Characteristic of left invertible semigroups and admissibility of observation operators

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In this paper we discuss the characteristic property of the left invertible semigroups on general Banach spaces and admissibility of the observation operators for such semigroups. We obtain a sufficient and necessary condition about their generators. Further, for the left invertible and exponentially stable semigroup in Hilbert space we show that there is an equivalent norm under which it is contractive. Based on these results we prove that for any observation operator satisfying the resolvent condition is admissible for the left invertible semigroup if its range is finite-dimensional. In addition we prove that any observation operator satisfying the resolvent condition can be approximated by the admissible observation operators. Finally we give a sufficient condition of exact observability of the left invertible semigroup.

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1. Introduction

Let \( \mathcal{H} \) and \( \mathbb{Y} \) be Hilbert spaces. Consider an infinite-dimensional system of the form

\[
\begin{align*}
    \dot{x}(t) &= Ax(t), & t > 0 \\
    x(0) &= x_0 \in \mathcal{D}(A) \\
    y(t) &= Cx(t),
\end{align*}
\]

where \( x(t) \in \mathcal{H} \) is the state of the system at time \( t \) and \( y(t) \in \mathbb{Y} \) is the output of the system. \( \mathcal{H} \) is said to be the state space and \( \mathbb{Y} \) the output space. \( C \) is the called the observation operator. Here \( A \) and \( C \) are linear operators that may be unbounded but generally we assume that \( A \) generates a \( C_0 \) semigroup \( T(t), \ t \geq 0 \) on \( \mathcal{H} \). In order to guarantee that the output function \( y(t) \) belongs to \( L^2_{\text{loc}}([0, \infty), \mathbb{Y}) \), the notion of admissible observation operators was introduced by Weiss [1].

Definition 1.1. Let \( \mathcal{H} \) and \( \mathbb{Y} \) be Hilbert spaces and \( T(t) \) be a \( C_0 \) semigroup on \( \mathcal{H} \) with generator \( A, C \in \mathcal{B}(D(A), \mathbb{Y}) \) is said to be admissible for \( T(t) \) if, for some (and hence any) \( t > 0 \), there exists \( K_t > 0 \) such that

\[
\int_0^t \|CT(s)x\|^2 \, ds \leq K^2_t \|x\|^2, \quad \forall x \in D(A).
\]  

An extension concept of admissibility is as follows.

Definition 1.2. Let \( \mathcal{H} \) and \( \mathbb{Y} \) be Hilbert spaces and \( T(t) \) be a \( C_0 \) semigroup on \( \mathcal{H} \) with generator \( A, C \in \mathcal{B}(D(A), \mathbb{Y}) \) is said to be infinite-time admissible for \( T(t) \) if there exists a constant \( K > 0 \) such that

\[
\int_0^\infty \|CT(s)x\|^2 \, ds \leq K^2 \|x\|^2, \quad \forall x \in D(A).
\]  

One can prove that, if \( T(t) \) is exponentially stable, the notion of admissibility and infinite-time admissibility are equivalent. Let \( \|T(t)\| \leq M_1 e^{\alpha t}, \forall t \geq 0 \). If \( C \) is admissible for \( T(t) \), then there exists a constant \( M > 0 \) such that

\[
\|CR(\lambda, A)\| \leq \frac{M}{\sqrt{\omega \lambda}}, \quad \forall \lambda \omega > \omega.
\]  

Thus Weiss conjectured that the above inequality is also a sufficient condition for admissibility of observation operator \( C \) (see, e.g., [2,13]). This is known as the Weiss conjecture. In what follows, we say that \( C \) satisfying (1.4) is a Weiss class operator.

Since the admissibility of \( C \) is a fundament of system analysis, many mathematicians have studied this issue including its dual in the past two decades. Some of them are interested in proving or disproving the admissibility of the Weiss class operators; for instance, [4–7] and [8] show that the Weiss class operators are admissible for some semigroups, while [9,10] and [11] show that there is a Weiss class operator that is not admissible. Here we mention an important work due to Zwart [6]; he proved that almost all Weiss class operators are admissible. For more detailed information about them please see the papers mentioned above and the references therein. Others are interested
in showing whether or not a system is exactly observable under the assumption of admissibility; for instance, see [12–14] and the references therein. An important subject in this aspect is the Hautus Test for infinite-dimensional systems. Here we refer to [12], in which the authors translated the admissibility of $C$ into the generation problem of $C_0$ semigroup in the extended space: $\mathcal{H}_e = \mathcal{H} \oplus L^2(0, \infty; Y)$, in which they define the operator $A_e$ by

$$A_e \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax \\ -y \end{bmatrix},$$

$$D(A_e) = \{ (x, y)^T \in \mathcal{H}_e | y \in H^1(0, \infty), x \in D(A), y(0) = Cx \}.$$ 

They showed that, if $T(t)$ is exponentially stable and $C$ is admissible and exactly observable for $T(t)$, then the extended operator $A_e$ generates a $C_0$ semigroup $S_e(t)$ that satisfies the condition that there exist positive constants $c$ and $M$ such that

$$c \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| \leq \left\| S_e(t) \begin{bmatrix} x \\ y \end{bmatrix} \right\| \leq M \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|, \quad \forall t \geq 0, \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{H}_e. \quad (1.5)$$

This implies that $S_e(t)$ is a left invertible semigroup.

Neerven, in [15], showed that for the isometries semigroups, the above implies

$$\left\| (\lambda I - A_e) \begin{bmatrix} x \\ y \end{bmatrix} \right\| \geq c_1|\Re \lambda| \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| - \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| \in D(A_e), \quad \Re \lambda < 0. \quad (1.6)$$

But Jacob and Zwart, in [11], showed that (1.6) is not a sufficient condition of a left invertible semigroup for $S_e(t)$.

Recently the authors of [14] introduced the zero-class admissibility of observation operators, and Jacob et al. [16] give a necessary and sufficient condition for a zero-class admissible observation operator. For an zero admissible observable operator, if the system is exactly observable then the semigroup $T(t)$, $t \geq 0$ is left invertible. However, we note that the existing results exclude admissibility of an observation operator for the left invertible semigroup, and to the authors’ best knowledge, there is no report on the characteristic of a generator of the left invertible semigroup on general Banach spaces. Motivated by this, in the present paper we discuss the characteristic property of the left invertible semigroup, the admissibility of observation operators for it as well as the exact observability.

Our first result gives the characteristic of the left invertible semigroup on a Banach space (see Theorem 2.4 in Section 2).

**Result 1.** Let $T(t)$, $t \geq 0$ be a $C_0$ semigroup on a complex Banach space $X$ and $A$ be its generator. Then the following statements are equivalent.

1. $T(t)$, $t \geq 0$ is a left invertible semigroup;
2. There exists an equivalent norm $\| \cdot \|_*$ on $X$ such that, for some real number $\alpha$, $-(A + \alpha I)$ is dissipative on $(X, \| \cdot \|_*)$.

Note that in a Banach space $X$ we always define an equivalent norm on $X$ such that an uniformly bounded semigroup becomes a contraction semigroup, but for a Hilbert space it is not. However, if $T(t)$ is an exponentially stable and left invertible semigroup on a Hilbert space, we can do it. The following result gives a precise description (see Theorem 3.1 in Section 3).
(1) $T(t), t \geq 0$ is a left invertible semigroup;
(2) There exist two constants $\alpha > 0$ and $c > 0$ such that
$$\|T(t)x\| \geq ce^{-\alpha t}\|x\|, \quad \forall x \in X, \quad t > 0; \quad (2.1)$$
(3) There exists a constant $t_0 > 0$ such that
$$\inf_{\|x\|=1, x \in X} \|T(t_0)x\| > 0. \quad (2.2)$$

For the left invertible semigroup, its point spectrum distributes as follows.

**Theorem 2.2.** Let $T(t), t \geq 0$ be a left invertible semigroup with generator $A$. If $\sigma_p(A) \neq \emptyset$, then there exist two constants $h_1 > 0$ and $h_2 > 0$ such that
$$-h_2 \leq \inf_{\lambda \in \sigma_p}(\lambda I - A) < \sup_{\lambda \in \sigma_p}(\lambda I - A) \leq h_1. \quad (2.3)$$

The following result is a general proposition for bounded linear operators, which is an extension result of [14].

**Theorem 2.3.** Let $S \in L(bounded linear operator)$ on a complex Banach space $X$ satisfying
$$\|Sx\| \geq \gamma\|x\|, \quad \forall x \in X \quad (2.4)$$
where $\gamma > 0$. If $\sigma(S) \neq X$, then $0 \in \sigma(S)$, the residual spectrum of $S$.

In what follows, we shall prove the inclusion relation
$$\{\lambda \in \mathbb{C} : |\lambda| < \gamma_0 \} \subset \sigma_r(S). \quad (2.5)$$

\textbf{Proof.} By (2.4), we have $\gamma_0 = \inf_{\|x\|=1}\|Sx\| \geq \gamma$. If $\sigma(S) \neq X$, then $0 \in \sigma_r(S)$, the residual spectrum of $S$.

By the definition of $\gamma_0$, it holds that $\|Sx\| \geq \gamma_0\|x\|, \forall x \in X$. For any $0 < \epsilon < \gamma_0$, and letting $\lambda \in \mathbb{C}$ satisfy $|\lambda| \leq \gamma_0 - \epsilon$, then we have
$$\|(\lambda I - S)x\| \geq \|Sx\| - |\lambda|\|x\| \geq \epsilon\|x\|, \quad \forall x \in X$$

which implies that $S_0 = \lambda I - S$ is injective and has closed range. It is sufficient to prove $\sigma(S_0) \neq X$ provided that $|\lambda| \leq \gamma_0 - \epsilon$. If this is not true, then there exists a $\lambda_0 \in \mathbb{C}$ with $|\lambda_0| \leq \gamma_0 - \epsilon$ such that $\sigma(S_0) = X$, which means that $S_0$ is invertible and $S_0^{-1}$ satisfies $\|S_0^{-1}\| \leq \epsilon^{-1}$. For $-\lambda_0 \leq \frac{\epsilon}{2}$, define operators by
$$R(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n[S_0^{-1}]^{n+1}.$$ 

Clearly, $R(\lambda) \in B(X)$ and $R(\lambda)(\lambda I - S) = I$; this shows that
$$\sigma(S_0) = X \quad (\lambda - \lambda_0) \leq \frac{\epsilon}{2};$$

If $|\lambda_0| \leq \frac{\epsilon}{2}$, we have $\sigma(S) = X$, which contradicts $\sigma(S) \neq X$. If $|\lambda_0| > \frac{\epsilon}{2}$, we can choose $\lambda_1 = (|\lambda_0| - \frac{\epsilon}{2})\frac{\lambda_0}{|\lambda_0|}$, then $R(\lambda_1) = X$ and
$$\|S_0^{-1}\| \leq \left\| \frac{1}{\gamma_0 - |\lambda_1|} \right\| \leq \frac{1}{\epsilon}.$$ 

Define operators for $|\lambda - \lambda_1| \leq \frac{\epsilon}{2}$ by
$$R_1(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_1)^n[S_0^{-1}]^{n+1},$$

then $R_1(\lambda) \in B(X)$ and $R_1(\lambda)(\lambda I - S) = I$. Repeating the previous process a finite number of times, we deduce that $\sigma(S) = X$, which contradicts the assumption. Therefore, $\sigma(S) \neq X$ for all $|\lambda| \leq \gamma_0 - \epsilon$. By arbitrariness of $\epsilon > 0$, we have $\sigma(S) \neq X$ for any $|\lambda| < \gamma_0$, which means that
$$\{\lambda \in \mathbb{C} : |\lambda| < \gamma_0 \} \subset \sigma_r(S).$$

The desired result is proved.

From the proof of Theorem 2.3, we can see that the following results are true.

**Corollary 2.1.** Let $S$ be a bounded linear operator on a complex Banach space $X$ satisfying
$$\|Sx\| \geq \gamma\|x\|, \quad \forall x \in X,$$
where $\gamma > 0$. If $\sigma(S) = X$, then
$$\{\lambda \in \mathbb{C} : |\lambda| < \gamma_0 \} \subset \rho(S).$$

**Corollary 2.2.** Let $T(t), t \geq 0$ be a left invertible semigroup. If there is a $t > 0$ such that $\sigma(T(t)) \neq X$, then
$$\{\lambda \in \mathbb{C} : |\lambda| < \epsilon(t) \} \subset \sigma_r(T(t)),$$
where $\epsilon(t) = \inf_{\|x\|=1}\|T(t)x\|$. If there is a $t > 0$ such that $\sigma(T(t)) = X$, then
$$\{\lambda \in \mathbb{C} : |\lambda| < \epsilon(t) \} \subset \rho(T(t)).$$

Note that, in Corollary 2.2, the number $\epsilon(t)$ is larger than $ce^{-\alpha t}$, where the positive constants $c$ and $a$ are given by Theorem 2.1. So it always holds that $\{\lambda \in \mathbb{C} : |\lambda| < ce^{-\alpha t}\} \subset \sigma_r(T(t))$ as $\sigma(T(t)) \neq X$. This relation will be used later.

To characterize the generator of a left invertible semigroup, we need the following lemma.

**Lemma 2.1.** Let $X$ be a complex Banach space and $X^*$ be its dual space. If function $f : (a, b) \rightarrow X$ satisfies the following two conditions:
(1) $f(t)$ is weakly differentiable at $t = s$ i.e. $\forall x^* \in X^*$, the function $(f'(t), x^*)$ is differentiable at $t = s$;
(2) The scalar function $\|f(t)\|$ is differentiable at $t = s$;

then we have
$$\frac{df(s)}{dt} = \frac{df(s)}{dt} = \Re(f'(s), x^*), \quad \forall x^* \in F(f(s))$$

where $F(x) = \{x^* \in X^* | x^*(x) = \|x\| = \|x^*\|\}$.

**Proof.** Let $\Delta s > 0$; then for every $y^* \in F(f(s))$, we have
$$\Re(f(s + \Delta s - f(s), y^*) = \Re(f(s + \Delta s), y^*) - \|f(s)\|^2 \leq \left(\|f(s + \Delta s)\| - \|f(s)\|\right)\|f'(s)\|.$$

Since $f(t)$ is weakly differentiable at $t = s$ and $\|f(t)\|$ is differentiable at $t = s$, we have
$$\Re(f'(t), y^*) \leq \|f(s)\| \frac{df(s)}{dt} \|f'(s)\|.$$

On the other hand, as $\Delta s \leq 0$, it is obvious that
$$\Re(f'(t), y^*) \geq \|f(s)\| \frac{df(s)}{dt} \|f'(s)\|.$$

Therefore, for each $y^* \in F(f(s))$,
$$\|f(s)\| \frac{df(s)}{dt} \|f'(s)\| \leq \Re(f'(s), y^*) \leq \|f(s)\| \frac{df(s)}{dt} \|f'(s)\|.$$

The differentiability of $\|f(t)\|$ implies $\frac{df}{dt} \|f(s)\| = \frac{df}{dt} \|f(s)\|$, so we have
$$\|f(s)\| \frac{df(s)}{dt} \|f'(s)\| = \Re(f'(s), y^*).$$

The desired result is proved. □

Using Lemma 2.1, we can give a sufficient and necessary condition for the generator of left invertible semigroups on Banach spaces.
Theorem 2.4. Let $T(t)$, $t \geq 0$ be a $C_0$ semigroup on a complex Banach space $X$ and $A$ be its generator. Then the following statements are equivalent.

(1) $T(t)$, $t \geq 0$ is a left invertible semigroup;
(2) There exists an equivalent norm $\| \cdot \|$ on $X$ such that, for some real number $\alpha$, $-(A + \alpha I)$ is dissipative on $(X, \| \cdot \|)$.

Proof. (1) $\Rightarrow$ (2) Suppose that $T(t)$, $t \geq 0$ is a left invertible semigroup on $X$ with norm estimate $\|T(t)\| \leq Me^{\alpha t}$, $t \geq 0$. By Theorem 2.1, there exist two constants $c > 0$ and $\alpha > 0$ such that

$$\|T(t)x\| \geq ce^{-\alpha t}\|x\|, \quad \forall t \geq 0.$$ 

Therefore

$$ce^{-\alpha t}\|x\| \leq \|T(t)x\| \leq Me^{\alpha t}\|x\|, \quad \forall x \in X, \quad t \geq 0.$$ 

We now define a new norm on $X$ by

$$\|x\|_* = \inf_{t \geq 0} \|e^{\alpha t}T(t)x\|, \quad \forall x \in X.$$ 

Obviously, it holds that

$$c\|x\| \leq \|x\|_* \leq M\|x\|,$$

which implies that $\|x\|_*$ is equivalent to the original norm $\| \cdot \|$ on $X$.

Putting $S_\alpha(t)x = e^{\alpha t}T(t)$, then $S_\alpha(t)$, $t \geq 0$ is also a left invertible semigroup with the generator $(A + \alpha I)$. Moreover, we have

$$\|S_\alpha(t)x\|_* = \inf_{t \geq 0} \|e^{\alpha t}T(s)S_\alpha(t)x\| = \inf_{t \geq 0} \|e^{\alpha t}T(s)x\| \geq \|x\|_*, \quad \forall t \geq 0,$$

which implies that $\|S_\alpha(t)x\|_*$ is an increasing function of $t$.

Now let $\forall x \in \mathcal{D}(A)$; then $S_\alpha(t)x \in \mathcal{D}(A)$, and for every $x^* \in \mathcal{F}(S_\alpha(t)x)$, we have

$$\Re((-A+\alpha I)S_\alpha(t)x,x^*) = \Re\left(\lim_{s \to 0^+} \frac{S_\alpha(t-s)x-S_\alpha(t)x}{s}, x^*\right) = \lim_{s \to 0^+} \Re\left(\frac{S_\alpha(t-s)x-S_\alpha(t)x}{s}\right) = \lim_{s \to 0^+} \frac{\|S_\alpha(t-s)x\|_* - \|S_\alpha(t)x\|_*}{s} \leq 0,$$

where we have used the relation $\|S(t-s)x\|_* \leq \|S(t)x\|_*$, so it holds that

$$\Re((-A+\alpha I)S_\alpha(t)x,x^*) \leq 0, \quad \forall x^* \in \mathcal{F}(S_\alpha(t)x), \quad t \geq 0.$$ 

Since $x \in \mathcal{D}(A)$, $\lim_{s \to 0} S_\alpha(t)x = x$, we have

$$\Re((-A+\alpha I)x,x^*) \leq 0, \quad \forall x^* \in \mathcal{F}(x).$$

This shows that $-(A+\alpha I)$ is dissipative in $(X, \| \cdot \|_*)$.

(2) $\Rightarrow$ (1) Suppose that there exists an equivalent norm $\| \cdot \|$ on $X$ such that, for some real number $\alpha$, $-(A+\alpha I)$ is dissipative in $(X, \| \cdot \|)$. Without loss of generality we can assume that $-(A+\alpha I)$ is dissipative in $(X, \| \cdot \|)$.

Set $S_\alpha(t) = e^{\alpha t}T(t)$; then $S_\alpha(t)$ is a $C_0$ semigroup with the generator $A+\alpha I$. If $x \in \mathcal{D}(A)$, $x^* \in X^*$, then the function $S_\alpha(t)x$, $t \geq 0$, is always differentiable in $t \geq 0$ and $\frac{d}{dt}(S_\alpha(t)x^*) = ((A+\alpha I)S_\alpha(t)x^*)$. On the other hand, $S_\alpha(t)x$ is continuously differentiable in $t \geq 0$; hence we have

$$\|S_\alpha(t+h)x\| - \|S_\alpha(t)x\| \leq \left\|\int_t^{t+h} S_\alpha(s)(A+\alpha I)x\, ds\right\| \leq \|(A+\alpha I)x\|\|h|M(t),$$

where $M(t) = \sup_{s \in [t,t+h]} \|S_\alpha(s)\|$. The inequality above shows that $\|S_\alpha(t)x\|$ is a Lipschitz continuous function, and hence $\|S_\alpha(t)x\|$ is differentiable almost everywhere. By Lemma 2.1, if $\|S_\alpha(t)x\|$ is differentiable at $t$, then we have

$$\|S_\alpha(t)\|\frac{d}{dt}\|S_\alpha(t)x\| = \Re((A+\alpha I)S_\alpha(t)x, x^*) \geq 0,$$

which means that $\|S_\alpha(t)x\|$ is a nondecreasing function. So we have

$$\|S_\alpha(t)x\| \geq \|x\|, \quad \forall x \in \mathcal{D}(A), \quad t \geq 0.$$ 

Since the $S_\alpha(t)$ are linear and bounded operators, we get

$$\|S_\alpha(t)x\| \geq \|x\|, \quad \forall x \in X, \quad t \geq 0;$$

this means that $S_\alpha(t)$, $t \geq 0$ is a left invertible semigroup. So is $T(t)$, $t \geq 0$. \hfill \Box

Applying Corollary 2.2 and the Spectral Mapping Theorem of semigroup (see [18]), we can get the following result.

Corollary 2.3. Let $T(t)$, $t \geq 0$ be a left invertible semigroup on Banach space $X$ with the generator $A$. If $\sigma(A) = \emptyset$, then $T(t)$, $t \geq 0$ can be embedded in a $C_0$ group.

As a consequence of Theorem 2.4, we have the corollary.

Corollary 2.4. Let $T(t)$, $t \geq 0$ be a $C_0$ semigroup on Banach space $X$ with the generator $A$. Then the following statements are equivalent.

(1) $T(t)$, $t \geq 0$ is a dissipative semigroup.
(2) $A$ and $-A$ are both dissipative operators and $\mathcal{R}(I-A) = X$.

As a direct result of Corollaries 2.3 and 2.4, we have the following corollary.

Corollary 2.5. Let $T(t)$, $t \in \mathbb{R}$ be a $C_0$ group on Banach space $X$ with the generator $A$. Then the following two assertions are equivalent.

(1) $T(t)$, $t \in \mathbb{R}$ is a dissipative group.
(2) $A$ and $-A$ are both dissipative operators and $\mathcal{R}(I \pm A) = X$.

The following theorem gives invariability of left invertible semigroups on Banach spaces under the bounded perturbation.

Theorem 2.5. Let $T(t)$, $t \geq 0$ be a left invertible semigroup on Banach space $X$ with generator $A$. If $B$ is a bounded linear operator on $X$, then the semigroup $S(t)$, $t \geq 0$ generated by $A+B$ is also a left invertible semigroup.

Proof. Let $A$ be the generator of $C_0$ semigroup $T(t)$ satisfying $\|T(t)\| \leq Me^{\alpha t}$ and $B$ be a bounded linear operator. By the perturbation theory of $C_0$ semigroups (see [18]), $A+B$ generates a $C_0$ semigroup $S(t)$, $t \geq 0$ satisfying $\|S(t)\| \leq Me^{(\omega+M\|B\|)t}$, $t \geq 0$. Moreover, $S(t)$ satisfies the integral equation

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x \, ds.$$ 

(2.6)

Using (2.6) we get the estimate

$$\left\|\int_0^t T(t-s)BS(s)x \right\| \leq M^2\|B\|e^{(\omega+M\|B\|)t}\|x\|, \quad \forall x \in X, \quad t \geq 0.$$ 

(2.7)

Since $T(t)$ is a left invertible semigroup, according to Theorem 2.1, there exist constants $c > 0$ and $\alpha > 0$ such that

$$\|T(t)x\| \geq ce^{-\alpha t}\|x\|, \quad \forall x \in X, \quad t > 0.$$ 

Let $M_1 = M^2\|B\|e^{(\omega+M\|B\|)t}$ and

$$t_0 < \min\left\{1, \frac{c}{2M_1}e^{-\alpha}\right\}, \quad x \in X;$$
then we have
\[
\|S(t_0)x\| \geq \|T(t_0)x\| - \left\| \int_0^t T(t - s)BS(s) \right\|
\geq (ce^{-\alpha t_0} - t_0 M_1)\|x\| \geq (ce^{-\alpha} - t_0 M_1)\|x\|
\geq \frac{1}{2} ce^{-\alpha}\|x\|, \quad \forall x \in X,
\]
which means that \(S(t), t \geq 0\) is a left invertible semigroup. The proof is then complete. \(\square\)

Applying Corollary 2.3 and Theorem 2.5, we can get the following result.

**Corollary 2.6.** Let \(T(t), t \geq 0\) be a left invertible semigroup on Banach space \(X\) with the generator \(A\). If \(\sigma(A) = \emptyset\) and \(B\) is a linear bounded operator, then \(A + B\) generates a \(C_0\) semigroup which can be embedded in a \(C_0\) group.

3. Admissibility of observation operator

In this section, we shall discuss the admissibility of observation operator for left invertible semigroups. In the existing results on the admissibility, a result shows that if \(B \in \mathcal{B}(U, D(A)^\beta)\) satisfies condition

\[
\|R(\lambda, A)B\| \leq \frac{M}{\sqrt{\lambda}}, \quad \forall \lambda > \alpha,
\]
then the control operator \(B\) is admissible for \(T(t)\). There is no result on the observation operator for the left invertible semigroups. However, the results for the contraction semigroup are much richer. The following proposition shows that the left invertible semigroup can become a contraction semigroup.

**Theorem 3.1.** Let \(T(t), t \geq 0\) be an exponentially stable and left invertible \(C_0\) semigroup on Hilbert space \(\mathcal{H}\). Then there exists an equivalent inner product on \(\mathcal{H}\) such that \(T(t)\) is an exponentially stable and contraction semigroup.

**Proof.** Let \(T(t), t \geq 0\) be an exponentially stable and left invertible \(C_0\) semigroup. Then there exist positive constants \(M, c, \alpha\) and \(\delta\) such that
\[
ce^{-\alpha t}\|x\| \leq \|T(t)x\| \leq Me^{-\delta t}\|x\|, \quad \forall t \geq 0, x \in \mathcal{H}.
\]
Define an inner product on \(\mathcal{H}\) by
\[
\langle x, y \rangle_1 = \int_0^\infty (T(t)x, T(t)y)dt, \quad \forall x, y \in \mathcal{H};
\]
then we have
\[
\|x\|_1^2 = \int_0^\infty \|T(t)x\|^2dt.
\]
Clearly, \(\|\cdot\|_1\) is an equivalent norm on \(\mathcal{H}\). In the sense of this norm, we have
\[
\|T(t)x\|_1^2 = \int_0^\infty \|T(s)T(t)x\|^2ds = \int_t^\infty \|T(s)x\|^2ds
\leq \int_0^\infty \|T(s)x\|^2ds = \|x\|_1^2.
\]
That is, \(T(t), t \geq 0\) is an exponentially stable and contraction semigroup on \(\mathcal{H}\).

For the contraction semigroups, the following result is due to Jacob and Partington [4, Theorem 1.3], which is probably one of the most important results in the area of admissibility.

**Lemma 3.1.** Let \(T(t), t \geq 0\) be a \(C_0\) semigroup of contraction on a separable Hilbert space with generator \(A\) and let \(\mathcal{C} \in \mathcal{B}(D(A), \mathcal{C})\). Then \(\mathcal{C}\) is infinite-time admissible if and only if \(\mathcal{C}\) satisfies the condition that there exists a constant \(M > 0\) such that
\[
\|CR(\lambda, A)\| \leq \frac{M}{\sqrt{\lambda}}, \quad \forall \lambda > 0.
\]

Let \(A\) be the generator of a left invertible \(C_0\) semigroup \(T(t)\), satisfying \(\|T(t)\| \leq M e^{\alpha t}\). We can choose a real \(\alpha > \omega\) such that \(T_0(t) = e^{-\alpha t}T(t)\) is exponentially stable. Applying Theorem 3.1 to \(T_0(t)\), we have that \(T_0(t)\) is an exponentially stable and contraction semigroup. As a direct result of Lemma 3.1, we have the following result.

**Theorem 3.2.** Let \(T(t), t \geq 0\) be the left invertible \(C_0\) semigroup on Hilbert space \(\mathcal{H}\). Let \(\mathcal{C} \in \mathcal{B}(D(A), \mathcal{C})\). Then \(\mathcal{C}\) is admissible if and only if \(\mathcal{C}\) satisfies the condition that, for positive constants \(M\) and \(\omega\),
\[
\|CR(\lambda, A)\| \leq \frac{M}{\sqrt{\lambda}}, \quad \forall \lambda > 0.
\]

Theorem 3.2 shows that the Weiss class operators are admissible for the left invertible semigroups if \(\mathcal{Y}\) is a finite-dimensional Hilbert space. However if \(\mathcal{R}(\mathcal{C})\) and hence \(\mathcal{Y}\) is infinite-dimensional, then the question of admissibility becomes more complicated. A result in [17] shows that there is a Weiss class operator \(C\) that is not admissible for the right shift semigroup in \(L^2(\mathbb{R}_+)\) if \(\mathcal{Y}\) is an infinite-dimensional Hilbert space. Therefore, the resolvent condition (1.4) is not a sufficient condition of admissibility for the left invertible semigroup. The following result shows that the class of admissible of observation operator is dense in Weiss class operators. The proof comes mainly from [6].

**Theorem 3.3.** Let \(T(t), t \geq 0\) be a \(C_0\) semigroup on Hilbert space \(\mathcal{H}\) and satisfy \(\|T(t)\| \leq M_1 e^{\omega t}\), \(\forall t \geq 0\). Let \(\mathcal{C} \in \mathcal{B}(D(A), \mathcal{Y})\), \(\mathcal{Y}\) be another infinite-dimensional Hilbert space. If \(\mathcal{C}\) satisfies the resolvent condition (1.4), then there exists a sequence of admissible observation operators, \(\{C_n\}_{n=1}^\infty\), such that
\[
\lim_{n \to \infty} C_nx = Cx, \quad \forall x \in D(A).
\]

**Proof.** Let \(\mathcal{H}\) and \(\mathcal{Y}\) be infinite-dimensional Hilbert spaces. Let \(T(t), t \geq 0\) be a \(C_0\) semigroup satisfying \(\|T(t)\| \leq M_1 e^{\omega t}\), \(\forall t \geq 0\), and \(\mathcal{C} \in \mathcal{B}(D(A), \mathcal{Y})\) satisfy the resolvent condition (1.4). Let \(\tau_n > 0\) with \(\tau_n \to 0\) as \(n \to \infty\), and define operators by
\[
C_nx = CT_n(t_0)x, \quad \forall x \in D(A), \quad n \in \mathbb{N}.
\]
Obviously, \(\lim_{n \to \infty} C_nx = Cx\).

In what follows, we prove that \(C_n\) is admissible for \(T(t)\). For any \(x \in D(A)\) and \(t > 0\), we have
\[
\int_0^t \|C_nT(t)X\|\, dt = \int_0^t \|C_nT_n(t_0)T(t)X\|^2\, dt
\leq \int_0^t \|C_nT_n(t_0)X\|^2\, dt \leq \int_{\tau_n}^{\tau_n + t} \left| \frac{1 - e^{-t}}{1 - e^{-\tau_n}} \right|^2 \times e^{2\alpha(t + \tau_n - t)}\|\mathcal{C}(t_0)X\|^2\, dt
\leq \frac{1}{(1 - e^{-\tau_n})^2}\|\mathcal{C}(t_0)X\|^2\, dt,
\]
where \(T_n(t) = e^{-\alpha t}T(t)\) with \(\alpha > \omega\).

Note that
\[
\int_0^\infty e^{-\alpha} (1 - e^{-t}) CT_n(t)X\, dt = CR(\alpha + 1 + is, A)R(is + A)x, \quad x \in D(A).
\]
The Parseval identity reads that
\[ \int_{-\infty}^{\infty} \| CR(\alpha + 1 + is, A)R(\alpha + is, A)x \|^2 \, ds = 2\pi \int_{0}^{\infty} \| (1 - e^{-t})CT_\alpha(t)x \|^2 \, dt, \]

and

\[ \int_{-\infty}^{\infty} \| R(\alpha + is, A)x \|^2 \, ds = 2\pi \int_{0}^{\infty} \| T_\alpha(t)x \|^2 \, dt, \]

where we have used that \( T_\alpha(t) \) is exponentially stable. Thus we have

\[ \int_{0}^{\tau} \| C_\alpha T_\alpha(t)x \|^2 \, dt \leq \frac{e^{2\alpha(t+\tau)}}{(1 - e^{-\alpha})^2} \int_{0}^{\infty} \| (1 - e^{-t})CT_\alpha(t)x \|^2 \, dt \]

\[ \leq \frac{e^{2\alpha(t+\tau)}}{(1 - e^{-\alpha})^2} \sup_{\alpha \in \mathbb{R}} \| CR(\alpha + 1 + is, A)x \| \int_{0}^{\infty} \| T_\alpha(t)x \|^2 \, dt \]

\[ \leq \frac{e^{2\alpha(t+\tau)}}{(1 - e^{-\alpha})^2} M^2 K^2 \| x \|^2 \]

where we have used the estimates

\[ \sup_{\alpha \in \mathbb{R}} \| CR(\alpha + 1 + is, A)x \| \leq M \| x \|. \]

\[ \int_{0}^{\infty} \| T_\alpha(t)x \|^2 \, dt \leq K^2 \| x \|^2. \]

Therefore, \( C_\alpha, n \in \mathbb{N} \) are admissible for \( T(t) \). The desired result follows. \( \square \)

Finally we close this section by giving a sufficient condition of exact observability for the left invertible semigroup.

**Theorem 3.4.** Let \( T(t), t \geq 0 \) be the left invertible semigroup on Hilbert space \( \mathcal{H} \). Let \( C \in \mathcal{B}(D(A), \mathcal{Y}) \) be a Weiss class operator. If there exists a \( \tau > 0 \) such that

\[ \| CT(\tau)x \| \geq \delta \| x \|, \quad \forall x \in D(A), \]

then the system \( (1.1) \) is exactly observable in finite time.

**Proof.** Suppose that a Weiss class operator \( C \in \mathcal{B}(D(A), \mathcal{Y}) \) satisfies the condition that there exists a constant \( \delta > 0 \) such that

\[ \| CT(\tau)x \| \geq \delta \| x \|, \quad \forall x \in D(A). \]

Then for any \( x \in D(A) \), it holds that

\[ \delta^2 \int_{0}^{\tau} \| T(t)x \|^2 \, dt \leq \int_{0}^{\tau} \| CT(\tau)T(t)x \|^2 \, dt \]

\[ \leq \int_{0}^{\infty} \| CT(\tau)x \|^2 \leq \frac{e^{4\alpha \tau}}{(1 - e^{-\alpha})^2} M^2 K^2 \| x \|^2. \]

Note that the semigroup \( T(t), t \geq 0 \) is left invertible. Denote

\[ \varepsilon = \inf_{t \in [0, \tau]} \inf_{|x|=1} \| T(t)x \|. \]

Then we get

\[ \delta^2 \varepsilon^2 \tau \| x \|^2 \leq \int_{0}^{\tau} \| T(t)x \|^2 \, dt \leq \int_{0}^{\tau} \| CT(\tau)x \|^2 \, dt \]

\[ = \int_{\tau}^{\infty} \| CT(\tau)x \|^2 \, dt. \]

The proof is then complete. \( \square \)

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**References**


