CLOSED-FORM EXPECTED QUEUEING TIMES OF A TANDEM SYSTEM WITH FINITE INTERMEDIATE BUFFER: A RECURSIVE APPROACH

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Abstract—In this paper we deal with a two-station tandem queueing system incorporating a finite intermediate buffer between the stations. Blocking occurs when a customer is momentarily stopped at the first station owing to a capacity limitation of the second station having been reached. We assume general input and exponentially distributed service times. We observe that it is possible to calculate the conditional expected queueing times at the first station (conditioned on the system state just before an arrival epoch), recursively, and we give closed-form expressions for them.

INTRODUCTION

We consider an open tandem queueing network consisting of two stations. Each station consists of a queue and a server. The first station has an infinite buffer capacity while the second station has finite buffer of maximum size $M$ (i.e., the maximum number of customers that can be simultaneously present at the second station is $M + 1$). Customers arriving according to a homogeneous process with parameter $\lambda$ are served in order of arrival by the two-station service system. When a customer's service at the first station is completed, he moves on to the second stage of the system, provided that there is at least one available space there at that time. If the second stage is fully occupied the customer will wait in the first station and block it until a customer from the second station departs. No customer can enter service at the first station while it is blocked. We assume the service times at both stations are independent and exponentially distributed with parameters $\mu_1$ and $\mu_2$, respectively.

Queueing networks with blocking have proved difficult to treat, in general, because of the fact that the output process at any node of a network is no longer Poisson. For this reason, closed-form solutions are not generally attainable. There is an extensive literature dealing with tandem queues with finite capacity. Most of the techniques employed for the analysis of such queueing systems are in the form of analytic approximations, numerical and simulation techniques. Some exact closed-form results of tandem queues with blocking have been reported in the literature, under different assumptions regarding the arrival process, service distribution, priority disciplines and feedback. A survey of exact and approximate results of tandem queues with blocking can be found in [1–8]. Finally, a bibliography of papers that report on analytic investigations of queueing networks with blocking, is given in [9]. It has not been previously observed, however, that it is possible to calculate the conditional expected queueing times (conditioned on the system state just before an arrival epoch) of a tandem queue recursively or to give closed-form expressions for them. It should be noted that, unlike the overwhelming majority of results mentioned above, our methodology does not require Poisson input.

EXPECTED QUEUEING TIMES: A RECURSIVE APPROACH

In order to begin a formal study of the model, it is convenient to define the state of the system and to adopt the following terminology and notation:
Let \((n, k, j)\) define the state of the system where:
- \(n = \) number of customers in the queue and in service at the first station, including the blocked customer (if any), \(n \geq 0;\)
- \(k = \) number of customers at the second stage of the system (buffer and second station), \(k = 0, 1, 2, \ldots, M + 1;\)
- \(j = \) number of blocked customers, \(j = 0, 1;\)

Define:

\(T_{nkj} = \) expected time in queue (in front of the first station) for a customer arriving to find the system in state \((n, k, j)\).

Consider a tagged customer who arrives to find the system in state \((n, k, j)\) and note that his expected queueing time at the first station is unaffected by subsequent arrivals. In fact, this expected queueing time depends only on the dynamics of the system as he moves up the queue by virtue of previous arrivals leaving the first station.

We derive relationships between the expected queueing times in terms of the "truncated" state of the system taking into account only customers that are ahead of the tagged customer, as such customers depart the first station.

In view of the fact that service times are exponential, and hence memoryless, and because an arriving customer has an expected queueing time that depends on the previous arrivals, we obtain the following recursive relationships for \(T_{nkj}\):

For \(n = 0:\)

\[ T_{0i0} = 0 \quad i = 0, 1, \ldots, M, M + 1. \]

For \(n \geq 1:\)

\[ T_{n00} = \frac{1}{\mu_1} + T_{n-1,1,0} \]

\[ T_{n10} = r_1 \left( \frac{1}{\mu_1} + T_{n-1,2,0} \right) + r_2 T_{n00} \]

\[ T_{n20} = r_1 \left( \frac{1}{\mu_1} + T_{n-1,3,0} \right) + r_2 T_{n10} \]

\[ T_{n20} = r_1 \left( \frac{1}{\mu_1} + T_{n-1,4,0} \right) + r_2 T_{n20} \]

\[ T_{n00} = r_1 \left( \frac{1}{\mu_1} + T_{n-1,1,1} \right) + r_2 T_{n00} \]

\[ T_{nM0} = r_1 \left( \frac{1}{\mu_1} + T_{n-1,M+1,0} \right) + r_2 T_{n,M-1,0} \]

\[ T_{n,M+1,0} = r_1 \left( \frac{1}{\mu_1} + T_{n,M+1,1} \right) + r_2 T_{n,M0} \]

\[ T_{n,M+1,1} = \frac{1}{\mu_1} + T_{n-1,M+1,0} \]

Note that \(r_1 = \mu_1/(\mu_1 + \mu_2)\) is the probability that the customer at the first station completes his service before the one at the second station does, and also that \(r_2 = 1 - r_1\).

Let \(A_i(z) = \sum_{n=0}^{\infty} T_{nio} z^n, i = 0, 1, \ldots, M + 1,\) be the \(z\)-transform for \(T_{n00}\). After some simplifications of the above set of equations, we obtain the following recursive:

\[ A_0(z) = \frac{1}{\mu_1} \frac{z}{1 - z} + z A_1(z), \]

\[ A_i(z) = \frac{r_1}{\mu_1} \frac{z}{1 - z} + r_1 \frac{z}{1 - z} A_{i+1}(z) + r_2 A_{i-1}(z), \quad i = 1, 2, \ldots, M; \]

\[ A_{M+1}(z) = \frac{1}{\mu_2} \frac{z}{1 - z} + r_1 \frac{z}{1 - z} A_{M+1}(z) + r_2 A_M(z). \]

This functional system of equations for \(A_i(z),\) which is associated with \(T_{n00},\) may be easily solved and the resulting expressions for \(A_i(z)\) may then be expanded into a power series to obtain \(T_{n00},\) the coefficient of \(z^n\) (by using partial fraction decomposition).
CONCLUSION

In recent years, there has been a growing interest in the development of methods to evaluate the stationary probability vector, the mean queueing times and the blocking probabilities for queueing networks with blocking. In this paper we develop an exact approach, based on the formulation and solution of a set of simultaneous recurrence relations, to obtain closed-form conditional expected queueing times of a two-station tandem queueing network model. The same methodology can be used to obtain results for other features of this problem, such as the conditional blocking probabilities [10].

REFERENCES