{(3)}], \ j = 1,2.$$

It is convenient to recall here the representation

$$b_{2,2}^{(3)}[w_2] = \int_{\Omega_p} \left\{ \varrho|\nabla w_2|^2 + |w_2|^2 \right\} dy = \int_{\Gamma_p} \left| \mathcal{D}_p^{1/2} w_2 \right|^2 ds, \ w_2 \in D[b_{2,2}^{(3)}],$$

(see (4.3)). Evidently, we have

$$D[b_2^{(2)}] = \sum_{j=1,2} \oplus D[b_{2,j}^{(3)}].$$

For $\mathbf{w} = (w_1, w_2)$, $w_j \in D[b_{2,j}^{(3)}]$, $j = 1,2$, set

$$a_2^{(4)}[\mathbf{w}] := a_2^{(3)}[w_1 + w_2],$$

$$b_2^{(2)}[\mathbf{w}] := b_2^{(2)}[w_1 + w_2] \equiv \sum_{j=1,2} b_{2,j}^{(3)}[w_j].$$

Thus we obtain

$$\sigma_{\mathrm{ess}}(F_2(k)) \equiv \sigma_{\mathrm{ess}}(a_2^{(3)}(k)/b_2^{(2)}) = \sigma_{\mathrm{ess}}(a_2^{(4)}(k)/b_2^{(3)}), k \in \mathbf{Z}, k \neq 0. \quad (7.2)$$

7.2. Now set

$$a_{2,1}^{(5)}[w_1] := k^2 \int_{\Omega_p} \varrho v_A^2 \left|\nabla w_1\right|^2 dy, w_1 \in D[b_{2,1}^{(3)}],$$

$$b_{2,1}^{(4)}[w_1] := \int_{\Omega_p} \varrho \left|\nabla w_1\right|^2 dy, w_1 \in D[b_{2,1}^{(3)}],$$

$$a_{2,2}^{(5)}[w_2; k] := k^2 \int_{\Gamma_p} \left\{ \left| \mathcal{D}_p^{1/2}(v_A w_2) \right|^2 + b_v^2 \left| \mathcal{N}_k^{1/2} \left(\frac{\partial w_2}{\partial \hat{s}} \right) \right|^2 \right\} ds, w_2 \in D[b_{2,2}^{(3)}],$$

$$b_{2,2}^{(4)}[w_2] := b_{2,2}^{(3)}[w_2] \equiv \int_{\Gamma_p} \left| \mathcal{D}_p^{1/2} w_2 \right|^2 ds, \ w_2 \in D[b_{2,2}^{(3)}],$$

$$a_2^{(5)}[\mathbf{w}; k] = \sum_{j=1,2} a_{2,j}^{(5)}[w_j; k], \ \mathbf{w} = (w_1, w_2), \ w_j \in D[b_{2,j}^{(3)}], \ j = 1,2,$$

$$b_2^{(4)}[\mathbf{w}] = \sum_{j=1,2} b_{2,j}^{(4)}[w_j], \ \mathbf{w} = (w_1, w_2), \ w_j \in D[b_{2,j}^{(3)}], \ j = 1,2.$$

The quadratic forms

$$a_2^{(4)}[\mathbf{w}; k] - a_2^{(5)}[\mathbf{w}; k] = -k^2 \left\{ \int_{\Omega_p} \left\{ v_A^2 \left|w_2\right|^2 + \mathrm{Re}\,\varrho \left(\nabla \left(v_A^2 \right), \nabla w_2 \right) \overline{w}_2 + \right.\right.$$

$$2\text{Re}\, \varrho \left(\nabla \left(v_A^2\right), \nabla w_2\right) \overline{w}_1 + 2\text{Re}\, v_A^2 w_1 \overline{w}_2 \Big\} \, dy +$$

$$\text{Re} \int_{\Gamma_p} v_A \left([\mathcal{D}_p, v_A] w_2\right) \overline{w}_2 \, ds \Big\}$$

and

$$b_2^{(3)}[\mathbf{w}] - b_2^{(4)}[\mathbf{w}] = \int_{\Omega_p} |w_1|^2 \, dy$$

are compact in $D[a_2^{(5)}(k)] \equiv D[b_2^{(4)}] = \sum_{j=1,2} \oplus D[b_{2,j}^{(4)}]$. Applying Lemma 5.2 combined with (7.2), we get

$$\sigma_{\text{ess}}(F_2(k)) \equiv \sigma_{\text{ess}}(a_2^{(3)}(k)/b_2^{(2)}) = \bigcup_{j=1,2} \sigma_{\text{ess}}(a_{2,j}^{(5)}(k)/b_{2,j}^{(4)}), \; k \in \mathbf{Z}, \; k \neq 0. \quad (7.3)$$

7.3. In this subsection we localize $\sigma_{\text{ess}}(a_{2,1}^{(5)}(k)/b_{2,1}^{(4)})$, $k \in \mathbf{Z}$, $k \neq 0$. We recall that we have

$$\frac{a_{2,1}^{(5)}[w_1; k]}{b_{2,1}^{(4)}[w_1]} = \frac{k^2 \int_{\Omega_p} \varrho v_A^2 |\nabla w_1|^2 \, dy}{\int_{\Omega_p} \varrho |\nabla w_1|^2 \, dy}, \; w_1 \in D[b_{2,1}^{(4)}].$$

Evidently, we have

$$\sigma_{\text{ess}}(a_{2,1}^{(5)}(k)/b_{2,1}^{(4)}) \subseteq \sigma(a_{2,1}^{(5)}(k)/b_{2,1}^{(4)}) \subseteq \bigcup_{y \in \overline{\Omega_p}} \left\{k^2 v_A^2(y)\right\} \equiv I_A(k), \; k \in \mathbf{Z}, \; k \neq 0.$$

Using the singular Weyl sequence described explicitly in [Rai 1, Subsection 5.2], we conclude that each $\lambda \in I_A(k)$ belongs to $\sigma_{\text{ess}}(a_{2,1}^{(5)}(k)/b_{2,1}^{(4)})$, $k \in \mathbf{Z}$, $k \neq 0$. Hence we have

$$\sigma_{\text{ess}}(a_{2,1}^{(5)}(k)/b_{2,1}^{(4)}) = I_A(k), \; k \in \mathbf{Z}, \; k \neq 0. \quad (7.4)$$

7.4. In this subsection we localize $\sigma_{\text{ess}}(a_{2,2}^{(5)}(k)/b_{2,2}^{(4)}), k \in \mathbf{Z}, k \neq 0$. We recall that we have

$$\frac{a_{2,2}^{(5)}[w_2; k]}{b_{2,2}^{(4)}[w_2]} = \frac{k^2 \int_{\Gamma_p} \left\{ \left|\mathcal{D}_p^{1/2} (v_A w_2)\right|^2 + b_v^2 \left|\mathcal{N}_k^{1/2} \left(\frac{\partial w_2}{\partial s}\right)\right|^2 \right\} ds}{\int_{\Gamma_p} \left|\mathcal{D}_p^{1/2} w_2\right|^2 ds}, \; w_2 \in D[b_{2,2}^{(4)}].$$

Substituting the functional variable w_2 for $\mathcal{D}_p^{-1/2} w$, $w \in L^2(\Gamma_p)$, we find that the operator generated by the quadratic-forms ratio $a_{2,2}^{(5)}(k)/b_{2,2}^{(4)}$ is unitarily equivalent to the operator $k^2 T$, $k \in \mathbf{Z}$, $k \neq 0$, where

$$T := T_1^* T_1 + T_2^* T_2$$

and

$$T_1 := \mathcal{D}_p^{1/2} v_A \mathcal{D}_p^{-1/2}, \quad T_2 := -i b_v \mathcal{N}_k^{1/2} \frac{d}{ds} \mathcal{D}_p^{-1/2}.$$

Using Lemmas 4.1–4.2 and the basic properties of the classical ΨDO (see [Shu]), we deduce that T is a classical ΨDO of order 0 whose principal symbol for $|\varsigma| \geq 1$ can be written as

$$v_A^2(s) + b_v^2 \varrho^{-1}(s) \equiv \left(b_p^2(s) + b_v^2\right)/\varrho(s), s \in \Gamma_p. \tag{7.5}$$

Hence, the operator T coincides up to a compact operator with the multiplier by the function (7.5). Applying Lemma 5.2, we get

$$\sigma_{\text{ess}}(a_{2,2}^{(5)}(k)/b_{2,2}^{(4)}) = \bigcup_{s\in\Gamma_p} \left\{k^2 \left(b_p^2(s) + b_v^2\right)/\varrho(s)\right\} \equiv I_V(k), \ k \in \mathbf{Z}, \ k \neq 0. \tag{7.6}$$

Combining (7.1)–(7.4) and (7.6), we obtain

$$\sigma_{\text{ess}}(F_2(k)) = I_A(k) \cup I_V(k), \ k \in \mathbf{Z}. \tag{7.7}$$

Finally, putting together (5.7), (6.1), (6.2) and (7.7), we come to (3.12), and whence to (2.4).

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