Optimal pension fund management under multi-period risk minimization

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Abstract

In this paper, a multi-period stochastic optimization model for solving a problem of optimal selection of a pension fund by a pension plan member is presented. In our model, a member of the pension plan is given a possibility to switch periodically between m types of the funds with different risk profile and so actively manage her risk exposure and expected return. Minimization the multi-period risk measure under a return constraint leads to an efficient frontier. A theoretical framework and the solution for the case of the pension system of Slovak Republic is presented.

Key words: pension fund management, large scale optimization, linear programming, multi-period risk measures

Several countries adopted a pension reform in recent decades. Pension systems in different countries are very different. They depend on a legal situation of the particular country. Description of pension systems of some countries and studies on pension fund management can be found e.g. in Pflug and Swietanowski (1999) (Austria), or Haneveld, Streutker, and van der Vlerk (2006) (Netherlands). In this paper, we will focus on the pension system of Slovak Republic. A detailed description of the Slovak pension reform can be found in Goliaš (2003), Melicherčík and Ungvarský (2004).

Since its reform in 2003, the pension system in Slovak Republic is based on three pillars: first - the mandatory non-funded (so-called pay-as-you-go system with a regular contribution rate of 9% from the gross salary), second - the mandatory fully funded (based on private savings of future pensioners) , and the third - voluntary fully funded pillar. Assets in the 2^{nd} pillar are managed in pension fund administration companies, and each of them is obliged to create three types of funds with different risk profile: Growth fund, Balanced fund and Conservative fund (see Table 1). Upon entrance of the 2^{nd} pillar, new participants choose one of the funds and are allowed to revise their decision and switch to other fund every 12 months. Hence, future pensioners are able to partially influence their final wealth and risk by balancing between the three fund types. The problem of determining an optimal strategy for these decisions was discussed and analyzed in a recent paper by Kilianová, Melicherčík and Ševčovič (2006) using the idea of maximization of saver's utility function.

In this paper we find the optimal strategy by minimizing the risk expectations arising from the balancing the funds. We solve a general problem with J funds assuming continuous balancing[1]. The optimization is applied to discretely generated scenarios of fund returns and is based on minimizing the average value-at-risk deviation ($AVaRD$) measure (known also as the conditional value-at-risk deviation $CVaRD$). We consider both a one-period and a multi-period $AVaRD$ measures. The experiments confirm the generally observed trend of investing from the most risky fund in the first years to less

[1]Notice that the current law requires discrete balancing in the sense that the future pensioner can choose only one fund each time. This has been addressed in recent paper by Kilianová, Melicherčík and Ševčovič (2006)

risky funds later.

The paper is organized as follows. In the first section we formulate the problem as a linear problem defined on a tree. We discuss natural constraints that have to be taken into account. We define the objective function as the static average value-at-risk deviation of the terminal wealth random variable. In the second section we formulate a model for a dynamic risk measure as an objective function, which is a linear program as well. However, when the decisions are not taken every year, a nonlinear constraint appears. In the third section, we pay attention to the numerical implementation of these two models. We present an iterative algorithm in order to overcome the problem with the nonlinear constraint. Then we discuss the data used for computation and we describe generation of the scenario tree. Finally, we discuss the results obtained for the Slovak pension system.

1 Terminal risk

We begin with a model for a future pensioner who is interested in his terminal wealth at the time of retirement T but does not care the level of his savings prior to it. Using a static risk measure we minimize the risk (uncertainty) of reaching his terminal wealth target.

1.1 Linear constraints

First, we describe natural constraints of the problem. Let $t \in \{0, ..., T\}$ denote the time and $j \in \{1, ..., J\}$ be different funds with stochastic returns r_t^j. By y_t we denote the ratio of the absolute value of the savings to the wage level in time t. This value is distributed into J funds in proportions $y_t^j, j \in \{1, ..., J\}$. We denote by (bold) $\mathbf{y}_t$ the vector $[y_t^1, ..., y_t^J]^\top$. Being $\mathbf{1}$ an unit vector, it holds: $y_t = \mathbf{y}_t^\top \mathbf{1} = \sum_{j=1}^J y_t^j$ and $y_t^j \geq 0$ for all t, j.

The growth rate ϱ_t of the wage w_t is assumed to be prescribed and follows a deterministic process $w_{t+1} = w_t(1 + \varrho_t)$. Let $s_t^j = \frac{1+r_t^j}{1+\varrho_t}$ and $\mathbf{s}_t = [s_t^1, ..., s_t^J]^\top$ be the adjusted fund returns. The process of the wealth evolution depends on balancing between funds and follows the formulae (1)-(4)

$$
\begin{aligned}
&(1) \qquad & \mathbf{y}_0^\top \mathbf{1} &= \tau\,, \\
&(2) \qquad & \mathbf{y}_t^\top \mathbf{1} &= \mathbf{y}_{t-1}^\top \mathbf{s}_t + \tau \quad \text{for all } t \in \{1, ..., T-1\}, \\
&(3) \qquad & \mathbf{y}_T^\top \mathbf{1} &= \mathbf{y}_{T-1}^\top \mathbf{s}_T\,, \\
&(4) \qquad & \mathbf{y}_t &\geq 0 \quad \text{for all } t \in \{1, ..., T\}
\end{aligned}
$$

where $\tau > 0$ denotes the yearly regular contribution as a percentage of the salary. The startup saving at $t = 0$ equals the first contribution τ. At the end of the saving ($t = T$), no contribution is added.

1.2 Objective function

Let us denote by V_T the *terminal wealth* random variable. There are several risk measures that can be used to measure the riskiness of V_T. A concept for appropriate one-period risk measures is given by Artzner et al. (1999).

Let F be the distribution function of the terminal wealth V_T and F^{-1} the corresponding quantile function $F^{-1}(p) = \inf\{u : F(u) \geq p\}$. The *value-at-risk* VaR at the confidence level α is defined as $VaR_\alpha(V_T) = F^{-1}(\alpha)$. Then the *average value-at-risk* (also known as *conditional value-at-risk* $CVaR$) at the level α is defined as $AVaR_\alpha(V_T) = \frac{1}{\alpha}\int_0^\alpha F^{-1}(p)dp$. Our aim is to maximize the $AVaR_\alpha$, or - equivalently - to minimize *the average value-at-risk deviation* defined by $AVaRD_\alpha(V_T) = \mathbb{E}(V_T) - AVaR_\alpha(V_T)$ under constraints (1)-(4) and a requirement of some minimal target wealth level μ, i.e. $\mathbb{E}(V_T) \geq \mu$.

1.3 Tree representation

Adjusted returns s_t^j of funds $j = 1, ..., J$, form a stochastic process in a discrete time t. It can be approximated by a scenario tree. We use the consecutive numbering of nodes. Let 0 be the root and $\mathcal{N} = \{0, 1, ..., N\}$ the set of all nodes in the tree, the last S of them being terminal nodes. Next, we denote by $\mathcal{T} = \{N - S + 1, ..., N\}$ the set of all terminal nodes, and $\mathcal{N}_0 = \{1, ..., N - S\}$. Each node n (except the root n=0) has exactly one predecessor n_-, each node $n \in \mathcal{N} \setminus \mathcal{T}$ has a set of successors $\{n\}^+$, and the node $n \in \mathcal{N}$ is located in a time stage $\xi(n) \in \{0, ..., T\}$.

With a slight abuse of notation, we adjust the notation for tree representation and we use y_n to denote the level of savings in the node n. Now the index represents the number of a node instead of the time t. We denote by y_n^j the amount of money (relativized to the wage level) invested in the fund j and we set $\mathbf{y}_n = [y_n^1, ..., y_n^J]^\top$. Clearly, $y_n = \mathbf{y}_n^\top \mathbf{1}$ and $y_n^j \geq 0$ for all n, j. Next, let $s_n^j = \frac{1+r_n^j}{1+\rho_{\xi(n)}}$ be the adjusted return of the fund j in the period from node n_- to n and $\mathbf{s}_n = [s_n^1, ..., s_n^J]^\top$. The terminal wealth random variable V_T is represented by the vector of possible values y_m, $m \in \mathcal{T}$, with corresponding probabilities p_m, $\sum_{m \in \mathcal{T}} p_m = 1$.

It follows from the paper by Rockafellar and Uryasev (2000) that the average value-at-risk can be rewritten as $AVaR_\alpha(V_T) = \max_{a \in \mathbb{R}} \{a - \frac{1}{\alpha}\mathbb{E}([V_T - a]^-)\}$ where $[g]^- = \max\{-g, 0\}$ is the negative part of g. Then the problem of minimizing the $AVaRD_\alpha(V_T)$ with targeting the terminal wealth is of the form

$$\min_{y,a,z} \left(\sum_{m \in \mathcal{T}} p_m (\mathbf{y}_m^\top \mathbf{1}) - a + \frac{1}{\alpha} \sum_{m \in \mathcal{T}} p_m z_{m-N+S} \right) \tag{5}$$

subject to

$$-a + \mathbf{y}_m^\top \mathbf{1} + z_{m-N+S} \geq 0, \quad z_{m-N+S} \geq 0, \quad \text{for all } m \in \mathcal{T}, \tag{6}$$

$$\sum_{m \in \mathcal{T}} p_m (\mathbf{y}_m^\top \mathbf{1}) \geq \mu\,, \tag{7}$$

$$\mathbf{y}_0^\top \mathbf{1} = \tau\,, \quad \mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n + \tau \quad \text{for all } n \in \mathcal{N}_0, \tag{8}$$

$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n \quad \text{for all } n \in \mathcal{T}, \tag{9}$$

$$\mathbf{y}_n \geq 0 \quad \text{for all } n \in \mathcal{N}, \tag{10}$$

where (7) represents the target wealth constraint, conditions (8)-(10) are the relations (1)-(4) adjusted to the tree notation. It is worth to note that (6) implies $z_{m-N+S} \geq [y_m - a]^- = \max(0, a - y_m)$. Therefore for the optimal solution (5)-(10) we have $z_{m-N+S} = [y_m - a]^-$. Indeed, if $z_{m-N+S} > [y_m - a]^-$ we can find a lower value of z_{m-N+S} yielding a lower value of the objective function. Therefore a solution $\mathbf{y}$ to (5)-(10) is an AVaRD minimizer subject to (6)-(10).

Problem (5)-(10) can be represented in a form of a linear program:

$$
\begin{aligned}
&\min_{\mathbf{x}=(a,\mathbf{z},\mathbf{y})} \mathbf{c}^\top \mathbf{x} \quad \text{subject to} \\
&\mathbf{A}_{ineq}\mathbf{x} \leq \mathbf{b}_{ineq}, \quad \mathbf{A}_{eq}\mathbf{x} = \mathbf{b}_{eq} \qquad (11)\\
&\mathbf{y}_n \geq 0, z_m \geq 0 \quad \text{for all } n \in \mathcal{N}, m \in \{1, ..., S\}.
\end{aligned}
$$

The vector of variables $\mathbf{x}$ has the length $vars = 1 + S + J(1 + N)$. The matrix $\mathbf{A}_{ineq}$ is of type $(1 + S) \times vars$ and it is a sparse matrix with $(2J + 2)S$ nonzero elements. The $(1 + N) \times vars$ matrix $\mathbf{A}_{eq}$ has $(1 + 2N)J$ nonzero elements.

2 Intermediate risk

Pension saving is a long-term investment where the terminal level of the savings is a main subject of pensioner's interest. However, in the case of saver's early death, the saved money becomes the subject of heritage and therefore it is reasonable to be also interested in the intermediate level of saver's assets rather than optimize the terminal wealth only.

The notation $\mathbf{y}_{\{n\}^+}$ stands for the set $\{\mathbf{y}_k, k \in \{n\}^+\}$ of vectors of wealth in nodes from the first to the last successor of $n \in \mathcal{N} \setminus \mathcal{T}$. According to Pflug (2006), a multi-period (dynamic) average value-at-risk deviation is defined as the sum of the conditional $AVaRD$ measures in nodes $n \in \mathcal{N} \setminus \mathcal{T}$, i.e. $\sum_{n\in\mathcal{N}\setminus\mathcal{T}} AVaRD_\alpha(y_{\{n\}^+})$. Each of the summands in this expression is calculated using conditional probabilities $pc(k) = p(k)/\sum_{l\in\{n\}^+} p(l)$ where $k \in \{n\}^+$, $n \in \mathcal{N} \setminus \mathcal{T}$. Hence, we minimize the term $\sum_{n\in\mathcal{N}\setminus\mathcal{T}} \big(\mathbb{E}(\mathbf{y}|n) - AVaR_\alpha(\mathbf{y}_{\{n\}^+})\big)$. The term $\mathbb{E}(\mathbf{y}|n)$ denotes the conditional mean in node n, i.e. the conditional mean of wealth in its successors $\{n\}^+$. The problem of minimizing the multi-period risk measured by the dynamic average value-at-risk deviation is of the form

$$
\min_{a,z,y} \sum_{n\in\mathcal{N}\setminus\mathcal{T}} \left(\sum_{k\in\{n\}^+} \big(pc(k)\mathbf{y}_k^\top \mathbf{1}\big) - a_n + \frac{1}{\alpha} \sum_{k\in\{n\}^+} pc(k) z_{kn} \right) \qquad (12)
$$

subject to

$$-a_n + \mathbf{y}_k^\top \mathbf{1} + z_{kn} \geq 0, \quad z_{kn} \geq 0, \quad \text{for all } k \in \{n\}^+, n \in \mathcal{N} \setminus \mathcal{T}, \tag{13}$$

$$\sum_{m \in \mathcal{T}} p_m \mathbf{y}_m^\top \mathbf{1} \geq \mu, \tag{14}$$

$$\mathbf{y}_0^\top \mathbf{1} = \tau, \quad \mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n + \tau \quad \text{for all } n \in \mathcal{N}_0, \tag{15}$$

$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n-}^\top \mathbf{s}_n \quad \text{for all } n \in \mathcal{T}, \tag{16}$$

$$\mathbf{y}_n \geq 0 \quad \text{for all } n \in \mathcal{N}. \tag{17}$$

Similarly to (5)-(10), it can be easily shown that the solution to (12)-(17) minimizes the dynamic AVaRD risk measure.

The above optimization problem can be represented as a linear program having the matrix form (11) with vector of variables $\mathbf{x}$ of length $vars = 1 + N - S + N + 3 * (1 + N)$. Both matrices A_{ineq} and A_{eq} have a sparse structure with size $(1 + N) \times vars$ and they have $JS + (J + 2)N$ resp. $(1 + 2N)J$ nonzero elements.

2.1 A nonlinear constraint

A typical active life-span of a future pensioner is 40 years. Since the complexity of a tree with 40 periods makes it computationally hard to handle (hardware limits), we reduce the dimesion of the problem assuming the saver makes the decision only in years $t_0 = 0, t_1, ..., t_{\widetilde{T}}$, represented by the tree stages $0, 1, ..., \widetilde{T}$.

Let l_k denote the length of the period $[t_{k-1}, t_k], k \in \{1, ..., \widetilde{T}\}$. The basic problems (5)-(10), (12)-(17) are good models if $l_k = 1$ for all k. However, savings are appreciated l_k-times during the period $[t_{k-1}, t_k]$. Furthermore, they are l_k-times credited by contribution τ (see Figure 1) that are each time allocated into funds j so that funds' weights keep constant at least till time t_k: if $\tau_n = [\tau_n^1, ..., \tau_n^J]^\top$ is the vector of yearly contributions transferred to funds during the period $[t_{\xi(n)}, t_{\xi(n)+1}]$, from the node n to any of its successors from the set $\{n\}^+$, $n \in \mathcal{N} \setminus \mathcal{T}$, then

$$\frac{\tau_n^j}{\tau} = \frac{y_n^j}{\mathbf{y}_n^\top \mathbf{1}}. \tag{18}$$

Of course, $\tau = \tau_n^\top \mathbf{1}$ for all $n \in \mathcal{N} \setminus \mathcal{T}$. Problems (5)-(10), (12)-(17) have to be modified in constraints (8), (15), concerning the appreciation of the wealth in nonterminal stages:

$$\mathbf{y}_n^\top \mathbf{1} = \mathbf{y}_{n_-}^\top \mathbf{s}_n + \tau_{n_-}^\top \sum_{i=0}^{l_{\xi(n)}-1} (\mathbf{s}_n)^{i/l_{\xi(n)}} \quad \text{for all } n \in \mathcal{N}_0 \,. \tag{19}$$

The power $(\mathbf{s}_n)^{i/l_{\xi(n)}}$ of the vector of yearly adjusted returns is meant componentwise, i.e. $(\mathbf{s}_n)^{i/l_{\xi(n)}} = [(s_n^1)^{i/l_{\xi(n)}}, ..., (s_n^J)^{i/l_{\xi(n)}}]^\top$. Constraint (19) makes the modified problems (5)-(10), (12)-(17) nonlinear.

3 Numerical implementation

In this section we implement (5)-(10) and (12)-(17) for the case of the Slovak pension system with three funds. In order to overcome the nonlinearity comprised in constraint (19) we use an iterative algorithm for solving the problems. We discuss the data used for computation and describe generation of the scenario tree. We implement this problem in MATLAB and we use the built-in `linprog` function for calculating the linear optimization problem.

3.1 An iterative algorithm

In Section 2.1 we showed that there is a nonlinear constraint in implementation of the problems (5)-(10) resp. (12)-(17). We overcome the problem with the nonlinearity by linearization of the problem using the following iterative algorithm:

1. fix $\tau_n^j = \tau / J$ (the starting point) for all $n \in \mathcal{N} \setminus \mathcal{T}, j \in \{1, ..., J\}$;
2. solve the linear problem (5)-(10) resp. (12)-(17), with constraint (8) resp. (15), modified to (19) with fixed parameters τ_n^j;
3. obtain optimal y_n^j for all n, j. Compute new τ_n^j for all n, j by (18);
4. repeat steps 2, 3, 4, until defined accuracy is achieved.

As a stopping criterion we take a value of the difference between optimal values of the objective function in two consecutive iterations, $\epsilon = |\mathbf{c}^\top \mathbf{x} - \mathbf{c}^\top \tilde{\mathbf{x}}|$. Here $\mathbf{c}$ is the vector representing the linear objective function (different for each problem), $\mathbf{x}$ denotes the solution from the new iteration and $\tilde{\mathbf{x}}$ the solution from the old one. In numerical experiments described in the next subsections, this iterative algorithm converges typically in five steps for the tolerance $\epsilon = 0.001$.

3.2 Data discussion

Table 1 explains the composition structure of the three funds of the Slovak pension system according to regulatory requirements. The funds differ mainly in proportions of high risk and low risk instruments (equity vs. fixed income). Table 2 contains historical returns and standard deviations for stocks represented by the S&P index and bonds represented by 10-years government bonds. The correlation coefficient of stock and bond returns calculated from historical data is $corr = -0.1151$. The data for the expected wage growth ϱ are shown in Table 3. The regular contribution to the account is set by the law at the level of $\tau = 9\%$ of the gross salary.

Furthermore, we take into account additional regulatory restrictions on fund investments: 15 years prior to retirement it is not allowed to invest to the Growth fund, and the last 7 are reserved for Conservative fund only. Therefore we decided to assume 6 decision points and splitted the investment horizon into the following periods : years $1-10, 11-18, 19-25, 26-29, 30-33, 33-40$, with lengths $[l_1, ..., l_6] = [10, 8, 7, 4, 4, 7]$. The closer the end of saving horizon the more money is accumulated, thus higher risk exposure faced, hence more frequent the balancing decision. The saver makes his/her decision at the beginning of each period and it is kept for the rest of it. For the reason of regulatory restriction on the last 7 years of saving, the last period can be omitted from the optimization. We implement the regulatory restrictions for periods 4 and 5 by adding a simple constraint $y_n^1 = 0$ (no Growth fund is allowed) for all nodes n belonging to the corresponding time stages to problem (5)-(10) resp. (12)-(17).

Since we consider only the first 33 years of saving in the optimization, we set $\widetilde{T} = 5$, and consider $\mu = 4, 4.5$, and 5 to be our target wealth at the end of the year 33. The wealth at the time of retirement is then calculated as the wealth at year 33 adjusted for the bond return and contributions in the last period. The value-at-risk confidence level is considered to be $\alpha = 0.05$.

3.3 Scenario tree generation

The values in the tree nodes $n \in \mathcal{N} \setminus \{0\}$ are triple vectors $[s_n^1, s_n^2, s_n^3]$ representing the adjusted fund returns during the period $[t_{\xi(n)-1}, t_{\xi(n)}]$ from node n_- to n, used in appreciation relations for $\mathbf{y}_n$. Since the periods represent several years, the triples $[s_n^1, s_n^2, s_n^3]$ are the returns for the whole period of length $l_{\xi(n)}$, not for one year only. The yearly returns needed in (19) for the appreciation of τ_{n_-} are then obtained simply as the $l_{\xi(n)} - th$ root of the components of $[s_n^1, s_n^2, s_n^3]$.

Values of s_n^j are calculated using their definitions as described in Section 1.3 where the denominator is modified to $1 + \rho_n^{avg}$ in order to represent the average wage growth during the period, i.e. $s_n^j = \frac{1+r_n^j}{(1+\rho_n^{avg})^{l_{\xi(n)}}}$ where $\rho_n^{avg} = \mathbb{E}([\rho_{l_{\xi(n)-1}+1}, ..., \rho_{l_{\xi(n)}}])$. Here the fund returns r_n^j for the period $[t_{\xi(n)-1}, t_{\xi(n)}]$ from node n_- to n are calculated from scenarios of stock and bond returns $r_n^{(s)}$ and $r_n^{(b)}$ for the given period and using the corresponding weights given in Table 1, i.e. $r_n^1 = 0.8r_n^{(s)} + 0.2r_n^{(b)}, r_n^2 = 0.5r_n^{(s)} + 0.5r_n^{(b)}, r_n^1 = r_n^{(b)}$. Hence, scenarios of s_n^j are determined by scenarios for the corresponding stock and bond returns.

We assume the log of stock and bond prices follow a bivariate Brownian motion and their returns are correlated with the correlation coefficient *corr* (see Tab.2). For the purpose of performing simulations and generating scenarios, we discretize the components of the bivariate Brownian motion using a 3-point discretization.

In this way, we obtain $K_s = 3$ scenarios for stock returns and $K_b = 3$ scenarios for bond returns. Combinations of them determine fund returns $r_{\{n^+\}}^j$ and thereby also values $s_{\{n^+\}}^j$ in the $K_s * K_b = 9$ successors of each node $n \in \mathcal{N} \setminus \mathcal{T}$.

4 Results

The output of our program is the triple set of values $\mathbf{y}_n = [y_n^1, y_n^2, y_n^3]^\top$ for nodes $n \in \mathcal{N}$. These values are then recalculated to corresponding weights $w_n^j = y_n^j / \sum_{i=1}^3 y_n^i$, $j \in \{1, 2, 3\}$, for each nonterminal node $n \in \mathcal{N} \setminus \mathcal{T}$ (there is no need to know the weights in the terminal time stage, because the savings are not distributed to different funds anymore). The results are in the form of a matrix of size 3 funds times the number of nonterminal nodes including root.

Since the number of nonterminal nodes $n \in \mathcal{N} \setminus \mathcal{T}$ is of the order 10^3, we prefer a graphical representation of the results. Figure 2 shows the averaged weights of the pension fund portfolio during the time. Averaging takes place over all nodes in the same stage; the weights are the node probabilities. One can observe that the first fund has a decreasing character in time whereas there is an evidence of the increasing weight of the less risky fund, the Conservative one. This is in accordance with intuitive perception as the higher amount of money is more sensitive to changes in the fund returns and a risk averse investor tends to lower the risk exposure in later times. This is also in accordance with the results in the paper by Kilianová, Melicherčík and Ševčovič (2006). One can also observe increasing share of the risky fund if the target wealth level μ is increased. Figure 3 shows the decreasing trend of the total share of stocks in the portfolio consisting of the three funds, for target wealth $\mu = 4.5$ and the static average value-at-risk risk measure.

Table 4 shows the resulting risk profile of the particular experiment. The results confirm the natural expectation that a higher return is accompanied by a higher risk. The difference between the results of minimizing the static and the dynamic risk measure is supported by observation that in the second case there is a stronger motivation to allocate the money into the most secure fund as soon as possible. This is explained by the fact that, in this case, the risk in *all* time stages is accounted for. Thus a higher portion of the savings is pushed to less risky strategy.

5 Conclusions

The goal of this paper was to apply the risk measures concept to the pension saving problem. We have presented a model for determining optimal balancing strategies between pension funds with different risk profiles. We showed that the problem of pension saving leads to a linear optimization program when minimizing the average value-at-risk deviation, in both a static and a dynamic case. We presented results for the Slovak pension system. Experimentally they confirmed the generally observed trend of investing from the most risky fund in the first years to less risky funds later, which was confirmed also in a recent paper based on utility approach. This trend is also in accord with intuition and to the governmental restrictions.

6 Tables

Fund type	Stocks	Bonds and money market instruments
Growth Fund (1)	up to 80%	at least 20%
Balanced Fund (2)	up to 50%	at least 50%
Conservative Fund (3)	no stocks	100%

Table 1: Limits for investment for the pension funds in Slovak Republic.

	Return	StDev
S&P	0.1028	0.1690
bonds	0.0516	0.0082

Table 2: Historical return and its standard deviation for the S&P index and 10-years government bonds (Jan 1996 - June 2002). The correlation coefficient between stock and bond returns is -0.1151.

Period	2006-08	2009-14	2015-21	2022-24	2025-50
wage growth $(1+\varrho_t)$	1.075	1.070	1.065	1.060	1.050

Table 3: The expected wage growth in Slovak Republic. Source: Slovak Savings Bank (SLSP).

obj. function	$\mu_{33}=4$	$\mu_{33}=4.5$	$\mu_{33}=5$
static AVaRD	1.5246	2.1063	2.8320
dynamic AVaRD	1.9995	2.5567	3.0593

Table 4: The $AVaRD$ of the terminal wealth in year 33 after solving the problems (5)-(10) and (12)-(17) for different target wealth μ.

7 Figures

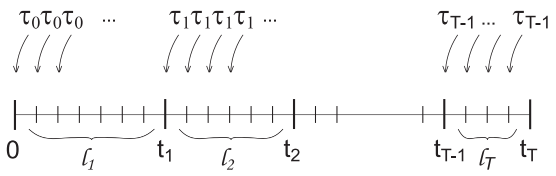

Figure 1: Transfer of the regular contribution τ in periods $[t_{k-1}, t_k]$ with lengths $l_k, k = 1, ..., T$. For simplification of the illustration, the τ contribution indexed is by the time t instead of the node n.

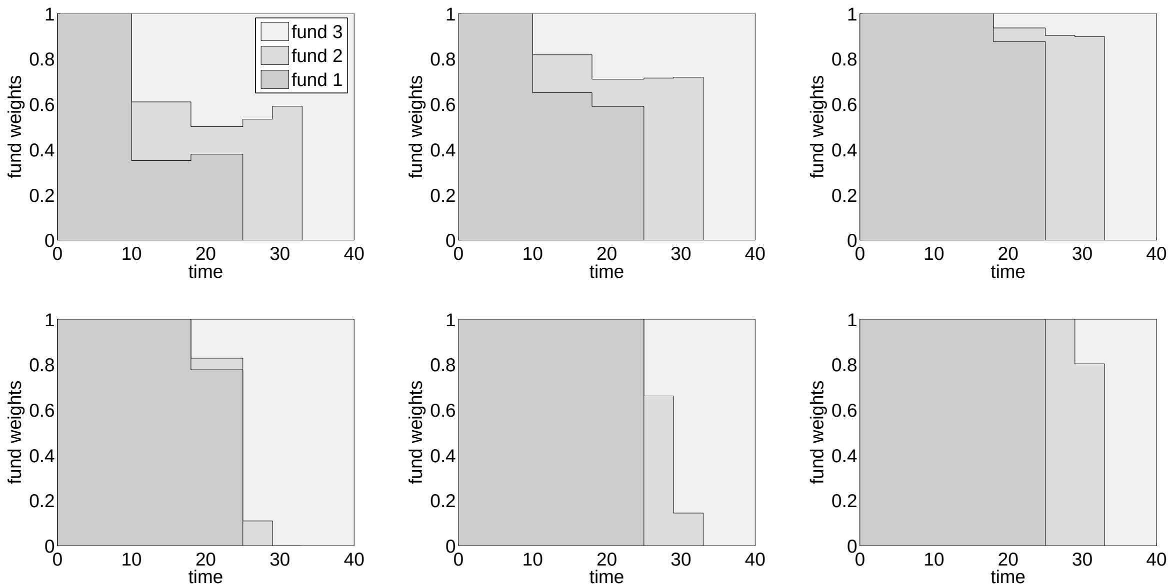

Figure 2: Averaged optimal weights of funds $1, 2, 3$ for the one-period AVaRD (top) and the multi-period AVaRD (bottom), and for the target wealth (after 33 years) $\mu = 4$ (left), $\mu = 4.5$ (middle) and $\mu = 5$ (right).

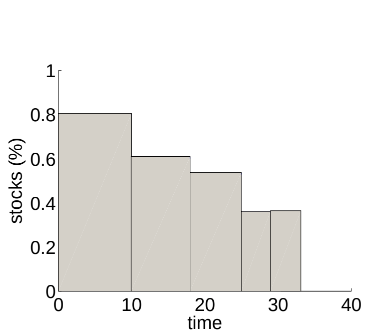

Figure 3: The decreasing trend of the portion of stocks in the portfolio, for static average value-at-risk and $\mu = 4.5$.

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