The Integrated Density of States for a Random Schrödinger Operator in Strong Magnetic Fields. II. Asymptotics near Higher Landau Levels

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Abstract. We consider the three-dimensional Schrödinger operator with strong constant magnetic field and random electric potential. We investigate the asymptotic behaviour of its integrated density of states near the qth Landau level, for any fixed $q > 1$.

1 Introduction

In this paper we consider the three-dimensional Schrödinger operator with constant magnetic field and random scalar potential, and analyze the asymptotic behaviour of its integrated density of states (IDOS) as the intensity b of the magnetic field tends to infinity. The paper should be regarded as a continuation of [K.R] where we studied the asymptotics as $b \to \infty$ of the IDOS near the *first* Landau level, while here we consider the same type of asymptotics near the qth Landau level, $q > 1$. Here we recall briefly the basic definitions from [K.R].

Let $\mathbf{b} := (0, 0, b)$, $b > 0$, $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Introduce the unperturbed self-adjoint Schrödinger operator

$$H_0(b) := \left(i\nabla + \frac{\mathbf{b} \wedge \mathbf{x}}{2}\right)^2 \equiv \left(i\frac{\partial}{\partial x} - \frac{by}{2}\right)^2 + \left(i\frac{\partial}{\partial y} + \frac{bx}{2}\right)^2 - \frac{\partial^2}{\partial z^2}, \tag{1.1}$$

defined originally on $C_0^\infty(\mathbb{R}^3)$, and then closed in $L^2(\mathbb{R}^3)$. We have

$$\sigma(H_0(b)) = [b, +\infty), b > 0, \tag{1.2}$$

where $\sigma(H_0(b))$ denotes the spectrum of the operator $H_0(b)$ (see e.g. [A.H.S]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $V_\omega(\mathbf{x})$, $\omega \in \Omega$, $\mathbf{x} \in \mathbb{R}^3$, be a real random field. We assume that V_ω is $\mathbb{G}^3$-ergodic with $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$ (see [K, Section 3.1]). In other words, there exists an ergodic group of measure preserving automorphisms $\mathcal{T}_\mathbf{k} : \Omega \to \Omega$, $\mathbf{k} \in \mathbb{G}^3$, such that $V_\omega(\mathbf{x} + \mathbf{k}) = V_{\mathcal{T}_\mathbf{k}\omega}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$ and $\omega \in \Omega$. We recall that ergodicity of a group G of automorphisms of Ω means that the G-invariance of a given set $\mathcal{A} \in \mathcal{F}$ implies either $\mathbb{P}(\mathcal{A}) = 1$ or $\mathbb{P}(\mathcal{A}) = 0$.

For $\mathbf{x} \in \mathbb{R}^3$ we write $\mathbf{x} = (X, z)$ with $X \in \mathbb{R}^2$, $z \in \mathbb{R}$. Hence, z is the variable along the magnetic field $\mathbf{b} = (0, 0, b)$, while X runs over the plane perpendicular to $\mathbf{b}$.

We suppose that V_ω is $\mathbb{G}$-ergodic with $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$ in the direction of the magnetic field, i.e. that the subgroup $\{\mathcal{T}_{\mathbf{k}} | \mathbf{k} = (0,0,k),\ k \in \mathbb{G}\}$ is ergodic. Further, we assume that the realizations of V_ω are almost surely uniformly bounded, i.e.

$$c_0 := \mathrm{ess} - \sup_{\omega \in \Omega} \sup_{\mathbf{x} \in \mathbb{R}^3} |V_\omega(\mathbf{x})| < \infty. \tag{1.3}$$

Finally, we suppose that the realizations of V_ω are almost surely continuous.

Let T be a selfadjoint operator in a Hilbert space. Denote by $P_{\mathcal{I}}(T)$ its spectral projection corresponding to the interval $\mathcal{I} \subset \mathbb{R}$. Set $N(\lambda; T) := \mathrm{rank}\, P_{(-\infty,\lambda)}(T)$, $\lambda \in \mathbb{R}$. If $T = T^*$ is compact, put $n_\pm(s;T) := \mathrm{rank}\, P_{(s,+\infty)}(\pm T)$, $s > 0$. Finally, if T is a linear compact operator in Hilbert space which is not necessarily self-adjoint, set $n_*(s;T) := \mathrm{rank}\, P_{(s^2,+\infty)}(T^*T)$, $s > 0$.

On $D(H_0(b))$ define the perturbed Schrödinger operator $H(b,\omega) := H_0(b) + V_\omega$. On the Sobolev space $\mathrm{H}^2\left(\left(-\frac{R}{2}, \frac{R}{2}\right)^3\right)$ with Dirichlet boundary conditions, define the operator $H_{0,R}^D(b) := \left(i\nabla + \frac{\mathbf{b}\wedge\mathbf{x}}{2}\right)^2$. Then there exists a non-random non-decreasing function $\mathcal{D}_b : \mathbb{R} \to \mathbb{R}_+$ such that almost surely

$$\lim_{R\to\infty} R^{-3}\, N(\mu; H_{0,R}^D(b) + V_\omega) = \mathcal{D}_b(\mu), \tag{1.4}$$

provided that $\mu \in \mathbb{R}$ is a continuity point of $\mathcal{D}_b$ (see [N], [H.L.M.W]). The function $\mathcal{D}_b(\mu)$, $\mu \in \mathbb{R}$, is called the IDOS for the operator $H(b,\omega)$.
In this paper we consider the asymptotic behaviour as $b \to \infty$ of $\mathcal{D}_b(\lambda_2 + (2q-1)b) - \mathcal{D}_b(\lambda_1 + (2q-1)b)$, the parameters λ_1, $\lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, and $q \in \mathbb{N}_* := \{1, 2, \ldots\}$, being fixed. Recall that the numbers $\{(2q-1)b\}_{q=1}^\infty$ are called Landau levels.

2 Statement of Main Result

Let $h_{0,R} := -\frac{d^2}{dz^2}$ be the self-adjoint operator defined on $\mathrm{H}^2\left(\left(-\frac{R}{2}, \frac{R}{2}\right)\right)$ with Dirichlet boundary conditions.

Proposition 2.1 ([K, Chapter7], [P.Fi, Chapter III]) *Let $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$. Let $f_\omega(z)$, $\omega \in \Omega$, $z \in \mathbb{R}$, be a real $\mathbb{G}$-ergodic random field whose realizations are almost surely uniformly bounded and continuous. Then for each $\lambda \in \mathbb{R}$ the limit*

$$\varrho(\lambda; f) := \lim_{R\to\infty} R^{-1}\, N(\lambda; h_{0,R} + f_\omega) \tag{2.1}$$

exists almost surely. Moreover, the function $\varrho(\lambda; f)$ is non-random, and continuous with respect to $\lambda \in \mathbb{R}$.

Our assumptions concerning V_ω guarantee that the random field $f_\omega = V_\omega(X, .)$ depending on the parameter $X \in \mathbb{R}^2$, satisfies the hypotheses of Proposition 2.1.

Moreover, if $\mathbb{G} = \mathbb{Z}$, then the function $\varrho(\lambda; V(X,.))$ is periodic with respect to $X \in \mathbb{R}^2$, while in the case $\mathbb{G} = \mathbb{R}$ the quantity $\varrho(\lambda; V(X,.))$ is independent of $X \in \mathbb{R}^2$ (see [K.R]). For $\lambda \in \mathbb{R}$ set

$$k(\lambda) = k(\lambda; V) := \begin{cases} \int_{(-\frac{1}{2},\frac{1}{2})^2} \varrho(\lambda, V(X,.))\, dX & \text{if} \quad \mathbb{G} = \mathbb{Z}, \\ \varrho(\lambda, V(0,.)) & \text{if} \quad \mathbb{G} = \mathbb{R}. \end{cases}$$

Obviously, $k(\lambda)$ is non-decreasing and continuous with respect to λ.

Theorem 2.1. *Let $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$. Let V_ω be a real $\mathbb{G}^3$-ergodic random field whose realizations are almost surely uniformly bounded and continuous. Assume in addition that V_ω is $\mathbb{G}$-ergodic in the direction of the magnetic field. Then for each $q \in \mathbb{N}_*$, and $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, we have*

$$\lim_{b \to \infty} b^{-1} \left(\mathcal{D}_b(\lambda_2 + (2q-1)b) - \mathcal{D}_b(\lambda_1 + (2q-1)b) \right) = \frac{1}{2\pi} \left(k(\lambda_2) - k(\lambda_1) \right). \quad (2.2)$$

Theorem 2.1 contains the asymptotics as $b \to \infty$ of the IDOS $\mathcal{D}_b$ near the qth Landau level, $q \geq 1$. Since (2.2) has been proved in [K.R] for $q = 1$, we shall prove it here for $q > 1$. The methods applied in this paper are similar to the ones used in [K.R], and are based on the Birman-Schwinger principle (see [B, Lemma 1.1]), a suitable version of the Kac-Murdock-Szegö theorem (see [R, Lemma 3.2]), and the Birkhoff-Khintchine ergodic theorem. However, the analysis near higher Landau levels is more complicated since the first Landau level coincides with lower bound of the spectrum of $H_0(b)$ (see (1.2)), while the higher Landau levels $(2q-1)b$, $q > 1$, are internal points of $\sigma(H_0(b))$. The proof of Theorem 2.1 can be found in Section 4, while Section 3 contains preliminary estimates.

3 Preliminary Estimates

3.1. Let $H_{0,R}^N(b)$ be the self-adjoint operator generated in $L^2\left(\left(-\frac{R}{2}, \frac{R}{2}\right)^3\right)$ by the closed quadratic form $\int_{(-\frac{R}{2},\frac{R}{2})^3} \left| i\nabla u + \frac{\mathbf{b} \wedge \mathbf{x}}{2} u \right|^2 d\mathbf{x}$, $u \in \mathrm{H}^1\left(\left(-\frac{R}{2}, \frac{R}{2}\right)^3\right)$.

Lemma 3.1. ([N, Theorem 1], [H.L.M.W, Theorem 3.1]) *Let $\mathbb{G} = \mathbb{Z}$ or $\mathbb{G} = \mathbb{R}$. Assume that V_ω is a real $\mathbb{G}^3$-ergodic random field whose realizations are almost surely uniformly bounded and continuous. Let $\mu \in \mathbb{R}$ be a continuity point of $\mathcal{D}_b$. Then almost surely*

$$\mathcal{D}_b(\mu) = \lim_{R \to \infty} R^{-3} N(\mu; H_{0,R}^N(b) + V_\omega). \quad (3.1)$$

Set $\chi_R(\mathbf{x}) := \mathbf{1}_{(-\frac{R}{2},\frac{R}{2})^3}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$. On $D(H_0(b))$ introduce the operator $H_0(b) + (V_\omega - \mu)\chi_R$ with $b > 0$, $\omega \in \Omega$, $\mu \in \mathbb{R}$, $R > 0$.

Proposition 3.1 *Under the hypotheses of Lemma 3.1 almost surely*

$$\mathcal{D}_b(\mu) = \lim_{R\to\infty} R^{-3}\, N(0; H_0(b) + (V_\omega - \mu)\chi_R). \tag{3.2}$$

Proof: By the minimax principle,

$$N(\mu; H_{0,R}^D + V_\omega) = N(0; H_{0,R}^D + V_\omega - \mu) \leq N(0; H_0(b) + (V_\omega - \mu)\chi_R). \tag{3.3}$$

Set $\mathcal{O}_R := \mathbb{R}^3 \setminus \left[-\frac{R}{2}, \frac{R}{2}\right]^3$. Denote by $\tilde{H}_{0,R}^N$ the self-adjoint operator generated by the quadratic form $\int_{\mathcal{O}_R} \left|i\nabla u + \frac{\mathbf{b}\wedge\mathbf{x}}{2}u\right|^2 d\mathbf{x}$, defined initially for $u \in C_0^\infty\left(\overline{\mathcal{O}_R}\right)$, and then closed in $L^2(\mathcal{O}_R)$. The minimax principle implies

$$N(0; H_0(b) + (V_\omega - \mu)\chi_R) \leq N(0; H_{0,R}^N + V_\omega - \mu) + N(0; \tilde{H}_{0,R}^N). \tag{3.4}$$

Since $\tilde{H}_{0,R}^N \geq 0$, we have $N(0; \tilde{H}_{0,R}^N) = 0$. Hence, (3.4) can be re-written as

$$N(0; H_0(b) + (V_\omega - \mu)\chi_R) \leq N(0; H_{0,R}^N + V_\omega - \mu) = N(\mu; H_{0,R}^N + V_\omega). \tag{3.5}$$

Combining (3.3) and (3.5) with (1.4) and (3.1), we get (3.2). $\diamondsuit$

Remark. Proposition 3.1 is very similar to [K.R, Proposition 4.1]. However, the proof presented here is much simpler because now we dispose of Lemma 3.1.

Introduce the compact Birman-Schwinger-type operators

$$T(\mu) = T_{b,\omega,R}(\mu) := H_0(b)^{-1/2}(V_\omega - \mu)\chi_R H_0(b)^{-1/2}, \ \mu \in \mathbb{R}, \tag{3.6}$$

$$\tilde{T} = \tilde{T}_{b,R} := H_0(b)^{-1/2}\chi_R H_0(b)^{-1/2}, \tag{3.7}$$

so that we have $T(\mu) = T(0) - \mu\tilde{T}$.

Corollary 3.1 *Under the assumptions of Lemma 3.1 almost surely*

$$\mathcal{D}_b(\mu) = \lim_{R\to\infty} R^{-3}\, n_-(1; T_{b,\omega,R}(\mu)). \tag{3.8}$$

Proof: It suffices to recall (3.2), and to apply the Birman-Schwinger principle. $\diamondsuit$

3.2. Let $\mathcal{H}_0(b) := \left(i\frac{\partial}{\partial x} - \frac{by}{2}\right)^2 + \left(i\frac{\partial}{\partial y} + \frac{bx}{2}\right)^2$ be the selfadjoint operator defined originally on $C_0^\infty(\mathbb{R}^2)$, and then closed in $L^2(\mathbb{R}^2)$. The spectrum of $\mathcal{H}_0(b)$ coincides with the set of the Landau levels, i.e. $\sigma(\mathcal{H}_0(b)) = \bigcup_{q=1}^\infty \{(2q-1)b\}$. Fix $q \geq 1$. Denote by $p_q = p_{q,b} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ the orthogonal projection onto the eigenspace of $\mathcal{H}_0(b)$ associated with the qth Landau level $(2q-1)b$. In other words, $p_q w = w$ implies $w \in D(\mathcal{H}_0(b))$ and $\mathcal{H}_0(b)w = (2q-1)bw$. It is well-known that

$$(p_q w)(x,y) = \int_{\mathbb{R}^2} \mathcal{P}_q(x,y;x',y')w(x',y')\, dx'dy', \quad w \in L^2(\mathbb{R}^2),$$

with

$$\mathcal{P}_q(x,y;x',y') :=$$
$$\frac{b}{2\pi}e^{-\frac{b}{4}\left[(x-x')^2+(y-y')^2+2i(xy'-yx')\right]}L_{q-1}\left(\frac{(x-x')^2+(y-y')^2}{2}\right) \tag{3.9}$$

where $L_s(\xi) := \frac{1}{s!}e^{\xi}\frac{d^s}{d\xi^s}\left(\xi^s e^{-\xi}\right)$, $s \geq 0$, $\xi \in \mathbb{R}$, is the Laguerre polynomial of order s. Note that we have

$$\mathcal{P}_q(x,y;x,y) = \frac{b}{2\pi},\ (x,y) \in \mathbb{R}^2,\ \forall q \in \mathbb{N}_*. \tag{3.10}$$

Define the orthogonal projection $P_q = P_{q,b} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ by $P_q{:=}\int_{\mathbb{R}}^{\oplus} p_q\,dz$, i.e.

$$(P_q u)(x,y,z) = \int_{\mathbb{R}^2}\mathcal{P}_q(x,y;x',y')u(x',y',z)\,dx'dy', \quad u \in L^2(\mathbb{R}^3).$$

Then P_q commutes with $H_0(b)$ and $\frac{\partial}{\partial z}$, and we have

$$H_0(b)P_q u = \left(-\frac{\partial^2}{\partial z^2} + (2q-1)b\right)P_q u, \quad u \in D(H_0(b)), \tag{3.11}$$

(see (1.1)). For $\gamma > 0$ define the operator

$$r(\gamma) := \left(-\frac{\partial^2}{\partial z^2} + \gamma\right)^{-1/2}, \tag{3.12}$$

bounded and selfadjoint in $L^2(\mathbb{R}^3)$. Evidently,

$$(r(\gamma)^2 u)(x,y,z) = \frac{1}{2\sqrt{\gamma}}\int_{\mathbb{R}} e^{-\sqrt{\gamma}|z-z'|}u(x,y,z')\,dz', \quad u \in L^2(\mathbb{R}^3). \tag{3.13}$$

Moreover, the operators P_q and $r(\gamma)$ commute.

3.3. In this subsection we estimate a quantity which yields the main asymptotic term as $b \to \infty$ of $\mathcal{D}_b(\lambda_2 + (2q-1)b) - \mathcal{D}_b(\lambda_1 + (2q-1)b)$.

Proposition 3.2 *Let the hypotheses of Theorem 2.1 hold. Then for every* $\lambda \in \mathbb{R}$, $q \in \mathbb{N}_*$, $s > 0$, *and* $\gamma > 0$ *almost surely*

$$\lim_{b\to\infty}\lim_{R\to\infty}\frac{1}{bR^3}n_-(s; r(\gamma)P_q(V_\omega - \lambda - \gamma)\chi_R P_q r(\gamma)) = \frac{1}{2\pi}k\left(\frac{\lambda+\gamma}{s} - \gamma; \frac{1}{s}V\right). \tag{3.14}$$

Idea of the proof: The proof is analogous to the one of [K.R, Corollary 4.3] which corresponds to $q = 1$ and $\gamma = 1$. By (3.9), (3.10), and (3.13), the extension to general q and γ can be carried out in a quite straightforward manner. $\diamondsuit$

Let $\phi : [b_0, +\infty) \to \mathbb{R}_+$, $b_0 > 0$, be a strictly decreasing function such that $\phi(b_0) < 1$ and $\lim_{b\to\infty} b\phi(b) = 0$.

Corollary 3.2 *Let the hypotheses of Theorem 2.1 hold. Then for every $\lambda \in \mathbb{R}$, $q \in \mathbb{N}_*$, almost surely*

$$\lim_{b\to\infty}\lim_{R\to\infty} b^{-1}R^{-3}n_-(1\pm\phi(b); P_qT(\lambda+(2q-1)b)P_q) = \frac{1}{2\pi}k(\lambda;V), \qquad (3.15)$$

the operator $T(\mu)$ being defined in (3.6).

Proof: First of all note that

$$P_qT(\lambda+(2q-1)b)P_q = r((2q-1)b)P_q(V_\omega-\lambda-(2q-1)b)\chi_RP_qr((2q-1)b)$$

(see (3.6), (3.11) and (3.12)). Then the Birman-Schwinger principle entails

$$n_-(1\pm\phi(b); P_qT(\lambda+(2q-1)b)P_q)) =$$

$$N\left(0; -\frac{\partial^2}{\partial z^2}+(2q-1)b+\varphi_\pm(b)P_q(V_\omega-\lambda-(2q-1)b)\chi_RP_q\right), \qquad (3.16)$$

where $\varphi_\pm(b) := 1/(1\pm\phi(b))$. Note that $\lim_{b\to\infty}\varphi_\pm(b) = 1$, $\lim_{b\to\infty} b(1-\varphi_\pm(b)) = 0$. Let us re-arrange the terms appearing at the right-hand side of (3.16). We have

$$-\frac{\partial^2}{\partial z^2}+(2q-1)b+\varphi_\pm(b)P_q(V_\omega-\lambda-(2q-1)b)\chi_RP_q = -\frac{\partial^2}{\partial z^2}+\gamma+E_{q,b}+W^\pm_{q,b}, \quad (3.17)$$

where $\gamma > 0$ is a fixed number, $E_{q,b} := ((2q-1)b-\gamma)(1-P_q\chi_RP_q)$, and $W^\pm_{q,b} := P_q\{V_\omega-\lambda-\gamma-(1-\varphi_\pm(b))(V_\omega-\lambda-(2q-1)b)\}\chi_RP_q$. Fix $\delta > 0$, and assume that b is so large that we have

$$P_q(V_\omega-\lambda-\gamma-\delta)\chi_RP_q \leq W^\pm_{q,b} \leq P_q(V_\omega-\lambda-\gamma+\delta)\chi_RP_q. \qquad (3.18)$$

Assume as well that $b > \gamma/(2q-1)$ so that the operator $E_{q,b}$ is non-negative. By the minimax principle, (3.16) – (3.18) imply

$$N\left(0; -\frac{\partial^2}{\partial z^2}+\gamma+E_{q,b}+P_q(V_\omega-\lambda-\gamma+\delta)\chi_RP_q\right) \leq$$

$$n_-(1\pm\phi(b); P_qT(\lambda+(2q-1)b)P_q) \leq N(0; -\frac{\partial^2}{\partial z^2}+\gamma+P_q(V_\omega-\lambda-\gamma-\delta)\chi_RP_q). \qquad (3.19)$$

By the Birman-Schwinger principle,

$$N\left(0; -\frac{\partial^2}{\partial z^2}+\gamma+P_q(V_\omega-\lambda-\gamma-\delta)\chi_RP_q\right) =$$

$$n_-(1; r(\gamma)P_q(V_\omega-\lambda-\gamma-\delta)\chi_RP_qr(\gamma)). \qquad (3.20)$$

It follows from (3.14) with $s = 1$ that

$$\lim_{b\to\infty}\lim_{R\to\infty}\frac{1}{bR^3}n_-(1; r(\gamma)P_q(V_\omega - \lambda - \gamma - \delta)\chi_R P_q r(\gamma)) = \frac{1}{2\pi}k(\lambda+\delta; V). \quad (3.21)$$

The second inequality in (3.19), and (3.20) - (3.21) imply that for any $\delta > 0$

$$\limsup_{b\to\infty}\limsup_{R\to\infty}\frac{1}{bR^3}n_-(1\pm\phi(b); P_qT(\lambda+(2q-1)b)P_q) \le \frac{1}{2\pi}k(\lambda+\delta; V). \quad (3.22)$$

Now assume that $\gamma > 0$ is so large that almost surely $V_\omega(\mathbf{x})-\lambda-\gamma+\delta \le 0$ for every $\mathbf{x}\in\mathbb{R}^3$. Then we have $P_q(V_\omega-\lambda-\gamma+\delta)\chi_R P_q = -S^*S$ with $S := U\chi_R P_q$, and $U := \sqrt{\gamma+\lambda-V_\omega-\delta}$. Introduce the operator $\tilde{r}(\gamma) := \left(-\frac{\partial^2}{\partial z^2}+\gamma+E_{q,b}\right)^{-1/2}$, bounded and self-adjoint in $L^2(\mathbb{R}^3)$. By the Birman-Schwinger principle,

$$N\left(0; -\frac{\partial^2}{\partial z^2}+\gamma+E_{q,b}+P_q(V_\omega-\lambda-\gamma+\delta)\chi_R P_q\right) = n_+(1; \tilde{r}(\gamma)S^*S\tilde{r}(\gamma)) =$$

$$n_*(1; S\tilde{r}(\gamma)) = n_*(1; \tilde{r}(\gamma)S^*) = n_+(1; S\tilde{r}(\gamma)^2S^*). \quad (3.23)$$

Applying the resolvent identity $\tilde{r}(\gamma)^2 = r(\gamma)^2 - r(\gamma)^2E_{q,b}\tilde{r}(\gamma)^2$, we get

$$n_+(1; S\tilde{r}(\gamma)^2S^*) \ge n_+(1+\varepsilon; Sr(\gamma)^2S^*) - n_+(\varepsilon; Sr(\gamma)^2E_{q,b}\tilde{r}(\gamma)^2S^*),\ \forall\varepsilon>0. \quad (3.24)$$

Let us estimate the first term at the right-hand side of (3.24). We have

$$n_+(1+\varepsilon; Sr(\gamma)^2S^*) = n_+(1+\varepsilon; r(\gamma)S^*Sr(\gamma)) =$$

$$n_-(1+\varepsilon; r(\gamma)P_q(V_\omega-\lambda-\gamma+\delta)\chi_R P_q r(\gamma)),\ \forall\varepsilon>0. \quad (3.25)$$

By (3.14) with $s = 1+\varepsilon$,

$$\lim_{b\to\infty}\lim_{R\to\infty} b^{-1}R^{-3}n_-(1+\varepsilon; r(\gamma)P_q(V_\omega-\lambda-\gamma+\delta)\chi_R P_q r(\gamma))$$

$$= \frac{1}{2\pi}k\left(\frac{\lambda+\gamma-\delta}{1+\varepsilon}-\gamma; \frac{V}{1+\varepsilon}\right). \quad (3.26)$$

Let us now estimate the second term at the right-hand side of (3.24). We have

$$Sr(\gamma)^2E_{q,b}\tilde{r}(\gamma)^2S^* = ((2q-1)b-\gamma)U\chi_R P_q r(\gamma)^2(1-P_q\chi_R P_q)\tilde{r}(\gamma)^2S^*.$$

The operators $((2q-1)b-\gamma)U$ and $\tilde{r}(\gamma)^2S^*$ are uniformly bounded with respect to R. Therefore,

$$n_+(\varepsilon; Sr(\gamma)^2E_{q,b}\tilde{r}(\gamma)^2S^*) \le n_*(\eta; \chi_R P_q r(\gamma)^2(1-P_q\chi_R P_q)) \quad (3.27)$$

where $\eta > 0$ is independent of R. Further,

$$n_*(\eta; \chi_R P_q r(\gamma)^2(1-P_q\chi_R P_q)) \le \eta^{-2}\|\chi_R P_q r(\gamma)^2(1-P_q\chi_R P_q)\|_{HS}^2, \quad (3.28)$$

$\|.\|_{HS}$ being the Hilbert-Schmidt norm. A straightforward calculation yields

$$\lim_{R\to\infty} R^{-3}\|\chi_R P_q r(\gamma)^2(1-P_q\chi_R P_q)\|_{HS}^2 = 0. \tag{3.29}$$

Putting together (3.27)–(3.29), we obtain

$$\lim_{R\to\infty} R^{-3}n_+(\varepsilon; Sr(\gamma)^2 E_{q,b}\tilde{r}(\gamma)^2 S^*) = 0, \ \forall\varepsilon > 0. \tag{3.30}$$

Combining the first inequality in (3.19) with (3.23)–(3.26) and (3.30), we get

$$\liminf_{b\to\infty}\liminf_{R\to\infty} b^{-1}R^{-3}n_-(1\pm\phi(b); P_qT(\lambda+(2q-1)b)P_q) \geq$$

$$\frac{1}{2\pi}k\left(\frac{\lambda+\gamma-\delta}{1+\varepsilon}-\gamma; \frac{V}{1+\varepsilon}\right). \tag{3.31}$$

Letting $\varepsilon\downarrow 0$ in (3.31), and then $\delta\downarrow 0$ in (3.22) and (3.31), and taking into account the boundedness of V_ω and the continuity of k, we arrive at (3.15). $\diamondsuit$

3.4. In this subsection we estimate certain quantities which do not contribute to the main asymptotic term as $b\to\infty$ of $\mathcal{D}_b(\lambda_2+(2q-1)b) - \mathcal{D}_b(\lambda_1+(2q-1)b)$. Introduce the orthogonal projections $P_q^- := \sum_{s=1}^{q-1} P_s$, $q>1$, $\tilde{P}_q^- := \sum_{s=1}^{q} P_s$, $q\geq 1$, and $P_q^+ := \sum_{s=q+1}^{\infty} P_s$, $q\geq 1$.

Proposition 3.3 *Assume that (1.3) holds. Let $\mu_-, \mu_+ \in \mathbb{R}$, $\mu_- < \mu_+$, $q>1$. Then almost surely*

$$\lim_{b\to\infty}\limsup_{R\to\infty} b^{-1}R^{-3}(n_-(1-\phi(b); P_q^-T(\mu_+ + (2q-1)b)\chi_R P_q^-)$$

$$-n_-(1+\phi(b); P_q^-T(\mu_- + (2q-1)b)\chi_R P_q^-) = 0. \tag{3.32}$$

Proof. Assume that b is so large that $(2q-1)b+\mu_\pm \pm c_0 > 0$, c_0 being defined in (1.3). Set $\psi_\pm(b) := (1\mp\phi(b))/((2q-1)b+\mu_\pm\pm c_0)$. Evidently,

$$n_-(1-\phi(b); P_q^-T(\mu_+ + (2q-1)b)\chi_R P_q^-) \leq n_+(\psi_+(b); P_q^-\tilde{T}P_q^-), \tag{3.33}$$

$$n_-(1+\phi(b); P_q^-T(\mu_- + (2q-1)b)\chi_R P_q^-) \geq n_+(\psi_-(b); P_q^-\tilde{T}P_q^-), \tag{3.34}$$

$\tilde{T}$ being defined in (3.7). Obviously, $P_q^-\tilde{T}P_q^-$ is a trace-class operator, and

$$\lim_{R\to\infty} R^{-3}\mathrm{Tr}\,(P_q^-\tilde{T}P_q^-)^l = \frac{b}{(2\pi)^2}\sum_{s=1}^{q-1}\int_{\mathbb{R}}\frac{d\zeta}{(\zeta^2+(2s-1)b)^l}, \ l\in\mathbb{N}_*.$$

Applying the Kac-Murdock-Szegö theorem (see [R, Lemma 3.2]), we get

$$\lim_{R\to\infty} R^{-3}n_+(\varepsilon; P_q^-\tilde{T}P_q^-) = \frac{b}{(2\pi)^2}\sum_{s=1}^{q-1}\mathrm{meas}\,\{\zeta\in\mathbb{R}|(\zeta^2+(2s-1)b)^{-1}>\varepsilon\} =$$

$$\frac{b}{2\pi^2}\sum_{s=1}^{q-1}\left(\varepsilon^{-1}-(2s-1)b\right)_+^{1/2}, \ \forall\varepsilon>0, \forall b>0. \tag{3.35}$$

Combining (3.33)–(3.35), we get

$$\limsup_{R\to\infty} R^{-3}(n_-(1-\phi(b); P_q^- T(\mu_+ + (2q-1)b)\chi_R P_q^-)$$

$$-n_-(1+\phi(b); P_q^- T(\mu_- + (2q-1)b)\chi_R P_q^-)) \le$$

$$\frac{b}{2\pi^2}\sum_{s=1}^{q-1}\left\{(\psi_+(b)^{-1}-(2s-1)b)_+^{1/2}-(\psi_-(b)^{-1}-(2s-1)b)_+^{1/2}\right\}. \tag{3.36}$$

Rationalizing, we find that for each $s=1,\ldots,q-1$, we have

$$\lim_{b\to\infty}\left\{(\psi_+(b)^{-1}-(2s-1)b)_+^{1/2}-(\psi_-(b)^{-1}-(2s-1)b)_+^{1/2}\right\}=0. \tag{3.37}$$

Now, (3.32) follows from (3.36) and (3.37). ♢

Proposition 3.4 *Assume that (1.3) holds. Fix $\lambda\in\mathbb{R}$. Then here exists $b_*>0$ independent of R, such that $b>b_*$ implies*

$$n_-(1\pm\phi(b)); P_q^+ T(\lambda+(2q-1)b)\chi_R P_q^+) = 0, \ \forall R>0. \tag{3.38}$$

Proof: It suffices to note that $\lim_{b\to\infty}(1\pm\phi(b))=1$, and $\limsup_{b\to\infty}||P_q^+ T(\lambda+(2q-1)b)\chi_R P_q^+|| \le \lim_{b\to\infty}\frac{c_0+|\lambda|+(2q-1)b}{(2q+1)b} = \frac{2q-1}{2q+1} < 1$. ♢

Proposition 3.5 *For each $\varepsilon>0$ and $q<1$ we have*

$$\lim_{R\to\infty} R^{-3} n_\pm(\varepsilon; 2\mathrm{Re}\,(P_q^-\tilde{T}P_q + \tilde{P}_q^-\tilde{T}P_q^+)) = 0. \tag{3.39}$$

Proof: Write the estimates $n_\pm(\varepsilon; 2\mathrm{Re}(P_q^-\tilde{T}P_q+\tilde{P}_q^-\tilde{T}P_q^+)) \le 2n_*(\varepsilon/2; P_q^-\tilde{T}P_q) + 2n_*(\varepsilon/2; \tilde{P}_q^-\tilde{T}P_q^+) \le 8\varepsilon^{-2}\left(||P_q^-\tilde{T}P_q||_{HS}^2 + ||\tilde{P}_q^-\tilde{T}P_q^+||_{HS}^2\right)$, and verify by direct calculation that $\lim_{R\to\infty} R^{-3}||P_q^-\tilde{T}P_q||_{HS}^2 = \lim_{R\to\infty} R^{-3}||\tilde{P}_q^-\tilde{T}P_q^+||_{HS}^2 = 0$. ♢

4 Proof of Theorem 2.1

Fix $q>1$, $\lambda_1,\lambda_2\in\mathbb{R}$, $\lambda_1<\lambda_2$. In order to prove (2.2), it suffices to show that for each sequence $\{b_j\}_{j\ge1}$ such that $b_j\to\infty$ as $j\to\infty$, we have

$$\lim_{j\to\infty} b_j^{-1}\left(\mathcal{D}_{b_j}(\lambda_2+(2q-1)b_j) - \mathcal{D}_{b_j}(\lambda_1+(2q-1)b_j)\right) = \frac{1}{2\pi}\left(k(\lambda_2)-k(\lambda_1)\right). \tag{4.1}$$

Fix four sequences $\{\lambda_{l,m}^{\pm}\}_{m\geq 1}$, $l = 1,2$, such that $\lambda_{l,m}^{-} < \lambda_l < \lambda_{l,m}^{+}$, $m \geq 1$, $\lim_{m\to\infty} \lambda_{l,m}^{\pm} = \lambda_l$, and $\lambda_{l,m}^{\pm} + (2q-1)b_j$ are continuity points of $\mathcal{D}_{b_j}$ for all $m \geq 1$ and $j \geq 1$, $l = 1,2$. Then by Corollary 3.1 we have

$$\limsup_{j\to\infty} b_j^{-1}\left(\mathcal{D}_{b_j}(\lambda_2 + (2q-1)b_j) - \mathcal{D}_{b_j}(\lambda_1 + (2q-1)b_j)\right) \leq$$

$$\limsup_{j\to\infty} \lim_{R\to\infty} b_j^{-1} R^{-3}(n_-(1; T(\lambda_{2,m}^{+} + (2q-1)b_j)) - n_-(1; T(\lambda_{1,m}^{-} + (2q-1)b_j))), \tag{4.2}$$

$$\liminf_{j\to\infty} b_j^{-1}\left(\mathcal{D}_{b_j}(\lambda_2 + (2q-1)b_j) - \mathcal{D}_{b_j}(\lambda_1 + (2q-1)b_j)\right) \geq$$

$$\liminf_{j\to\infty} \lim_{R\to\infty} b_j^{-1} R^{-3}(n_-(1; T(\lambda_{2,m}^{-} + (2q-1)b_j)) - n_-(1; T(\lambda_{1,m}^{+} + (2q-1)b_j))). \tag{4.3}$$

Further, note that the elementary operator inequalities

$$T(\lambda + (2q-1)b) \geq P_q T(\lambda + (2q-1)b + 2c_0\delta)P_q +$$
$$P_q^- T(\lambda + (2q-1)b + c_0(1+\delta^{-1}))P_q^- + P_q^+ T(\lambda + (2q-1)b + c_0(1+\delta^{-1}))P_q^+ -$$
$$2(\lambda + (2q-1)b)\mathrm{Re}(P_q^+ \tilde{T} \tilde{P}_q^- + P_q^- \tilde{T} P_q),$$
$$T(\lambda + (2q-1)b) \leq P_q T(\lambda + (2q-1)b - 2c_0\delta)P_q +$$
$$P_q^- T(\lambda + (2q-1)b - c_0(1+\delta^{-1}))P_q^- + P_q^+ T(\lambda + (2q-1)b - c_0(1+\delta^{-1}))P_q^+ -$$
$$2(\lambda + (2q-1)b)\mathrm{Re}(P_q^+ \tilde{T} \tilde{P}_q^- + P_q^- \tilde{T} P_q),$$

are valid for each $\lambda \in \mathbb{R}$, $b > 0$, and $\delta > 0$. Therefore,

$$n_-(1; T(\lambda + (2q-1)b)) \leq n_-(1-\phi(b); P_q T(\lambda + (2q-1)b + 2c_0\delta)P_q) +$$
$$n_-(1-\phi(b); P_q^- T(\lambda + (2q-1)b + c_0(1+\delta^{-1}))P_q^-) +$$
$$n_-(1-\phi(b); P_q^+ T(\lambda + (2q-1)b + c_0(1+\delta^{-1}))P_q^+) +$$
$$n_+(\phi(b); 2(\lambda + (2q-1)b)\mathrm{Re}(P_q^+ \tilde{T} \tilde{P}_q^- + P_q^- \tilde{T} P_q)),$$

and

$$n_-(1; T(\lambda + (2q-1)b)) \geq n_-(1+\phi(b); P_q T(\lambda + (2q-1)b - 2c_0\delta)P_q) +$$
$$n_-(1+\phi(b); P_q^- T(\lambda + (2q-1)b - c_0(1+\delta^{-1}))P_q^-) +$$
$$n_-(1+\phi(b); P_q^+ T(\lambda + (2q-1)b - c_0(1+\delta^{-1}))P_q^+) -$$
$$n_-(\phi(b); 2(\lambda + (2q-1)b)\mathrm{Re}(P_q^+ \tilde{T} \tilde{P}_q^- + P_q^- \tilde{T} P_q)),$$

the numerical function ϕ being introduced before Corollary 3.2. Hence, we get

$$\limsup_{j\to\infty} \lim_{R\to\infty} b_j^{-1} R^{-3}(n_-(1; T(\lambda_{2,m}^{+} + (2q-1)b_j)) - n_-(1; T(\lambda_{1,m}^{-} + (2q-1)b_j))) \leq$$

$$\limsup_{j\to\infty}\limsup_{R\to\infty} b_j^{-1}R^{-3}(n_-(1-\phi(b); P_qT(\lambda_{2,m}^+ + (2q-1)b_j + 2c_0\delta)P_q)$$
$$-n_-(1+\phi(b); P_qT(\lambda_{1,m}^- + (2q-1)b_j - 2c_0\delta)P_q))+$$
$$\limsup_{j\to\infty}\limsup_{R\to\infty} b_j^{-1}R^{-3}(n_-(1-\phi(b); P_q^-T(\lambda_{2,m}^+ + (2q-1)b_j + c_0(1+\delta^{-1}))P_q^-)$$
$$-n_-(1+\phi(b); P_q^-T(\lambda_{1,m}^- + (2q-1)b_j - c_0(1+\delta^{-1}))P_q^-))+$$
$$\limsup_{j\to\infty}\limsup_{R\to\infty} b_j^{-1}R^{-3}n_-(1-\phi(b); P_q^+T(\lambda_{2,m}^+ + (2q-1)b_j + c_0(1+\delta^{-1})P_q^+)+$$
$$\limsup_{j\to\infty}\limsup_{R\to\infty} b_j^{-1}R^{-3}(n_+(\phi(b); 2(\lambda_{2,m}^+ + (2q-1)b_j)\mathrm{Re}(P_q^+\tilde{T}\tilde{P}_q^- + P_q^-\tilde{T}P_q))$$
$$+n_-(\phi(b); 2(\lambda_{1,m}^- + (2q-1)b)\mathrm{Re}(P_q^+\tilde{T}\tilde{P}_q^- + P_q^-\tilde{T}P_q))), \tag{4.4}$$

and

$$\liminf_{j\to\infty}\lim_{R\to\infty} b_j^{-1}R^{-3}(n_-(1; T(\lambda_{2,m}^- + (2q-1)b_j)) - n_-(1; T(\lambda_{1,m}^+ + (2q-1)b_j))) \geq$$
$$\liminf_{j\to\infty}\liminf_{R\to\infty} b_j^{-1}R^{-3}(n_-(1+\phi(b); P_qT(\lambda_{2,m}^- + (2q-1)b_j - 2c_0\delta)P_q)$$
$$-n_-(1-\phi(b); P_qT(\lambda_{1,m}^+ + (2q-1)b_j + 2c_0\delta)P_q))-$$
$$\liminf_{j\to\infty}\liminf_{R\to\infty} b_j^{-1}R^{-3}(n_-(1-\phi(b); P_q^-T(\lambda_{1,m}^+ + (2q-1)b_j + c_0(1+\delta^{-1}))P_q^-)$$
$$-n_-(1+\phi(b); P_q^-T(\lambda_{2,m}^- + (2q-1)b_j - c_0(1+\delta^{-1}))P_q^-))-$$
$$\liminf_{j\to\infty}\liminf_{R\to\infty} b_j^{-1}R^{-3}n_-(1-\phi(b); P_q^+T(\lambda_{1,m}^+ + (2q-1)b_j + c_0(1+\delta^{-1}))P_q^+)-$$
$$\liminf_{j\to\infty}\liminf_{R\to\infty} b_j^{-1}R^{-3}(n_+(\phi(b); 2(\lambda_{1,m}^+ + (2q-1)b_j)\mathrm{Re}(P_q^+\tilde{T}\tilde{P}_q^- + P_q^-\tilde{T}P_q))$$
$$+n_-(\phi(b); 2(\lambda_{2,m}^- + (2q-1)b)\mathrm{Re}(P_q^+\tilde{T}\tilde{P}_q^- + P_q^-\tilde{T}P_q))). \tag{4.5}$$

Employing Corollary 3.2, we find that the first term at the right-hand side of (4.4) is equal to $\frac{1}{2\pi}\left(k(\lambda_{2,m}^+ + 2c_0\delta)) - k(\lambda_{1,m}^- - 2c_0\delta)\right)$, while the first term at the side of (4.5) is equal to $\frac{1}{2\pi}\left(k(\lambda_{2,m}^- - 2c_0\delta)) - k(\lambda_{1,m}^+ + 2c_0\delta)\right)$. Assume that $\delta > 0$ is small enough, and apply Proposition 3.3 to the second terms at the right-hand sides of (4.4) and (4.5) in order to check that these terms vanish. Similarly, utilize Proposition 3.4 (respectively, Proposition 3.5) in order to verify that the third (respectively, fourth) terms at the right-hand sides of (4.4) and (4.5) vanish. Putting together (4.2) – (4.5), we get

$$\limsup_{j\to\infty} b_j^{-1}\left(\mathcal{D}_{b_j}(\lambda_2 + (2q-1)b_j) - \mathcal{D}_{b_j}(\lambda_1 + (2q-1)b_j)\right) \leq$$
$$\frac{1}{2\pi}\left(k(\lambda_{2,m}^+ + 2c_0\delta)) - k(\lambda_{1,m}^- - 2c_0\delta)\right), \tag{4.6}$$
$$\liminf_{j\to\infty} b_j^{-1}\left(\mathcal{D}_{b_j}(\lambda_2 + (2q-1)b_j) - \mathcal{D}_{b_j}(\lambda_1 + (2q-1)b_j)\right) \geq$$
$$\frac{1}{2\pi}\left(k(\lambda_{2,m}^- - 2c_0\delta)) - k(\lambda_{1,m}^+ + 2c_0\delta)\right). \tag{4.7}$$

Letting $m \to \infty$, and $\delta \downarrow 0$ in (4.6)-(4.7), we obtain (4.1).

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