Expansion of solution of an inverted pendulum system with time delay

Ya-Xuan Zhang\textsuperscript{a}, Zhong-Jie Han\textsuperscript{b, *}, Gen-Qi Xu\textsuperscript{b}

\textsuperscript{a} College of Science, Civil Aviation University of China, Tianjin 300300, China
\textsuperscript{b} Department of Mathematics, Tianjin University, Tianjin 300072, China

\section*{A R T I C L E   I N F O}

Keywords:
Expansion
Time delay
Spectrum
Eigenvector
Semigroup
Inverted pendulum

\section*{A B S T R A C T}

In this paper, the solution expansion for an inverted pendulum system with time delay is studied. The linearized model of this nonlinear system near its equilibrium is derived on the assumption that a unique equilibrium exists in it. Then the asymptotic expressions of its eigenvalues and the eigenvalues’ corresponding eigenvectors are obtained. Moreover, although the set of these eigenvectors does not form a Schauder basis for the state space, the solution of this model still can be expressed by these eigenvectors in the form of infinite series under certain conditions. Finally, a simulation is provided to support these results.

\section*{1. Introduction}

It is well-known that the inverted pendulum, as a "man–machine" system, can describe a kind of inherently unstable systems and has wide applications in many areas, for example, human body self-balancing explanation (see \cite{2–6}) and robot design technology (see \cite{7, 8}). Presently, because of the wide applications of the inverted pendulum, its control strategy has become more attractive and many promising results have been obtained. For example, Gawthrop and Wang \cite{9} considered the intermittent predictive control of an inverted pendulum; \cite{10} designed the backstepping control strategy to balance an inverted pendulum; \cite{12, 13} controlled a kind of inverted pendulums by neural networks theory; \cite{11, 14–16} studied the stabilization of inverted pendulums using fuzzy control theory. However, the results mentioned above mainly concerned control problems of inverted pendulums. Few results have been obtained for such systems’ solution structure, especially solution expansion, which usually plays an important role in the engineering calculation.

In this paper, we consider a simple model of an inverted pendulum involving time delay, which is shown in Fig. 1. In fact, the stability analysis of this kind of inverted pendulum has been discussed in \cite{1}. The aim of this paper is to analyze the spectral properties of the system and expand the solution of the system near its equilibrium by its eigenvectors.

The mathematical model of the inverted pendulum system under consideration has the following form:

\[
\begin{pmatrix}
\frac{1}{2}m\ell^2 & \frac{1}{2}m\ell \cos q_1(t) \\ \frac{1}{2}m\ell \cos q_1(t) & m
\end{pmatrix}
\begin{pmatrix}
\ddot{q}_1(t) \\ \ddot{q}_2(t)
\end{pmatrix}
- \left(\begin{pmatrix}
\frac{1}{2}mg \sin q_1(t) \\ \frac{1}{2}m\ell \dot{q}_1(t) \sin q_1(t)
\end{pmatrix}
\right)
= \left(\begin{pmatrix}
0 \\ Q(t)
\end{pmatrix}\right),
\]

where \(m\) is the mass of the homogeneous rod; \(\ell\) is its length; \(g = 9.8 \text{ m/s}^2\) stands for the gravitational acceleration; \(q_1(t)\) is its tilt angle from the vertical and \(q_2(t)\) is the frictional force. We choose the following control force given in Stépán \cite{24}:

\[
Q(t) = b_1\dot{q}_1(t - \tau) + b_0q_1(t - \tau),
\]

where \(b_0\) and \(b_1\) are positive parameters chosen properly.

\* This research is supported by the Natural Science Foundation of China grant NSFC-60874034 and the Seed Foundation of Tianjin University.

\* Corresponding author.

E-mail address: zjhan@tju.edu.cn (Z.-J. Han).

0096-3003/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2011.01.023
Eliminating $q_2(t)$ in (1.1) leads to the following equation:

$$(4 - 3 \cos^2 q_1(t)) \ddot{q}_1(t) + 3 \dot{q}_1^2(t) \sin q_1(t) \cos q_1(t) - \frac{6g}{l} \sin q_1(t) + \frac{6}{ml} (b_1 \dot{q}_1(t - \tau) + b_0 q_1(t - \tau)) \cos q_1(t) = 0. \quad (1.2)$$

Obviously, $q_1 = 0$ is an equilibrium of the above system. From [24], the system (1.2) can be linearized at zero with respect to the variation $x$ of $q_1$ as follows:

$$\ddot{x}(t) - \frac{6g}{l} x(t) + \frac{6}{ml} b_1 \dot{x}(t - \tau) + \frac{6}{ml} b_0 x(t - \tau) = 0. \quad (1.3)$$

Set $x_1 := x$, $x_2 := \dot{x}$. We rewrite (1.3) in the following matrix form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{6g}{l} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\frac{d}{m} b_0 & -\frac{d}{m} b_1 \end{pmatrix} \begin{pmatrix} x_1(t - \tau) \\ x_2(t - \tau) \end{pmatrix}. \quad (1.4)$$

Assume that the initial condition is as follows:

$$\begin{cases} y(0) = (x_1(0), x_2(0)), \\
y(\zeta) = \phi(\zeta), \quad \zeta \in [-\tau, 0], \end{cases} \quad (1.5)$$

where $\phi(\zeta)$, $\zeta \in [-\tau, 0]$ is a given function. Then, (1.4) together with (1.5) leads to the following retarded system:

$$\begin{cases} \dot{y}(t) = A_0 y(t) + A_1 y(t - \tau), & t > 0, \\
y(0) = (x_1(0), x_2(0))^T, \\
y(\zeta) = \phi(\zeta), & \zeta \in [-\tau, 0], \end{cases} \quad (1.6)$$

where

$$y(t) := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A_0 := \begin{pmatrix} 0 & 1 \\ \frac{6g}{l} & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} 0 & 0 \\ -\frac{d}{m} b_0 & -\frac{d}{m} b_1 \end{pmatrix}.$$ 

Set $z(t, \theta) := y(t - \theta \tau)$. We have $z(t, 1) = y(t - \tau)$, $z(t, 0) = y(t)$ and

$$\frac{\partial z(t, \theta)}{\partial t} = -\frac{1}{\tau} \frac{\partial z(t, \theta)}{\partial \theta}.$$ 

Thus, (1.6) is equivalent to the following equations:

$$\begin{cases} \dot{y}(t) = A_0 y(t) + A_1 z(t, 1), \\
\frac{\partial z(t, \theta)}{\partial t} = -\frac{1}{\tau} \frac{\partial z(t, \theta)}{\partial \theta}, \\
z(t, 0) = y(t), \\
z(0, \theta) = y(-\theta \tau) = \phi(-\theta \tau), & \theta \in [0, 1]. \end{cases} \quad (1.7)$$

In this paper, we shall analyze the eigenvalues of the system (1.7) and expand the solution of the system by their corresponding eigenvectors. By calculations, we obtain that the characteristic equation of the system (1.7) follows the form in Hagen [20], in which the zeros of characteristic equations were studied. The techniques in [20] were employed to derive characteristic equations of the size structured population model (see [22, 23]). Similar to [20], we obtain the asymptotic expressions of the spectrum of the system. The main difficulty we encounter in this paper is that the set of the eigenvectors of the system does not form a basis for the state space. However, we still expand the solution of the system (1.7) according to its eigenvectors by a result in [17].

Fig. 1. An inverted pendulum.
The remaining parts of this paper are organized as follows. In Section 2, the system (1.7) is formulated in a Hilbert state space setting and the well-posedness is proved. In Section 3, a complete asymptotic analysis of the spectrum of the system operator is given and its corresponding eigenvectors are obtained. In Section 4, the spectrum and its corresponding eigenvectors of the adjoint operator of the system operator are discussed in the similar way. Then we prove that the set of eigenvectors of the system can not form a basis for the state space. However, in Section 5, the semigroup generated by the system operator is proved to be eventually differentiable. Moreover, it is shown that the solution of (1.7) can be expanded by its eigenvectors. In fact, the expansion converges absolutely in the sense of norm. In Section 6, a simulation is presented to support these results. Finally, a concluding remark is provided.

2. Well-posedness of the system

In this section, we shall study the well-posedness of the system (1.7). We begin with formulating this system in an appropriate Hilbert space setting.

Set the space

\[ \mathcal{H} := C^2 \times L^2([0, 1], C^2) \]

equipped with an inner product: for \((y_1, z_1)^\top, y_2, z_2)^\top \in \mathcal{H}, i = 1, 2,

\[ ((y_1, z_1)^\top, (y_2, z_2)^\top)_{\mathcal{H}} = (y_1, y_2)_{L^2} + \int_0^1 (z_1(s), z_2(s))_{C^2} ds. \]

Obviously, \( \mathcal{H} \) is a Hilbert space.

Then, define the system operator \( A \) in \( \mathcal{H} \) as follows:

\[ A \begin{pmatrix} y \\ z \end{pmatrix} := \begin{pmatrix} A_0 & A_1 \delta \\ 0 & -\frac{1}{\tau} D \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \]

where \( \delta z := z(1) \), \( Df := \frac{df}{dt} \) and

\[ \mathcal{D}(A) := \{(y, z) \in C^2 \times H^1([0, 1], C^2) | z(0) = y \}. \]

With the above notations, (1.7) can be rewritten as an evolutionary equation in \( \mathcal{H} \)

\[
\begin{cases}
\frac{dU(t)}{dt} = AU(t), & t > 0 \\
U(0) = U_0,
\end{cases}
\]

where \( U(t) = (y(t), z(t, 0))^\top \) and \( U(0) = (y(0), \phi(\cdot c)) \). The following result holds.

**Lemma 2.1.** Let \( A \) and \( \mathcal{H} \) be defined as before. Then there exists a new product \((\cdot, \cdot)_{\mathcal{H}}\), which is equivalent to \((\cdot, \cdot)_{\mathcal{H}}\), such that

\[ \Re(\mathcal{A}U, U) \leq M(U, U), \quad \forall U \in \mathcal{D}(A), \text{ for some } M > 0, \]

i.e., \( A - \mathcal{M} \) is a dissipative operator in \( \mathcal{H} \).

**Proof.** Firstly, it is easy to check that \( A \) is a densely defined closed linear operator in \( \mathcal{H} \). Then, define the new inner product of \( \mathcal{H} \): for \( X = (y_1, z_1(\cdot)) \), \( Y = (y_2, z_2(\cdot)) \),

\[ (X, Y) := (y_1, y_2)_{L^2} + \int_0^1 q(\theta)(z_1(\theta), z_2(\theta))_{C^2} d\theta, \]

where \( q(\theta) := \tau \|A_1\|^2 + 1. \) Obviously, \((\cdot, \cdot)_{\mathcal{H}}\) is equivalent to \((\cdot, \cdot)_{\mathcal{H}}\). Under this new inner product of \( \mathcal{H} \), we shall show that \( A - \mathcal{M} \) is dissipative in \( \mathcal{H} \) for some \( M > 0 \). In fact, for any real \( X = (y, z(\cdot))^\top \in \mathcal{D}(A) \), we have

\[
\begin{align*}
(AX, X) &:= (A_0y + A_1z(1), y)_{L^2} + \int_0^1 q(\theta) \left(-\frac{1}{\tau} \frac{dz(\theta)}{d\theta}, z(\theta)\right)_{C^2} d\theta \\
&\leq \|A_0\| \|y\|_{C^2}^2 + \|A_1\| \|z(1)\|_{C^2} \|y\|_{C^2} + \frac{1}{2\tau} \int_0^1 q(\theta) \frac{dz(\theta)}{d\theta}, z(\theta)\right)_{C^2} d\theta \\
&\leq \left(\|A_0\| + \frac{1}{2} + \frac{1}{2\tau} q(0)\right) \|y\|_{C^2}^2 + \left(\|A_1\|^2 - \frac{1}{2\tau} q(1)\right) \|z(1)\|_{C^2}^2 + \frac{1}{2\tau} \int_0^1 \frac{q(\theta)q(\theta)}{q(\theta)} \|z(\theta)\|_{C^2}^2 d\theta \\
&\leq M(\|y\|_{C^2}^2 + \int_0^1 q(\theta)\|z(\theta)\|_{C^2}^2) = M(\mathcal{X}, \mathcal{X}),
\end{align*}
\]

where \( M := \max\{\|A_0\| + \frac{1}{2} + \frac{1}{2\tau}, \|A_1\|^2\} \). Therefore, \( A - \mathcal{M} \) is dissipative in \( \mathcal{H} \). \( \square \)
Lemma 2.2. Let \(A\) and \(\mathcal{H}\) be defined as before. Set
\[
\Delta(\lambda) := \lambda I_{22} - A_0 - A_1 e^{-i\lambda}, \quad \lambda \in \mathbb{C},
\]  
(2.2)
where \(I_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). If \(\lambda\) satisfies \(\det \Delta(\lambda) \neq 0\), then \(\lambda \in \rho(A)\). Moreover, \(R(\lambda, A)\) is compact and given as follows:
\[
\begin{cases}
(\lambda I - A)^{-1} Y = X = (y, \bar{z}(0))^T, \quad \forall Y = (y_1, y_2)^T \in \mathcal{H}, \\
y = \Delta(\lambda)^{-1} y_1 + A_1 \int_0^1 e^{-i\lambda(1-t)} \tau y_2(r) dr, \\
\bar{z}(0) = e^{-i\lambda t} \bar{z} + \int_0^t e^{-i\lambda(\tau-t)} \tau y_2(\gamma) d\gamma.
\end{cases}
\]  
(2.3)

Proof. For \(\lambda \in \mathbb{C}\) and \(Y = (y_1, y_2)^T \in \mathcal{H}\), let us consider the following resolvent problem:
\[
(\lambda I - A) X = Y, \quad X = (\bar{y}, \bar{z})^T \in \mathcal{D}(A),
\]
i.e.,
\[
\begin{align*}
\dot{y} - A_0 y - A_1 \bar{z}(1) &= y_1, \\
\dot{z} + \frac{1}{\mathcal{D}(\theta)} &= y_2, \quad \theta \in [0, 1], \\
\bar{z}(0) &= \bar{y}.
\end{align*}
\]  
(2.4)
Solving (2.4), we obtain that when \(\Delta(\lambda) \neq 0\), there is a unique \((\bar{y}, \bar{z})^T\) such that \((\bar{y}, \bar{z})^T \in \mathcal{D}(A)\) and
\[
R(\lambda, A) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \Delta(\lambda)^{-1} y_1 + A_1 \int_0^1 e^{-i\lambda(1-t)} \tau y_2(r) dr \\ e^{-i\lambda t} \bar{z} + \int_0^t e^{-i\lambda(\tau-t)} \tau y_2(\gamma) d\gamma \end{pmatrix}.
\]  
(2.5)

From (2.5), we have that \(R(\lambda, A)\) is bounded and \(R(\lambda, A) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{D}(A) \subset \mathcal{H}\). Hence, \(\lambda \in \rho(A)\). Furthermore, note that Sobolev embedding theorem (see [26]) asserts that \(\mathcal{D}(A)\) is a compact subspace of \(\mathcal{H}\). Therefore, \(R(\lambda, A)\) is compact on \(\mathcal{H}\). □

As a consequence of Lemma 2.2, the following result holds.

Corollary 2.1. Let \(A\) and \(\mathcal{H}\) be defined as before. Then
\[
\sigma(A) = \sigma_\rho(A) = \{ \lambda \in \mathbb{C} | \det \Delta(\lambda) = 0 \}.
\]

Proof. The compactness of the resolvent operator implies that the spectrum of \(A\) only consists of isolated eigenvalues of \(A\), i.e., \(\sigma(A) = \sigma_\rho(A)\). On one hand, we have known from Lemma 2.2 that if \(\det \Delta(\lambda) \neq 0\), then \(\lambda \in \rho(A)\). Thus, \(\sigma(A) \subset \{ \lambda \in \mathbb{C} | \det \Delta(\lambda) = 0 \}\). On the other hand, for \(\lambda \in \mathbb{C}\), which satisfies \(\det \Delta(\lambda) = 0\), we have that \(\Delta(\lambda) \xi = 0\) has at least one nonzero solution \(\xi \in \mathbb{C}^2\). Then \((\xi, e^{-i\lambda t} \xi)^T \in \mathcal{D}(A)\) and satisfies
\[
(\lambda I - A)(\xi, e^{-i\lambda t} \xi)^T = (0, 0)^T,
\]
which implies \(\lambda \in \sigma_\rho(A)\). Hence, \(\{ \lambda \in \mathbb{C} | \det \Delta(\lambda) = 0 \} \subset \sigma_\rho(A)\). Therefore, \(\sigma(A) = \sigma_\rho(A) = \{ \lambda \in \mathbb{C} | \det \Delta(\lambda) = 0 \}\). □

By Lemmas 2.1 and 2.2, according to the Lumer–Phillips theorem (see [19]), we have the following result.

Theorem 2.1. Let \(A\) and \(\mathcal{H}\) be defined as before. Then \(A - M I\) generates a \(C_0\) semigroup of contractions on \(\mathcal{H}\). Therefore, \(A\) generates a \(C_0\) semigroup on \(\mathcal{H}\). Hence, the system (2.1) is well-posed.

3. Spectral analysis of \(A\)

In this section, we shall give a complete asymptotic analysis of eigenvalues and their corresponding eigenvectors of \(A\). In Corollary 2.1, we have proved that \(\sigma(A) = \{ \lambda \in \mathbb{C} | \det \Delta(\lambda) = 0 \}\), where
\[
\Delta(\lambda) = \lambda I_{22} - A_0 - A_1 e^{-i\lambda} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \frac{6\varepsilon}{\mu} & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{6\varepsilon}{\mu} b_0 e^{-i\lambda} \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{6\varepsilon}{\mu} b_1 e^{-i\lambda} \end{pmatrix}.
\]
Hence,
\[
\det \Delta(\lambda) = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ \frac{6\varepsilon}{\mu} & 0 \end{vmatrix} - \begin{vmatrix} 0 \\ -\frac{6\varepsilon}{\mu} b_0 e^{-i\lambda} \end{vmatrix} = \lambda e^{-i\lambda} \begin{vmatrix} 0 \\ -\frac{6\varepsilon}{\mu} b_0 e^{-i\lambda} \end{vmatrix} + \begin{vmatrix} \frac{6\varepsilon}{\mu} b_1 e^{-i\lambda} \\ 0 \end{vmatrix}.
\]
It is easy to check that \(\det \Delta(\lambda)\) is an entire function on \(\mathbb{C}^2\). Note that for any \(\lambda \in \mathbb{C}^2\) with \(\varepsilon < \Re \lambda < \beta\), when \(|\lambda| \to +\infty\), we have \(|\det \Delta(\lambda)| \to +\infty\). Therefore, \(\det \Delta(\lambda)\) has finite zeros in the strip \(\varepsilon < \Re \lambda < \beta\). Moreover, \(|\det \Delta(\lambda)| \to +\infty\) as \(\Re \lambda \to +\infty\).
Now let us determine the asymptotic zeros of det$\Lambda(\lambda)$ as $\Re \lambda \to -\infty$. Set det$\Lambda(\lambda) = 0$. Then we have $\lambda^2 e^{i\tau} - \frac{6}{m^2} e^{i\tau} + \frac{6}{m^2} b_1 + \frac{6}{m^2} b_0 = 0$, which leads to $\lambda^2 e^{i\tau} = -\frac{6}{m^2} b_1 + O(1)$. Hence the asymptotic zeros of det$\Lambda(\lambda)$ are entirely determined by

$$\lambda e^{i\tau} + \frac{6}{m^2} b_1 = 0.$$ 

For convenience, set

$$f(z) := z e^{i\tau} - b, \quad \tau > 0,$$

where $b := -\frac{6}{m^2} b_1$. Note that $f(z)$ can be considered as a particular case of the general form in [20]. As a result, some techniques in Hagen [20] are used here to obtain the asymptotic description of the spectrum of $A$. Since $m$, $\ell$, $b_1$ are all positive constants, we have that $b < 0$. Then we obtain the following result.

**Lemma 3.1.** Let $f(z)$ be defined by (3.1). Then we have

1. when $b = -\frac{1}{\ell} e^{-1}$, $f(z)$ has a real zero uniquely;
2. when $b < -\frac{1}{\ell} e^{-1}$, $f(z)$ has no real zero;
3. when $-\frac{1}{\ell} e^{-1} < b < 0$, $f(z)$ has two real zeros which are all negative.

**Proof.** Since only the real zeros of $f(z)$ are discussed in this lemma, we can restrict $f(z)$ on $\Re$. A direct calculation yields

$$f'(z) = (1 + \tau z) e^{i\tau}.$$ 

Hence, $f(z)$ is strictly monotone decreasing in $(-\infty, -\frac{1}{\tau})$ and strictly monotone increasing in $(-\frac{1}{\tau}, +\infty)$. Then $z = -\frac{1}{\tau}$ is the only minimum of $f(z)$, and $f(-\frac{1}{\tau}) = -\frac{1}{\ell} e^{-1} - b$. Moreover, we have $\lim_{z \to -\infty} f(z) = -b$, $\lim_{z \to +\infty} f(z) = +\infty$, and $f(0) = -b > 0$. Thus, the desired result follows. $\square$

**Lemma 3.2.** Let $f(z)$ be defined by (3.1). Then the set of zeros of $f(z)$ is $\Lambda = \{z_n, \overline{z_n}\}_{n \in \mathbb{N}} \cup \Upsilon$, where $\Upsilon$ is the set of possible real zeros of $f(z)$ and the expressions of $z_n$ are given as follows:

$$z_n = \frac{1}{\tau} \left( \ln(-b) - \ln \omega_n \right) + i \left( \omega_n - \frac{\ln \omega_n}{\tau^2 \omega_n} \right) + O\left(\frac{1}{n}\right),$$

where $\omega_n := \frac{2n}{\tau} \pi$, $n = 1, 2, \ldots$

**Proof.** Since $\tau$ and $b$ are real constants, the zeros of $f(z)$ distribute symmetrically with respect to real axis. Therefore, the set of zeros of $f(z)$ can be written as follows:

$$\Lambda = \{z_n, \overline{z_n}\}_{n \in \mathbb{N}} \cup \Upsilon.$$

Set $\xi := x + iy \in \Lambda$ with $y > 0$, then $\xi e^{i\tau} = b$. Hence,

$$e^{i\tau}(\cos \tau y + i \sin \tau y)(x + iy) = b,$$

which leads to $e^{i\tau}(\cos \tau y - y \sin \tau y) = b$ and $e^{i\tau}(x \sin \tau y + y \cos \tau y) = 0$. Thus,

$$x = -\frac{y \cos \tau y}{\sin \tau y},$$

$$e^{i\tau} = -\frac{b \sin \tau y}{y}.$$ 

By (3.3), we obtain $x = \frac{1}{\tau} \left[ \ln(-b \sin \tau y) - \ln y \right]$, which together with (3.4) leads to

$$\ln(-b \sin \tau y) - \ln y + \frac{\tau y \cos \tau y}{\sin \tau y} = 0.$$ 

Since $b = -\frac{6}{m^2} b_1 < 0$, we have $\sin \tau y > 0$, which implies $y \in \left(\frac{2n-1}{\tau} \pi, \frac{2n}{\tau} \pi\right), \quad n \geqslant 1$.

Set

$$\psi(y) := \ln(-b) + \ln(\sin \tau y) - \ln y + \frac{\tau y \cos \tau y}{\sin \tau y}, \quad \forall y \in \left(\frac{2n-2}{\tau} \pi, \frac{2n-1}{\tau} \pi\right), \quad n \geqslant 1.$$
A direct calculation leads to

$$\psi(y) = \frac{2\tau y \sin \tau y \cos \tau y - \sin^2 \tau y - \tau^2 y^2}{y \sin^2 \tau y} < 0, \quad \forall y \in \left(\frac{2n - 2}{\tau}, \frac{2n - 1}{\tau}\right), \quad n \geq 1.$$  \hfill (3.6)

Meanwhile, we obtain that

$$\lim_{y \to (-\frac{2}{\tau}e^{i\pi})} \psi(y) = \ln(-b) - \ln \left(\frac{2n - 2}{\tau}\right) + \lim_{y \to (-\frac{2}{\tau}e^{i\pi})} \left[\ln(\sin \tau y) + \frac{\tau y \cos \tau y}{\frac{\sin \tau y}{y}}\right] = +\infty$$  \hfill (3.7)

and

$$\lim_{y \to (-\frac{2}{\tau}e^{i\pi})} \psi(y) = \ln(-b) - \ln \left(\frac{2n - 1}{\tau}\right) + \lim_{y \to (-\frac{2}{\tau}e^{i\pi})} \left[\ln(\sin \tau y) + \frac{\tau y \cos \tau y}{\frac{\sin \tau y}{y}}\right] = -\infty.$$  \hfill (3.8)

Therefore, there exists a unique $y_n \in \left(\frac{2n - 2}{\tau}, \frac{2n - 1}{\tau}\right)$, such that $\psi(y_n) = 0$, $n = 1, 2, \ldots$

For each $n \in \mathbb{N}$, set $x_n := \frac{1}{\tau} \ln(-b_n)$, then $\lambda_n = x_n + iy_n$ is a zero of $f(z)$. Hence $\Lambda$ contains infinite number of points. Note that when $y_n > |b|$, $x_n = \frac{1}{\tau} \ln(-b_n) < 0$ and

$$x_n \to -\infty, \quad y_n \to +\infty \text{ as } n \to +\infty.$$  \hfill (3.9)

From (3.4) and (3.5), we have $e^{\tau e^{i\pi}} = b \cos \tau y_n$. This together with (3.9) yields that $\cos y_n < 0$ and $\cos y_n \to 0$ as $n \to +\infty$. Hence, for $y_n \in \left(\frac{2n - 2}{\tau}, \frac{2n - 1}{\tau}\right)$, we have

$$y_n \to \frac{2n - 2}{\tau}, \quad \text{as } n \to +\infty.$$  \hfill (3.10)

Set $y_n := \frac{2n - 2}{\tau e^{i\pi}}$, where $\epsilon_n \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\epsilon_n \to 0$ as $n \to +\infty$. Substituting $y_n$ into $\psi(y)$ yields

$$\psi(y_n) = \ln(-b) + \ln \cos \epsilon_n - \ln y_n - \frac{\tau y_n \sin \epsilon_n}{\cos \epsilon_n} = 0,$$

which implies $\sin \epsilon_n = -\frac{\ln^{2n\frac{1}{2}}}{(2n - 2)\pi} + O(n^{-1})$ as $n \to +\infty$. Hence, we have

$$\epsilon_n = -\frac{\ln^{2n\frac{1}{2}}}{(2n - 2)\pi} + O(n^{-1}).$$

Therefore,

$$y_n \to \frac{2n - 2}{\tau} - \frac{\ln^{2n\frac{1}{2}}}{(2n - 2)\pi} + O(n^{-1}).$$  \hfill (3.11)

Substituting (3.11) into $x_n = \frac{1}{\tau} \ln(-b) + \ln(\sin(\tau y_n)) - \ln y_n$ yields

$$x_n = \frac{1}{\tau} \left[\ln(-b) - \ln \frac{2n - 2}{\tau} + O\left(\frac{\ln^2 n}{n^2}\right)\right].$$  \hfill (3.12)

Then the desired result follows from (3.11) and (3.12). \hfill \Box

By this lemma, we have the following theorem.

**Theorem 3.1.** Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then $\sigma(\mathcal{A})$ is given as follows:

$$\sigma(\mathcal{A}) = \{\lambda_n, \bar{\lambda}_n\}_{n \in \mathbb{N}} \cup \bar{Y},$$  \hfill (3.13)

where $\lambda_n = \xi_n + O\left(\frac{1}{n}\right)$ and $\bar{\xi}_n, n = 1, 2, \ldots$ are given by (3.2), and $\bar{Y}$ is the set of possible real eigenvalues of $\mathcal{A}$.

**Proof.** According to Corollary 2.1, for $\lambda \in \mathbb{C}$, $\lambda \in \sigma(\mathcal{A})$ if and only if $\det(\Delta(\lambda)) = 0$. Note that

$$\det(\Delta(\lambda)) = e^{-i\Delta} \left[\lambda^2 e^{i\Delta} - \frac{6g}{\ell} e^{i\Delta} + \left(\frac{6}{m^2} b_1 \lambda + \frac{6}{m^2} b_0\right)\right].$$  \hfill (3.14)

Since $m, g, \ell, b_0, b_1$ are all real constants, we also have $\det(\Delta(\lambda)) = 0$ when $\det(\Delta(\lambda)) = 0$. Therefore, the spectrum of $\mathcal{A}$ distributes in conjugate pairs on the complex plane.

When $\Re \lambda < 0$ and $|\Re \lambda|$ is sufficiently large, $\det(\Delta(\lambda))$ has the asymptotic expression

$$\frac{1}{\lambda} e^{i\Delta} \det(\Delta(\lambda)) = \left(\lambda e^{i\Delta} - b\right) + O\left(\frac{1}{\lambda}\right).$$
We have

\[ 0 = \det \Delta(\lambda) = \lambda e^{-i\tau} \left[ (\lambda e^{i\tau} - b) - \frac{6g}{\ell \lambda} e^{i\tau} + \frac{6}{m \ell \lambda} b_0 \right]. \]

For \( \lambda e^{-i\tau} \to \infty \), we obtain that

\[(\lambda e^{i\tau} - b) - \frac{6g}{\ell \lambda} e^{i\tau} + \frac{6}{m \ell \lambda} b_0 = 0.\]

Let \( \tilde{\lambda}_n = \tilde{\epsilon}_n + \epsilon_n \) be a zero of \( \det \Delta(\lambda) = 0 \), where \( \tilde{\epsilon}_n \) is the zero of \( f(z) = b \) given by (3.2). Then

\[ \tilde{\lambda}_n e^{i\tau z} = (\tilde{\epsilon}_n + \epsilon_n) e^{i(\tilde{\epsilon}_n + \epsilon_n)} = \left( b + \tilde{\epsilon}_n \frac{b}{\tilde{\epsilon}_n} \right) e^{i\tau n}. \]

Since \( \frac{1}{\tilde{\epsilon}_n} = O(n^{-1}) \), we have

\[
0 = (b + \tilde{\epsilon}_n \frac{b}{\tilde{\epsilon}_n}) e^{i\tau n} - \frac{6g}{\ell \tilde{\epsilon}_n + \epsilon_n} e^{i(\tilde{\epsilon}_n + \epsilon_n)} - b + \frac{6}{m \ell \tilde{\epsilon}_n + \epsilon_n} b_0 = \left( b + \tilde{\epsilon}_n \frac{b}{\tilde{\epsilon}_n} \right) e^{i\tau n} - \frac{6g}{\ell \tilde{\epsilon}_n + \epsilon_n} \tilde{\epsilon}_n e^{i\tau n} - b + \frac{6}{m \ell \tilde{\epsilon}_n + \epsilon_n} b_0.
\]

By the above calculation, we have that \( \tilde{\epsilon}_n = O(n^{-1}) \). Therefore, \( \tilde{\lambda}_n = \tilde{\epsilon}_n + O(n^{-1}) \). The proof is complete. \( \square \)

**Remark 3.1.** The asymptotic expressions of the spectrum of \( A \) also can be derived from [20]. In fact, the zeros of the characteristic equation of the general form \( \alpha^t e^{i\tau n} - b = 0 \) is studied in Hagen [20]. The characteristic equation \( f(z) = 0 \) of our problem given by (3.1) is a special case of this form.

Now let us consider the eigenvectors of \( A \). We have the following result.

**Theorem 3.2.** Let \( A \) and \( H \) be defined as before. Then each \( \lambda \in \sigma(A) \) is simple. Furthermore, its corresponding eigenvector is of the form:

\[ \Phi_n = (\tilde{y}_n, e^{-i\tau n} y_n)^T, \]

where \( \tilde{y}_n \) is a non-zero solution to equation \( \Delta(\tilde{\lambda}_n) y = 0 \) and given by \( \tilde{y}_n = (1, \tilde{\lambda}_n)^T \).

**Proof.** For \( \lambda \in \sigma(A) \), we have

\[ \det \Delta(\lambda) = \lambda \left[ \lambda + \frac{6g}{m \ell} b_0 e^{-i\tau} \right] - \frac{6g}{\ell} e^{i\tau} + \frac{6}{m \ell} b_0 e^{-i\tau} = 0. \]

Since \( \frac{\partial}{\partial \lambda} \det \Delta(\lambda) \neq 0 \) as \( \det \Delta(\lambda) = 0 \), the zeros of \( \det \Delta(\lambda) \) are simple ones.

Now let \( \lambda \in C \) be an eigenvalue of \( A \), and \( \Phi = (\tilde{x}_0, \tilde{x}_1)^T \in D(A) \) be an eigenvector corresponding to \( \lambda \). Then \( A \Phi - \lambda \Phi \), i.e.

\[
\begin{cases}
A_0 \tilde{x}_0 + A_1 \tilde{x}_1(1) = \lambda \tilde{x}_0, \\
- \frac{1}{\tau} \frac{\partial \tilde{x}_0}{\partial \theta} = \lambda \tilde{x}_1(\theta), \\
\tilde{x}_1(0) = \tilde{x}_0.
\end{cases}
\]

From the second equation in (3.15), we have \( \tilde{x}_1(\theta) = e^{-i\tau \theta} \tilde{x}_0, \ \theta \in [0, 1] \). Then substituting the expression of \( \tilde{x}_1(\theta) \) into the first equation in (3.15) leads to

\[ \Delta(\lambda) \tilde{x}_0 = \left( \begin{array}{cc} \lambda & -1 \\ -\frac{6g}{\ell} + \frac{6}{m \ell} b_0 e^{-i\tau} & \lambda + \frac{6}{m \ell} b_1 e^{-i\tau} \end{array} \right) \tilde{x}_0 = 0. \]

Since \( \tilde{x}_0 \neq 0 \), solving the equation above, we get the form of \( \tilde{x}_0 \) as follows:

\[ \tilde{x}_0 = (1, \tilde{\lambda})^T. \]

Therefore, the eigenvector of \( A \) corresponding to \( \tilde{\lambda}_n \) is \( \Phi_n = (\tilde{y}_n, e^{-i\tau n} y_n)^T \), where \( \tilde{y}_n = (1, \tilde{\lambda}_n)^T. \) \( \square \)
4. Non-basis property of eigenvectors of $A$

In this section, we shall prove that the set of eigenvectors of $A$ can not form a basis for the state space. For this aim, let us discuss the spectral property of the adjoint operator of $A$. A direct calculation yields that the adjoint operator of $A$ is as follows.

$$\mathcal{D}(A^*) = \left\{ (x_2, y_2(\theta))^T \in \mathcal{H} | y_2 \in H^1([0, 1], \mathbb{C}^2), \partial y_2 = y_2(1) = \tau A^*_j x_2 \right\} \quad (4.1)$$

and

$$A^*(x_2, y_2(\theta)) = \begin{pmatrix} A^*_j x_2 \\ 0 \\ \frac{\delta_0}{1 - r} \end{pmatrix}, \quad (4.2)$$

where $\delta_0(0) = g(0), \forall \theta \in [0, 1],$ and $A^*_j, j = 0, 1$ denote the conjugate transposes of $A_j$.

Similar to the discussion in the last section, we obtain the following result.

**Lemma 4.1.** Let $A^*$ and $\mathcal{D}(A^*)$ be defined as before. For any $\lambda \in \mathbb{C},$ set

$$\Delta^*(\lambda) = \mathcal{I} A_2 - A_2 - e^{-\tau i} A_1^*.$$ 

If $\det \Delta^*(\lambda) \neq 0,$ we have $\lambda \in \rho(A^*).$ Furthermore, the resolvent of $A^*$ is compact and given as follows:

$$\left\{ \begin{array}{l}
(\mathcal{I} - A^*)^{-1} X = Y, Y = (\tilde{y}, \tilde{z}(\theta))^T \in \mathcal{D}(A^*), \\
\tilde{y} = (\Delta^*(\lambda))^{-1} \tilde{x}_1 + \int_0^1 e^{-r \tau (1 - r) A_2} (1 - r) dr,
\end{array} \right.$$ 

By the similar calculation in the proof of Theorem 3.2, we obtain the following spectral property of $A^*$.

**Theorem 4.1.** Let $A^*$ be given by (4.1) and (4.2). Then the spectrum of $A^*$ is

$$\sigma(A^*) = \overline{\sigma(A)} = \{ \lambda \in \mathbb{C} | \det \Delta^*(\lambda) = 0 \}.$$ 

For $\lambda \in \sigma(A^*),$ a corresponding eigenvector is

$$\Psi_\lambda = \left( y_n, \tau e^{-\tau i (1 - \eta)} A_2^* y_n \right)^T \in \mathcal{D}(A^*),$$ 

where

$$y_n = \left( \lambda_n + \frac{6}{m} b_1 e^{-\tau i}, 1 \right)^T.$$ 

Now, let us discuss the non-basis property of the eigenvectors of $A.$ In fact, the following result holds.

**Theorem 4.2.** Let $\lambda \in \sigma(A)$ be an eigenvalue of $A$ and $E(\lambda; A)$ be the corresponding Riesz spectral projection. Then for any $X \in \mathcal{H},$ we have

$$E(\lambda; A)X = (X, \Psi_\lambda)_H \Phi_\lambda,$$ 

where $\Phi_\lambda$ and $\Psi_\lambda$ are of the form in Theorems 3.2 and 4.1 with $(\Phi_\lambda, \Psi_\lambda)_H = 1.$ Furthermore, the eigenvector sequence $\{ \Phi_\lambda | \lambda \in \sigma(A) \}$ can not form a basis for the state space $\mathcal{H}.$

**Proof.** We mainly use the general strategy in [17] to prove this theorem. For any $\eta, \zeta \in \sigma(A)$ with $\eta \neq \zeta,$ let $\Phi_\eta$ be an eigenvector of $A$ corresponding to $\eta,$ and $\Psi_\zeta$ be an eigenvector of $A^*$ corresponding to $\zeta.$ Then we have

$$\lambda \in (\Phi_\eta, \Psi_\zeta)_H = (\Phi_\eta, A^* \Psi_\zeta)_H = (\Phi_\eta, \tilde{\Psi}_\zeta)_H = (\Phi_\eta, \tilde{\Psi}_\zeta)_H = \zeta (\Phi_\eta, \Psi_\zeta)_H,$$

which implies that $\eta \neq \zeta$ if $(\Phi_\eta, \Psi_\zeta)_H = 0.$

Let $\lambda \in \sigma(A)$ and $E(\lambda; A)$ be the corresponding Riesz spectral projection. From Theorems 3.2 and 4.1, we obtain that $E(\lambda; A)_H$ is a one-dimensional subspace of $\mathcal{H}$ spanned by $\Phi_\lambda.$ Moreover, similar to [17], choose

$$\begin{align}
\Phi_\lambda &= \frac{1}{\sqrt{\| h \|_H^2(\zeta)}} \begin{pmatrix} \eta \lambda + \frac{6}{m} b_1 e^{-\tau i} \\ \eta \lambda + \frac{6}{m} b_1 e^{-\tau i} \end{pmatrix}, \\
\Psi_\zeta &= \frac{1}{\sqrt{\| h \|_H^2(\zeta)}} \begin{pmatrix} (\lambda + \frac{6}{m} b_1 e^{-\tau i}, 1) \\ (\lambda + \frac{6}{m} b_1 e^{-\tau i}, 1) \end{pmatrix},
\end{align}$$

where $\eta(\lambda) := 2 \lambda + \frac{6}{m} e^{-\tau i} (b_1 - \tau b_0 - \tau b_1).$ It is easy to check that $(\Phi_\lambda, \Psi_\zeta)_H = 1$ for $\lambda \in \sigma(A).$ Therefore, for any $X \in \mathcal{H},$ we have

$$E(\lambda; A)X = (X, \Psi_\zeta)_H \Phi_\lambda.$$ 

(4.9)
Thus, \( \|E(\lambda; A)\| = \|\Phi_1\|_N \|\Psi_1\|_N \).

By a direct calculation, we estimate that when \( \Re \lambda \to -\infty \),
\[
\|\Phi_1\|_N^2 = \frac{|\lambda|^2}{2\pi} + o(1),
\]
(4.10)
\[
\|\Psi_1\|_N^2 = \frac{6m_b|\Re \lambda|^2 |\lambda| e^{-2\Re \lambda t}}{2\tau^2b_1|\Re \lambda|^2} + 2|\Re \lambda|^2 |\lambda|^2 - \frac{6m b_1^2 k^2}{2m b_1} e^{-2\Re \lambda t} + o(1).
\]
(4.11)

Therefore,
\[
\|E(\lambda; A)\| = \|\Phi_1\|_N \|\Psi_1\|_N \equiv \frac{6m_b|\Re \lambda|^2 |\lambda| e^{-2\Re \lambda t}}{2\tau^2b_1|\Re \lambda|^2} + 2|\Re \lambda|^2 |\lambda|^2 - \frac{6m b_1^2 k^2}{2m b_1} e^{-2\Re \lambda t} \to +\infty, \quad \text{as} \quad \lambda \to -\infty,
\]
(4.12)
which implies that \( \{\Phi_1 \lambda \in \sigma(A)\} \) does not form a Schauder basis for the state space \( \mathcal{H} \). \( \square \)

5. Expansion of the solution of the system

In this section, we shall study the expansion of the solution of the system (2.1) according to its eigenvectors. In order to obtain the convergence of the expansion of the solution, we need the following notations and results (see [17]).

Let \( \{T(t)\}_{t \geq 0} \) be a \( C_0 \) semigroup on Banach space \( X \) and \( \lambda \) be its generator. Suppose that the spectrum of \( \mathcal{A} \) is discrete, i.e., \( \sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda_n; n \in \mathbb{N}\} \). For each \( \lambda_n \in \sigma(\mathcal{A}) \), \( E(\lambda_n; A) \) is the Riesz projection corresponding to \( \lambda_n \). Define a \( T(t) \)-spectral invariant subspace by
\[
\text{Sp}(\mathcal{A}) := \text{span}\left\{ \sum_{j=1}^{m} E(\lambda_j; A)x \middle| x \in X; \forall m \in \mathbb{N} \right\}
\]
and another \( T(t) \)-invariant subspace by
\[
\mathcal{M}_\infty := \{x \in X | E(\lambda; A)x = 0, \forall \lambda \in \sigma(A)\}.
\]

Obviously, \( \text{Sp}(\mathcal{A}) + \mathcal{M}_\infty \subseteq X \).

For each \( \lambda_n \in \sigma(\mathcal{A}) \), let \( m_n \) be the algebraic multiplicity of the root subspace corresponding to \( \lambda_n \), and define operators for each \( n \in \mathbb{N} \) by
\[
D_n := (\mathcal{A} - \lambda_n)E(\lambda_n; A), \quad D_n^0 = E(\lambda_n; A).
\]

In fact, \( D_n \) is a bounded linear operator on \( X \) and \( D_n^k = (\mathcal{A} - \lambda_n)^k E(\lambda_n; A) \), \( D_n^m = 0 \).

The following results come from [17].

Lemma 5.1. Let \( T(t) \) be a \( C_0 \) semigroup on a Banach space \( X \) and \( \mathcal{A} \) be its generator. Suppose that \( \mathcal{A} \) has discrete spectrum and satisfies following conditions:

(c1) there exist positive constants \( M_1, \rho_1, \rho_2 \) such that
\[
\sum_{k=0}^{m_n} \frac{\|D_n^k\|}{k!} \leq M_1 e^{-\rho_1 \Re \lambda_n} e^{\rho_2 t}, \quad \forall n \in \mathbb{N}, \quad t \geq 0.
\]
(c2) there exists a \( \tau_0 > 0 \), such that the series \( \sum_{i=1}^{\infty} e^{\tau_0 t} \) converges.

Then we can define two families of operators parameterized on \( [\tau_0 + \rho_1, \infty) \),
\[
T_1(t) : X \to \text{Sp}(A) \quad \text{and} \quad T_2(t) : X \to \mathcal{M}_\infty,
\]
such that

(1) \( T_1(t) \) is a compact operator, \( T_1(t) \) and \( T_2(t) \) are strongly continuous;
(2) \( T_j(t)T_s = T(t)T_j(t) = T_j(t+s), \) for \( t \geq \tau_0 + \rho_1, s \geq 0, j = 1, 2; \)
(3) \( T(t) \) has a decomposition
\[
T(t) = T_1(t) + T_2(t), \quad t \geq \tau_0 + \rho_1.
\]

In addition, if the following condition on the spectrum of \( \mathcal{A} \) holds:

(c3) there exist constants \( M_2 > 0 \) and \( \rho_2 > 0 \) such that
\[
|\lambda_n| \leq M_2 e^{-\rho_2 \Re \lambda_n},
\]
then, for each \( x \in X, T_1(t)x \) is differentiable in \( [\tau_0 + \rho_1 + \rho_2, \infty) \).
Lemma 5.2. Let $T(t)$ be a $C_0$ semigroup on a Banach space $X$ and $A$ be its generator. Suppose that conditions (c1)–(c3) in Lemma 5.1 hold. In addition, if one of the following conditions is fulfilled:

1. the generalized eigenvectors of $A$ are complete in $X$;
2. the restriction of the resolvent of $A$ to $M_{\infty}$ is an entire function with values in $X$ of finite exponential type $h$; then $T(t)$ is a differentiable semigroup for $t > \tau_1$, where
   \[
   \tau_1 := \max\{\tau_0 + \rho_1 + \rho_2, \tau_0 + \rho_1 + h\}.
   \]

Remark 5.1. Eventual differentiability is a very important property of semigroups. However, it is usually difficult to verify. In fact, Lemmas 5.1 and 5.2 give us a kind of sufficient conditions to check the eventual differentiability of semigroups. In addition, it should be noted that there are some other approaches to verify the eventual differentiability of semigroups, see for example, Hagen [21] and Batty [25].

In what following, we shall verify that $A$ satisfies the conditions in Lemmas 5.1 and 5.2.

Theorem 5.1. Let $A$ and $\mathcal{H}$ be defined as before. Then $A$ satisfies conditions (c1)–(c3) in Lemma 5.1, where $\rho_1 = 2\tau$, $\rho_2 = \tau$, $\rho_3 = 0$ and $\tau_0 > \tau$.

Proof. For any $\lambda \in \sigma(A)$, Theorem 3.2 says that $\lambda$ is a simple eigenvalue of $A$. Hence the algebraic multiplicity $m_\lambda = 1$. From (4.12), we obtain that $E(\lambda; A)$ has the estimate:

\[
\|E(\lambda; A)\| \approx \left(\frac{6b_1}{m^3 b_1} |\Re\lambda| e^{-2\pi |\Re\lambda|} + 2b_1 t |\Re(\lambda e^{-\pi |\Re\lambda|})| \right) e^{-2\pi |\Re\lambda|}.
\]

Note that

\[
\Re\lambda_n = \left(\ln(-1) - \ln(\omega_n)\right) = \left(\ln\left(\frac{6b_1}{m^3 b_1}\right) - \ln\left(\frac{2n - \frac{3}{2}}{\tau}\right)\right).
\]

When $n > \frac{1}{2} \left(\frac{6b_1}{m^3 b_1} \right)$, we have $\Re\lambda_n < 0$. Therefore, if $|\Re\lambda| > 1$,

\[
\sum_{k=0}^{m-1} \frac{t^k}{k!} |||D_k^n||| = \|E(\lambda; A)\| \leq \frac{6}{m^3} e^{-2\pi |\Re\lambda|}.
\]

So the condition (c1) holds, where $M_1 = \frac{6}{m^3}$, $\rho_1 = 2\tau$ and $\rho_2 = 0$.

From Theorem 3.2, we have known that $\sigma(A) = \{\lambda_n, n \in \mathbb{N}\} \cup \bar{Y}$, where $\lambda_n$ has the asymptotic expression:

\[
\lambda_n = \frac{1}{\tau} \left(\ln(-1) - \ln(\omega_n)\right) + i(\omega_n - \frac{\ln(\omega_n)}{\tau^2}) + O\left(\frac{1}{n}\right)
\]

and

\[
\omega_n = \frac{2n - \frac{3}{2}}{\tau}, \quad n = 1, 2, \ldots
\]

Since

\[
e^{\Re\lambda_n t_0} = e^{\Re(\ln(-1) - \ln(\omega_n))} = \left(\frac{-b}{W_{\lambda_n}}\right)^{\frac{\tau}{n}}
\]

taking $t_0 > \tau > 0$, we have

\[
\sum_{n=1}^{\infty} e^{\Re\lambda_n t_0} = \sum_{n=1}^{\infty} \left(\frac{-b}{W_{\lambda_n}}\right)^{\frac{\tau}{n}} = \sum_{n=1}^{\infty} \left(\frac{6b_1}{2n - \frac{3}{2}}\right)^{\frac{\tau}{n}} < \infty.
\]

Therefore, the condition (c2) holds for $\tau_0 > \tau$. Note that $\lambda_n$ is the solution of

\[
ie^{\lambda_n \tau} = 0 + O\left(\frac{1}{n}\right) = -\frac{6b_1}{m^3} b_1 + O\left(\frac{1}{n}\right).
\]

Thus we have

\[
|\Re\lambda_n| \leq \left|\lambda_n\right| \leq \left(\frac{6b_1}{m^3} b_1 + 1\right) e^{-\tau_0 |\Re\lambda_n|}.
\]

Hence the condition (c3) is also fulfilled. □
**Theorem 5.2.** Let $A$ be defined as before. Then its resolvent is a meromorphic function on $C$ of finite exponential type at most $2\tau$. Therefore, the condition 2) in Lemma 5.2 holds.

**Proof.** In Lemma 2.2, we have shown that for any $Y = (y_1, y_2) \in \mathcal{H}$ and $\lambda \in \rho(A)$, the resolvent of $A$ has the form:

$$R(\lambda, A)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \hat{y} \\ 2(\lambda) \end{pmatrix} = \begin{pmatrix} \Delta(\lambda)^{-1} [y_1 + A_1 \int_0^1 e^{-\tau(1-T)} y_2(\gamma)d\gamma] \\ e^{-\tau(1-T)} y_1 + \int_0^1 e^{-\tau(1-T)} y_2(\gamma)d\gamma \end{pmatrix},$$

where $X = (\hat{y}, \hat{2}(\lambda)) \in \mathcal{D}(A)$. The following estimate holds.

$$\left\| e^{-\tau(1-T)} y + \int_0^1 e^{-\tau(1-T)} y_2(\gamma)d\gamma \right\|_{L^2} \leq e^{-\alpha(\lambda)} \left\| y \right\|_{L^2} + \tau \left[ \int_0^1 \left| e^{-\tau(1-T)} y_2(\gamma) \right|^2 d\gamma \right]^{\frac{1}{2}} \leq e^{-\alpha(\lambda)} \left\| y \right\|_{L^2} + \tau \left\| y_2 \right\|_{L^2} \sqrt{1 - e^{-2\alpha(\lambda)}}.$$

Then we have

$$\left\| \hat{2}(\lambda) \right\|_{L^2} = \left\| e^{-\tau(1-T)} y + \int_0^1 e^{-\tau(1-T)} y_2(\gamma)d\gamma \right\|_{L^2} \leq \left( \int_0^1 \left[ 2 \left( e^{-\alpha(\lambda)} \left\| y \right\|_{L^2} + \tau \left\| y_2 \right\|_{L^2} \right) \left( 1 - e^{-2\alpha(\lambda)} \right) \right] d\lambda \right)^{\frac{1}{2}} \leq M_1 e^{\alpha(\lambda)} \left\| y \right\|_{L^2} + \left\| y_2 \right\|_{L^2},$$

where $M_1 > 0$ is a constant.

Furthermore,

$$\left\| y \right\|_{L^2} \leq \left\| \Delta^{-1}(\lambda) \right\| \left\| y_1 + A_1 \int_0^1 e^{-\tau(1-T)} y_2(\gamma)d\gamma \right\|_{L^2} \leq \left\| \Delta^{-1}(\lambda) \right\| \left( \left\| y_1 \right\|_{L^2} + \tau \left\| y_2 \right\|_{L^2} \sqrt{1 - e^{-2\alpha(\lambda)}} \right).$$

According to [18], we obtain that if $T$ is an invertible matrix, there exists a constant $\gamma > 0$, which is independent of $T$ and dependent only on the norm of $C$, such that

$$\left\| T^{-1} \right\| \leq \gamma \left\| T \right\|^{n-1}.$$
If \( \Re \lambda < 0 \) with \( |\lambda| \) large enough, we have
\[
\det \Delta(\lambda) = \lambda e^{-\tau \lambda} \left( \lambda e^{\tau \lambda} + \frac{6}{m^2} b_1 + o(1) \right).
\]
For \( |\lambda e^{\tau \lambda} + \frac{6}{m^2} b_1| > \delta > 0 \),
\[
\|\Delta^{-1}(\lambda)\|^2 \leq \gamma^2 \frac{\|\Delta(\lambda)\|^2}{|\det \Delta(\lambda)|^2} \leq \gamma^2 \frac{\tilde{M}_1(\lambda^2 + \kappa e^{-2\tau \lambda})}{|\lambda e^{\tau \lambda} + \frac{6}{m^2} b_1 + o(1)|^2} \leq \tilde{M}_3.
\]
Set \( \tilde{M}_4 := \max(\tilde{M}_2, \tilde{M}_3) \). For all \( \lambda \in \mathbb{C}^2 \) and \( |\lambda e^{\tau \lambda} + \frac{6}{m^2} b_1| > \delta > 0 \),
\[
\|\Delta^{-1}(\lambda)\| \leq \tilde{M}_4.
\]
Therefore,
\[
\|R(\lambda, A)Y\|^2_{H_0} = \|X\|^2_{H_0} = \|\tilde{y}\|^2_{\mathbb{C}^2} + \|\tilde{\tilde{z}}(\theta)\|^2_{\mathbb{C}^2} \leq M_2^2 e^{2\tau |\mathbb{R}|} \tilde{M}_4 \left( \|y_1\|^2_{\mathbb{C}^2} + \|y_2\|^2_{\mathbb{C}^2} \right) + M_2^2 e^{2\tau |\mathbb{R}|} (\|\tilde{y}\|^2_{\mathbb{C}^2} + \|y_2\|^2_{\mathbb{C}^2})
\leq M_2^2 e^{2\tau |\mathbb{R}|} \tilde{M}_4 \left( \|y_1\|^2_{\mathbb{C}^2} + \|y_2\|^2_{\mathbb{C}^2} \right) + M_2^2 e^{2\tau |\mathbb{R}|} \tilde{M}_2^2 \left( \|y_1\|^2_{\mathbb{C}^2} + \|y_2\|^2_{\mathbb{C}^2} \right) + \|y_2\|^2_{\mathbb{C}^2} \left[ M_2^2 e^{2\tau |\mathbb{R}|} \tilde{M}_4 + M_2^2 e^{2\tau |\mathbb{R}|} \tilde{M}_2^2 + M_2^2 e^{2\tau |\mathbb{R}|} \tilde{M}_4 + M_2^2 e^{2\tau |\mathbb{R}|} \tilde{M}_2^2 \right]
\leq \tilde{M} e^{4\tau |\mathbb{R}|} \|Y\|^2_{H_0}.
\]
Thus, \( \|R(\lambda, A)\| \leq \tilde{M} e^{2\tau |\mathbb{R}|} \), \( \tilde{M} > 0 \). \( \square \)

Set \( \rho_1 = 2\tau, \rho_2 = \tau, \rho_3 = 0, \tau_0 > \tau \) and \( h = 2\tau \), which is the exponential type of the resolvent of \( A \). By Lemma 5.2, the following result holds.

**Theorem 5.3.** Let \( A \) and \( \mathcal{H} \) be defined as before. Then the semigroup generated by \( A \) is eventually differentiable. Furthermore, when \( t > 5\tau \), the solution to the system (2.1) can be expanded according to its eigenvectors as follows:
\[
X(t) = T(t)X_0 + \sum_{j=1}^{m} e^{\mu_j(t)} (X_0, \Psi_{\mu_j})_H \Phi_j + \sum_{n=1}^{\infty} \left( e^{\mu_n(t)} (X_0, \Psi_{\mu_n})_H \Phi_n + e^{\mu_n(t)} (X_0, \Psi_{\mu_n})_H \Phi_n \right),
\]
where \( \zeta \) is the cardinality of \( \mathcal{Y} \), which is the set of the possible real spectra of \( A \); \( \Phi_j, \Psi_j \) is defined in Theorems 3.2 and 4.1, respectively.

### 6. Simulation

In this section, we shall conduct a simulation to support our results. Set the parameters in (1.7) as follows
- \( m = 1.4 \) kg, \( \ell = 1 \) m, \( \tau = 0.1 \) s, \( b_1 = 3 \), \( b_0 = 15 \), and \( g = 9.8 \) N/kg.

![Fig. 2. Spectrum and approximate solution.](image-url)
By Matlab scientific calculation, we get the numerical spectral distributions of $A$ that satisfies $|3\lambda| < 400$ (see the left figure in Fig. 2, in which the symbol "*" stands for the position of each spectrum). From this figure, we see clearly that all the spectra locate in the left half complex plane and distribute in conjugate pairs on the complex plane.

Set the initial condition of the system (2.1) $X_0 = ([0.2, 0.5], [0.5, -0.1])$. By simulation, we get the approximate numerical solution (Tilt Angle) of the system (2.1), given by the right figure in Fig. 2.

Note that the numerical solution (Tilt Angle) at the initial time $t = 0$ is $q_1 = 0.09$, which is not equal to the initial condition $q_0 = 0.2$. The reason is that our method simulates the solution of the system (2.1) effectively only when $t > 5\tau = 0.5 \text{ s}$ according to Theorem 5.3. Hence, our numerical solution can become available only under this condition.

However, although when $t < 5\tau = 0.5 \text{ s}$ we do not know whether the result in Theorem 5.3 is true, we can get the solution by iterative method in the interval $[0, 5\tau]$. The iterative method usually works not very well when $t$ is large, but it always gets the valid solution when $t$ is small. Hence these two methods can be combined to calculate the solution of this kind of systems involving time delay. Especially, when $t$ is large enough, our method can be applied directly to calculate numerical solution effectively.

7. Concluding remark

In this paper, we studied the expansion of the solution of a kind of inverted pendulum system with time delay according to its eigenvectors. In fact, we considered the following equations:

$$
\begin{pmatrix}
\frac{1}{2}m^2 \dot{q}_1(t) \\
\frac{1}{2}m\cos q_1(t)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2}m t^2 \\
\frac{1}{2}m\cos q_1(t)
\end{pmatrix}
\begin{pmatrix}
q_1(t) \\
\dot{q}_2(t)
\end{pmatrix} - \begin{pmatrix}
\frac{1}{2}mg \sin q_1(t) \\
\frac{1}{2}m\dot{q}_1(t) \sin q_1(t)
\end{pmatrix} = \begin{pmatrix}
0 \\
Q(t)
\end{pmatrix}
$$

with

$$Q(t) = b_1q_1(t - \tau) + b_0\dot{q}_1(t - \tau).$$

Under the following condition

$$t > 5\tau,$$

where $\tau$ is the time delay involved, we proved that the sum in (5.1) is convergent absolutely. Therefore, when $t > 5\tau$, the solution of this kind of systems can be expanded according to its eigenvectors as the form of (5.1). Besides, the method used in this paper can be applied to many other kinds of retarded different equations.

Acknowledgements

The authors thank the referees for their very constructive and useful comments and suggestions. The authors also thank Dr. Xiao-Qin Tang in Michigan State University and Mr. Bo-Jian Cao at Ford Motor Company for their helpful suggestions.

References


