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Chromaticity of A Family of K_4-homeomorph with Girth 9

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Abstract. For a graph G, let $P(G,\lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply χ-equivalent), denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. A graph G is chromatically unique (or simply χ-unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G. A K_4-homeomorph is a subdivision of the complete graph K_4. In this paper, we investigate the chromaticity of one family of K_4-homeomorph which has girth 9, and give sufficient and necessary condition for the graph in the family to be chromatically unique.

Keywords: Chromatic polynomial, chromaticity, K_4-homeomorph.
PACS: 02.10.Ox

INTRODUCTION

All graphs considered here are simple graphs. For such a graph G, let $P(G,\lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply χ-equivalent), denoted by $G{\sim}H$, if $P(G,\lambda)= P(H,\lambda)$. A graph G is chromatically unique (or simply χ-unique) if for any graph H such as $H{\sim}G$, we have $H \cong G$,i.e, H is isomorphic to G. A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(a,b,c,d,e,f)$ if the six edges of K_4 are replaced by the six paths of length a,b,c,d,e,f, respectively, as shown in Figure 1. So far, the chromaticity of K_4-homeomorph with girth g, where $3 \le g \le 7$ has been studied by many authors(see [1-5]). Also the study of the chromaticity of K_4-homeomorph with at least two paths of length 1 has been fulfilled (see [2,6-8]). Recently, Shi et al. [9] studied the chromaticity of one family of K_4-homeomorph with girth 8, i.e., $K_4(2,3,3,d,e,f)$. In [10], Shi has solved completely the chromaticity of K_4-homeomorph with girth 8. As we know, only the chromaticity of such graph with at least two paths of length 1 has been obtained among all the K_4-homeomorph with girth 9. The chromaticity of K_4-homeomorph with exactly three paths of the same length has been obtained by Ren [11]. Recently, Catada-Ghimire and Hasni [12] investigated the chromaticity of K_4-homeomorph with exactly two paths of length 2. When referring to the chromaticity of K_4-homeomorph with girth 9, we know that six types of K_4-homeomorph need to be solved, that is, $K_4(1,2,6,d,e,f)$, $K_4(1,3,5,d,e,f)$, $K_4(1,4,4,d,e,f)$, $K_4(2,3,4,d,e,f)$, $K_4(1,2,c,3,e,3)$ and $K_4(1,3,c,2,e,3)$. Because the length of this paper will be too long and some details cannot be left out, in this paper, we consider the chromaticity K_4 (2, 3, 4, d, e, f) (Figure 2). The chromaticity of other types will be discussed in other paper.

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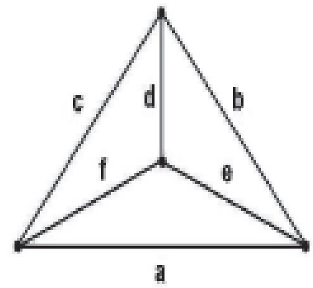

FIGURE 1. $K_4(a,b,c,d,e,f)$

PRELIMINARY RESULTS

In this section, we give some known results used in the sequel.

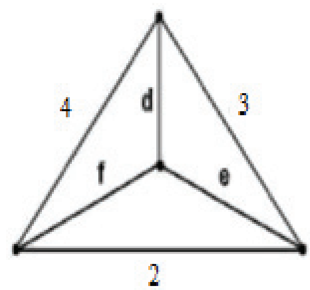

FIGURE 2. $K_4(2,3,4,d,e,f)$

Lemma 2.1 Assume that G and H are simply χ-equivalent. Then,
(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ (see [13]);
(2) G and H has the same girth and the same number of cycles with length equal to their girth (see[14]);
(3) If G is a K_4-homeomorph, then H must itself be a K_4-homeomorph (see [15]);
(4) Let $G = K_4(a,b,c,d,e,f)$ and $H = K_4(a',b',c',d',e',f')$, then
(i) min (a,b,c,d,e,f) = min (a',b',c',d',e',f') and the number of times that this minimum occurs in the list $\{a,b,c,d,e,f\}$ is equal to the number of times that his minimum occurs in the list $\{a',b',c',d',e',f'\}$ (see [16]);
(ii) if $\{a,b,c,d,e,f\} = \{a',b',c',d',e',f'\}$ as multisets, then $H \cong G$ (see [2]).

Lemma 2.2 (Hasni [17]) Let K_4-homeomorphs $K_4(2,3,4,d,e,f)$ and $K_4(1,2,6,d',e',f')$ be chromatically equivalent, then

$$K_4(2,3,4,1,7,s) \sim K_4(1,2,6,4,s,4),$$
$$K_4(2,3,4,7,1,5) \sim K_4(1,2,6,6,3,4),$$
$$K_4(2,3,4,1,5,8) \sim K_4(1,2,6,6,4,4),$$
$$K_4(2,3,4,1,7,4) \sim K_4(1,2,6,4,4,4),$$
$$K_4(2,3,4,10,6,1) \sim K_4(1,2,6,9,3,5),$$
$$K_4(2,3,4,6,6,1) \sim K_4(1,2,6,5,5,5),$$

where $s \geq 6$.

Lemma 2.3 (Ren [11]) Let $G = K_4(a,b,c,d,e,f)$ when exactly three of a,b,c,d,e,f are the same. Then G is not chromatically unique if and only if G is isomorphic to $K_4(s,s,s\text{-}2,1,2,s)$ or $K_4(s,s\text{-}2,s,2s\text{-}2,1,s)$ or $K_4(t,t,1,2t,t+2,t)$ or $K_4(t,t,1,2t,t+1,t)$ or $K_4(t,t+1,t,2t+1,1,t)$ or $K_4(1,t,1,t+1,3,1)$ or $K_4(1,1,t,2,t+2,1)$, where $s \geq 3$, $t \geq 2$.

Lemma 2.4 (Catada-Ghimire and Hasni [12]) A K_4-homeomorph graph with exactly two paths of length two is χ-unique if and only if it is not isomorphic to $K_4(1,2,2,4,1,1)$ or $K_4(4,1,2,1,2,4)$ or $K_4(1,s+2,2,1,2,s)$ or

$K_4(1,2,2,t+2,t+2,t)$ or $K_4(1,2,2,t,t+1,t+3)$ or $K_4(3,2,2,r,1,5)$ or $K_4(1,r,2,4,2,4)$ or $K_4(3,2,2,r,1,r+3)$ or $K_4(r+2,2,2,1,4,r)$ or $K_4(r+3,2,2,r,1,3)$ or $K_4(4,2,2,1,r+2,r)$ or $K_4(3,4,2,4,2,6)$ or $K_4(3,4,2,4,2,8)$ or $K_4(3,4,2,8,2,4)$ or $K_4(7,2,2,3,4,5)$ or $K_4(5,2,2,3,4,7)$ or $K_4(8,2,2,3,4,6)$ or $K_4(5,2,2,9,3,4)$ or $K_4(5,2,2,5,3,4)$ where $r \geq 3$, $s \geq 3$, $t \geq 3$.

MAIN RESULT

In this section, we present our main results. In the following, we only consider graphs with at most one path of length 1 and have girth 9.

Lemma 3.1 Let K_4-homeomorph $K_4(2,3,4,d,e,f)$ and $K_4(1,3,5,d',e',f')$ be chromatically equivalent, then

$$K_4(2,3,4,1,5,7) \sim K_4(1,3,5,2,8,3),$$
$$K_4(2,3,4,e+4,e,1) \sim K_4(1,3,5,e+3,2,e),$$
$$K_4(2,3,4,6,e,1) \sim K_4(1,3,5,5,e,2),$$

where $e \geq 6$.

Proof. Let G and H be two graphs such that $G \cong K_4(2,3,4,d,e,f)$ and $H \cong K_4(1,3,5,d',e',f')$. Since the girth of G is 9, there is at most a 1 among d, e or f. Let

$$Q(K_4(a,b,c,d,e,f)) = -(s+1)(s^a+s^b+s^c+s^d+s^e+s^f)+s^{a+d}+s^{b+f}+s^{c+e}+s^{a+b+e}+s^{b+d+c}+s^{a+c+f}+s^{d+e+f}.$$

Suppose $s = 1-\lambda$ and x is the number of edges in G. From [16], we have the chromatic polynomial of K_4-homeomorphs $K_4(a,b,c,d,e,f)$ as follows:

$$Q(K_4(a,b,c,d,e,f)) = (-1)^{x-1}\frac{s}{(s-1)^2}\Big[\big(s^2+3s+2\big)+Q\big(K_4(a,b,c,d,e,f)\big)-s^{x-1}\Big]$$

Hence $P(G) = P(H)$ if and only if $Q(G) = Q(H)$. We solve the equation $Q(G) = Q(H)$ to get all solutions. Let the lowest remaining power and the highest remaining power denoted by l.r.p. and h.r.p., respectively. As $G \cong K_4(2,3,4,d,e,f)$ and $H \cong K_4(1,3,5,d',e',f')$, then

$$Q(G) = -(s+1)(s^2+s^3+s^4+s^d+s^e+s^f)+s^{d+2}+s^{f+3}+s^{e+4}+s^{e+5}+s^{d+7}+s^{f+6}+s^{d+e+f}$$
$$Q(H) = -(s+1)(s+s^3+s^5+s^{d'}+s^{e'}+s^{f'})+s^{d'+1}+s^{f'+3}+s^{e'+5}+s^{e'+4}+s^{d'+8}+s^{f'+6}+s^{d'+e'+f'}$$

Since $K_4(2,3,4,d,e,f)$ has exactly one path of length 1, we have $\min\{d,e,f\} = 1$. From Lemma 2.1(1),

$$d+e+f = d'+e'+f' \tag{1}$$

$Q(G) = Q(H)$ yields

$$Q_1(G) = -s^3-s^4-s^d-s^e-s^f-s^{d+1}-s^{e+1}-s^{f+1}+s^{d+2}+s^{d+7}+s^{e+4}+s^{e+5}+s^{f+3}+s^{f+6}$$
$$Q_1(H) = -s-s^6-s^{d'}-s^{e'}-s^{f'}-s^{e'+1}-s^{f'+1}+s^{d'+8}+s^{e'+4}+s^{e'+5}+s^{f'+3}+s^{f'+6}$$

There are three cases to be considered, that is, $d = 1$ (Case A) or $e = 1$ (Case B) or $f = 1$ (Case C). We only show the detailed proof of Case A, that is the case $d = 1$.

<u>Case A</u> $d = 1$. We obtain the following after simplification.

$$Q_2(G) = -s^2-s^4-s^e-s^f-s^{e+1}-s^{f+1}+s^8+s^{e+4}+s^{e+5}+s^{f+3}+s^{f+6}$$
$$Q_2(H) = -s^6-s^{d'}-s^{e'}-s^{f'}-s^{e'+1}-s^{f'+1}+s^{d'+8}+s^{e'+4}+s^{e'+5}+s^{f'+3}+s^{f'+6}$$

After comparing the l.r.p. in $Q_2(G)$ and the l.r.p. in $Q_2(H)$, we have $d' = 2$ or $e' = 2$ or $f' = 2$.

Case 1 $d' = 2$. We have $e' \geq 4$ and $f' \geq 3$. From $Q_2(G)$ and $Q_2(H)$, we obtain the following after simplification.

$$Q_3(G) = -s^4-s^e-s^f-s^{e+1}-s^{f+1}+s^8+s^{e+4}+s^{e+5}+s^{f+3}+s^{f+6}$$
$$Q_3(H) = -s^6-s^{e'}-s^{f'}-s^{e'+1}-s^{f'+1}+s^{10}+s^{e'+4}+s^{e'+5}+s^{f'+3}+s^{f'+6}$$

Consider the l.r.p. in $Q_3(G)$ and the l.r.p. in $Q_3(H)$. then, we have $e' = 4$ or $f' = 4$.

Case 1.1 $e' = 4$. From $Q_3(G)$ and $Q_3(H)$, we obtain the following after simplification.

$$Q_4(G) = -s^e-s^f-s^{e+1}-s^{f+1}+s^{e+4}+s^{e+5}+s^{f+3}+s^{f+6}$$
$$Q_4(H) = -s^5-s^6-s^{f'}-s^{f'+1}+s^9+s^{10}+s^{f'+3}+s^{f'+6}$$

Consider the h.r.p. in $Q_4(G)$ and the h.r.p. in $Q_4(H)$, then either $e + 5 = f' + 6$ or $f + 6 = f' + 6$.

Case 1.1.1 $e + 5 = f' + 6$. Then, $e = f' + 1$. By Equation (1), $f = 4$. Simplifying $Q_4(G)$ and $Q_4(H)$, we obtain

$$Q_5(G) = -s^4-s^{e+1}+s^7+s^{e+4},$$
$$Q_5(H) = -s^6-s^{f'}+s^9+s^{f'+3}$$

We can see that $e = 5$ and $f' = 4$. Thus, $G \cong H$.

Case 1.1.2 $f + 6 = f' + 6$. So $f = f'$. By Equation (1), $e = 5$. Thus, $G \cong H$.

Case 1.2 $f' = 4$. From $Q_3(G)$ and $Q_3(H)$, we obtain the following after simplification.

$$Q_6(G) = -s^e-s^f-s^{e+1}-s^{f+1}+s^8+s^{e+4}+s^{e+5}+s^{f+3}+s^{f+6}$$
$$Q_6(H) = -s^5-s^6-s^{e'}-s^{e'+1}+s^7+s^{10}+s^{10}+s^{e'+4}+s^{e'+5}$$

We know that $e' \geq 4$. If $e' = 4$, we get the same conclusion as in Case 1.1.1. If $e' \geq 5$, by comparing the h.r.p. in $Q_6(G)$ and the h.r.p. in $Q_6(H)$, then either $e + 5 = e' + 5$ or $f + 6 = e' + 5$.

Case 1.2.1 $e + 5 = e' + 5$, then $e = e'$. From Equation (1), $f = 5$. We obtain that $Q_6(G) \neq Q_6(H)$, a contradiction.

Case 1.2.2 $f + 6 = e' + 5$, then $f + 1 = e'$. From Equation (1), $e = 6$. We obtain that $Q_6(G) \neq Q_6(H)$, a contradiction.

Case 2 $e' = 2$. Then, $d' \geq 4$ and $f' \geq 6$. From $Q_2(G)$ and $Q_2(H)$, we obtain the following after simplification.

$$Q_7(G) = -s^4-s^e-s^f-s^{e+1}-s^{f+1}+s^8+s^{e+4}+s^{e+5}+s^{f+3}+s^{f+6}$$
$$Q_7(H) = -s^3-s^{d'}-s^{f'}-s^{f'+1}+s^7+s^{d'+8}+s^{f'+3}+s^{f'+6}$$

Consider the term $-s^3$ in $Q_7(H)$. It cannot be cancelled with any positive term in $Q_7(H)$ since $d' \geq 4$ and $f' \geq 6$ and it cannot be cancelled with any negative term in $Q_7(G)$ as well since $f \geq 4$ and $e \geq 5$. Thus, a contradiction.

Case 3 $f' = 2$. Then $d' \geq 3$ and $e' \geq 6$. From $Q_2(G)$ and $Q_2(H)$, we obtain the following after simplification.

$$Q_8(G) = -s^4-s^e-s^f-s^{e+1}-s^{f+1}+s^{e+4}+s^{e+5}+s^{f+3}+s^{f+6}$$
$$Q_8(H) = -s^3-s^6-s^{d'}-s^{e'}-s^{e'+1}+s^5+s^{d'+8}+s^{e'+4}+s^{e'+5}$$

The term $-s^3 \in Q_8(H)$ cannot be cancelled with any term in $Q_8(G)$ and $Q_8(H)$, thus a contradiction.

Cases B and C can be proved similar to Case A.

This completes the proof of Lemma 3.1 □

Similar to Lemma 3.1, we can prove the following lemmas.

Lemma 3.2 Let K_4-homeomorph $K_4(2,3,4,d,e,f)$ and $K_4(1,4,4,d',e',f')$ be chromatically equivalent, then

$$K_4(2,3,4,1,7,4) \sim K_4(1,4,4,4,2,6),$$
$$K_4(2,3,4,1,5,8) \sim K_4(1,4,4,6,2,6).$$

Lemma 3.3 Let K_4-homeomorph $K_4(2,3,4,d,e,f)$ and $K_4(2,2,5,d',e',f')$ be chromatically equivalent, then

$$K_4(2,3,4,2,4,8) \sim K_4(2,2,5,4,3,7),$$
$$K_4(2,3,4,6,2,8) \sim K_4(2,2,5,3,9,4).$$

Lemma 3.4 If G is in the type of $K_4(2,3,4,d,e,f)$, and H is in the type of $K_4(2,3,4,d',e',f')$, then there is no graph G satisfying $G \sim H$ unless $G \cong H$.

Lemma 3.5 If G is in the type of $K_4(2,3,4,d,e,f)$, and H is in the type of $K_4(1,3,c',2,e',3)$, then there is no graph G satisfying $G \sim H$.

Lemma 3.6 If G is in the type of $K_4(2,3,4,d,e,f)$, and H is in the type of $K_4(1,2,c',3,e',3)$, then there is no graph G satisfying $G \sim H$.

Lemma 3.7 If G is in the type of $K_4(2,3,4,d,e,f)$, and H is in the type of $K_4(2,2,c',2,e',3)$, then there is no graph G satisfying $G \sim H$.

We now give our main result.

Theorem 3.1 K_4-homeomorphs $K_4(2,3,4,d,e,f)$ with girth 9 is not χ-unique if and only if it is isomorphic to $K_4(2,3,4,1,7,f)$, $K_4(2,3,4,7,1,5)$, $K_4(2,3,4,1,5,8)$, $K_4(2,3,4,1,5,7)$, $K_4(2,3,4,e+4,e,1)$, $K_4(2,3,4,6,e,1)$, $K_4(2,3,4,2,4,8)$, $K_4(2,3,4,6,2,8)$, where $f \geq 4$ ($f \neq 5$), $e \geq 6$.

Proof. It follows directly from Lemmas 3.1-3.7.

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