Almost periodic solutions for impulsive neural networks with delay

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Abstract

In the present paper the problems of existence and uniqueness of almost periodic solutions for impulsive cellular neural networks with delay are considered.

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1. Introduction

The theory of impulsive differential equations goes back to the works of Mil’man and Myshkis [1]. This theory is now being recognized to be not only richer than the corresponding theory of differential equations without impulses but also represents a more natural framework for mathematical modelling of many real-world phenomena. In recent years impulsive differential equations have been intensively researched (see the monographs [2–4]). Results related to the study of the existence of almost periodic solutions for such equations have been obtained in [5–7]. Now there also exist a well-developed qualitative theory of functional differential equations [8–11]. However, not so much has been developed in the direction of impulsive functional differential equations. Till now only few papers has been published on the topic. In the few publications dedicated to this subject, earlier works were done by Anokhin [12] and Gopalsamy and Zhang [13]. Recently, some qualitative properties (oscillation, asymptotic behavior and stability) are investigated by several authors (see [14–19]).

The researches on dynamics of neural networks systems have been arousing a great deal of interests for more than thirty years. In the last ten years, many researches focused on the study of dynamics of neural
network models with delay. In fact, neural networks often have times delay in reality, for example due to the finite switching speed of amplifiers in electronic neural networks. Stability of different classes of neural networks with time delay, such as Hopfield neural networks, cellular neural networks, bidirectional associative neural networks, Lotka-Volterra neural networks, has been extensively studied and various stability conditions have been obtained for these models of neural networks. See, for example, [20–23] and the references cited therein.

In this paper, we investigate the existence and attractiveness of almost periodic solution for an impulsive cellular neural networks with delay. The results obtained are a generalization of the results for the dynamics behavior of Hopfield neural networks with delay [20–23]. Such a generalization enables to study different type of classical problems as well as the problem of stability of the systems, respectively the problem of optimal control for this systems.

2. Preliminary notes

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space with elements \( x = \text{col}(x_1, x_2, \ldots, x_n) \) and norm \( |x| = \max_{i=1}^n |x_i| \), \( \mathbb{R} = (-\infty, \infty) \), \( \Omega \) be a domain in \( \mathbb{R}^n \), \( \Omega \neq \emptyset \), \( h > 0 \).

By \( B, B = \{ \tau_k \} : \tau_k \in \mathbb{R}, \tau_k < \tau_{k+1}, k \in \mathbb{Z}, \lim_{k \to \pm \infty} \tau_k = \pm \infty \} \) we denote the set of all sequences unbounded and strictly increasing.

We shall investigate the problem of existence of almost periodic solutions of the system of impulsive Hopfield neural networks with delay

\[
\begin{align*}
\dot{x}_i(t) &= \sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n x_{ij}(t)f_j(x_j(t-h)) + \gamma_i(t), \quad t \neq \tau_k, \quad i = 1, 2, \ldots, n, \\
\Delta x(t) &= A_k x(t) + I_k(x(t)) + \gamma_k, \quad t = \tau_k, \quad k \in \mathbb{Z},
\end{align*}
\]

where

(i) \( t \in \mathbb{R}, a_{ij}(t), x_{ij}(t) \in C(\mathbb{R}, \mathbb{R}), f_j(u) \in C(\mathbb{R}, \mathbb{R}) i = 1, 2, \ldots, n, j = 1, 2, \ldots, n \);  
(ii) \( \gamma_i(t) \in C(\mathbb{R}, \mathbb{R}), i = 1, 2, \ldots, n, I_k(x) \in C(\Omega, \mathbb{R}^n), \{ \tau_k \} \in B, k \in \mathbb{Z}; \) 
(iii) \( A_k \in \mathbb{R}^{n \times n}, \gamma_k \in \mathbb{R}^n, \Delta x(t) = x(t+0) - x(t-0). \)

Let \( t_0 \in \mathbb{R} \). Introduce the following notations:  
\( PC(t_0) \) is the space of all functions \( \phi : [t_0 - h, t_0] \to \Omega \) having points of discontinuity at \( 0, t_0 \) of the first kind and are left continuous at these points.

Let \( \phi_0 \in PC(t_0) \). Denote by \( x(t) = x(t; t_0, \phi_0), x \in \Omega \) the solution of system (1) satisfying the initial conditions

\[
\begin{align*}
x(t; t_0, \phi_0) &= \phi_0(t), \quad t_0 - h \leq t \leq t_0, \\
x(t_0 + 0; t_0, \phi_0) &= \phi_0(t_0).
\end{align*}
\]

The solution \( x(t) = x(t; t_0, \phi_0) \) of the initial value problem (1) and (2) is characterized by the following:

(a) For \( t_0 - h \leq t \leq t_0 \) the solution \( x(t) \) satisfied the initial conditions (2).

(b) For \( t > t_0 \) the solution \( x(t; t_0, \phi_0) \) of problem (1) and (2) is a piecewise continuous function with points of discontinuity of the first kind \( t = \tau_k, k \in \mathbb{Z} \) at which it is continuous from the left, i.e., the following relations hold:

\[
x(\tau_k - 0) = x(\tau_k), \quad x(\tau_k + 0) = x(\tau_k) + \Delta x(\tau_k) = x(\tau_k) + I_k(x(\tau_k)).
\]

(c) If for some integer \( j \) we have \( \tau_j < \tau_j + h < \tau_{j+1}, k \in \mathbb{Z} \), then in the interval \([\tau_j + h, \tau_{j+1}]\) the solution \( x(t) \) of problem (1) and (2) coincides with the solution of the problem

\[
\begin{align*}
\dot{y}_i(t) &= \sum_{j=1}^n a_{ij}(t)y_j(t) + \sum_{j=1}^n x_{ij}(t)f_j(x_j(t-h+0)) + \gamma_i(t), \\
y(t_j + h) &= x(t_j + h).
\end{align*}
\]
and if \( \tau_j + h \equiv \tau_k \) for \( j = 0, 1, 2, \ldots, k = 1, 2, \ldots \), then in the interval \([\tau_j + h, \tau_{k+1}]\) the solution \( x(t) \) coincides with the solution of the problem

\[
\begin{aligned}
\dot{y}_j(t) &= \sum_{j=1}^{n} a_{ij}(t)y_j(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t-h+0)) + \gamma_j(t), \\
y_j(\tau_j + h) &= x(\tau_j + h) + A_kx(\tau_j + h) + I_k(x(\tau_j + h) + \gamma_k).
\end{aligned}
\]

Since the solutions of problem (1) and (2) is a piecewise continuous function with points of discontinuity of the first kind \( t = \tau_k, \ k \in \mathbb{Z} \) and we adopt the following definitions for almost periodicity.

**Definition 1** [4]. The set of sequences \( \{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k \in \mathbb{Z}, j \in \mathbb{Z}, \{\tau_k\} \in B \) is said to be *uniformly almost periodic* if for arbitrary \( \varepsilon > 0 \) there exists relatively dense set of \( \varepsilon \)-almost periods common for any sequences.

**Definition 2** [4]. A piecewise continuous function \( \varphi : \mathbb{R} \rightarrow \mathbb{R}^n \) with discontinuity of first kind at the points \( \tau_k \) is said to be *almost periodic*, if:

1. the set of sequences \( \{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k \in \mathbb{Z}, j \in \mathbb{Z}, \{\tau_k\} \in B \) is uniformly almost periodic.
2. for any \( \varepsilon > 0 \) there exists a real number \( \delta > 0 \) such that if the points \( t' \) and \( t'' \) belong to one and the same interval of continuity of \( \varphi(t) \) and satisfy the inequality \( |t' - t''| < \delta \), then \( |\varphi(t') - \varphi(t'')| < \varepsilon \).
3. for any \( \varepsilon > 0 \) there exists a relatively dense set \( T \) such that if \( t \in T \), then \( |\varphi(t + \tau) - \varphi(t)| < \varepsilon \) for all \( t \in \mathbb{R} \) satisfying the condition \( |t - \tau_k| > \varepsilon, k \in \mathbb{Z} \).

Together with the system (1) we consider the linear system

\[
\begin{aligned}
\dot{x}(t) &= A(t)x(t), \quad t \neq \tau_k, \\
\Delta x(t) &= A_kx(t), \quad t = \tau_k, \quad k \in \mathbb{Z},
\end{aligned}
\]

(3)

where \( t \in \mathbb{R}, \ A(t) = (a_{ij}(t)), \ i = 1, 2, \ldots, n, j = 1, 2, \ldots, n. \)

Introduce the following conditions:

H1. \( A(t) \in C(\mathbb{R}, \mathbb{R}^n) \) and is almost periodic in the sense of Bohr.

H2. \( \det(E + A_k) \neq 0 \) and the sequence \( \{A_k\}, k \in \mathbb{Z} \) is almost periodic, \( E \in \mathbb{R}^{n \times n} \).

H3. The set of sequences \( \{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k \in \mathbb{Z}, j \in \mathbb{Z}, \{\tau_k\} \in B \) is uniformly almost periodic and there exists \( \theta > 0 \) such that \( \inf \tau_k^j = \theta > 0 \).

Recall [2] that if \( U_k(t, s) \) is the Cauchy matrix for the system

\[ \dot{x}(t) = A(t)x(t), \quad \tau_{k-1} < t \leq \tau_k, \quad \{\tau_k\} \in B \]

then the Cauchy matrix for the system (3) is in the form

\[
W(t, s) = \begin{cases}
U_k(t, s), & \tau_{k-1} < s \leq t \leq \tau_k, \\
U_{k+1}(t, \tau_k + 0)(E + A_k)U_k(t, s), & \tau_{k-1} < s \leq \tau_k < t \leq \tau_{k+1}, \\
U_{k+1}(t, \tau_k + 0)(E + A_k)U_k(\tau_k, \tau_k + 0) \cdots (E + A_2)U_1(\tau_0, s), & \tau_{k-1} < s \leq \tau_k < t \leq \tau_{k+1}.
\end{cases}
\]

**Lemma 1** [5]. *Let the following conditions be fulfilled:

1. Conditions H1–H3 are fulfilled.
2. For the Cauchy matrix \( W(t, s) \) of the system (3) there exist positive constants \( K \) and \( \lambda \) such that

\[ |W(t, s)| \leq Ke^{-\lambda(t-s)}, \quad t \geq s, \quad t, s \in \mathbb{R}. \]

Then for any \( \varepsilon > 0 \), \( t \in \mathbb{R}, \ s \in \mathbb{R}, \ t \geq s \), \( |t - \tau_k| > \varepsilon, \ |s - \tau_k| > \varepsilon, \ k \in \mathbb{Z} \) there exists a relatively dense set \( T \) of \( \varepsilon \)-almost periods of the matrix \( A(t) \) and a positive constant \( \Gamma \) such that for \( \tau \in T \) it follows:
Introduce the following conditions:

H4. The functions $f_j(u)$ are almost periodic in the sense of Bohr, and

$$0 < \sup_{r \in \mathbb{R}} |f_j(u)| < \infty, \quad f_j(0) = 0$$

and there exists $L_1 > 0$ such that for $u, v \in \mathbb{R}$

$$\max_j |f_j(u) - f_j(v)| < L_1|u - v|, \quad j = 1, 2, \ldots, n.$$

H5. The functions $x_j(t)$ are almost periodic in the sense of Bohr, and

$$0 < \sup_{r \in \mathbb{R}} |x_j(t)| = \exists x_j < \infty.$$

H6. The functions $\gamma_k(t), i = 1, 2, \ldots, n$ are almost periodic in the sense of Bohr, $\{\gamma_k\}_{k \in \mathbb{Z}}$ is almost periodic sequence and there exists $C_0 > 0$ such that

$$\max_j \left| \gamma_j(t) - \gamma_j(t0) \right| < C_0.$$

H7. The sequence of functions $I_k(x)$ is almost periodic uniformly with respect to $x \in \Omega$ and there exists $L_2 > 0$ such that

$$|I_k(x) - I_k(y)| \leq L_2|x - y|$$

for $k \in \mathbb{Z}$, $x, y \in \Omega$.

**Lemma 2** [4]. Let the conditions H1–H6 be fulfilled. Then for each $\varepsilon > 0$ there exist $\varepsilon_1, 0 < \varepsilon_1 < \varepsilon$ and relatively dense sets $T$ of real numbers and $Q$ of whole numbers, such that the following relations are fulfilled:

1. **(a)** $|A(t + \tau) - A(t)| < \varepsilon_1, \quad t \in \mathbb{R}, \quad \tau \in T$
2. **(b)** $|x_j(t + \tau) - x_j(t)| < \varepsilon_1, \quad t \in \mathbb{R}, \quad \tau \in T, \quad |t - \tau| > \varepsilon, \quad k \in \mathbb{Z}, \quad i, j = 1, 2, \ldots, n$
3. **(c)** $|f_j(t + \tau) - f_j(t)| < \varepsilon_1, \quad t \in \mathbb{R}, \quad \tau \in T, \quad |t - \tau| > \varepsilon, \quad k \in \mathbb{Z}, \quad j = 1, 2, \ldots, n$
4. **(d)** $|A_{k+q} - A_k| < \varepsilon_1, \quad q \in Q, \quad k \in \mathbb{Z}$
5. **(e)** $|\gamma_j(t + \tau) - \gamma_j(t)| < \varepsilon_1, \quad t \in \mathbb{R}, \quad \tau \in T, \quad |t - \tau| > \varepsilon, \quad k \in \mathbb{Z}, \quad j = 1, 2, \ldots, n$
6. **(f)** $|\gamma_{k+q} - \gamma_k| < \varepsilon_1, \quad q \in Q, \quad k \in \mathbb{Z}$
7. **(g)** $|\tau_k^q - \tau| < \varepsilon_1, \quad q \in Q, \quad \tau \in T, \quad k \in \mathbb{Z}$

**Lemma 3** [4]. Let the set of sequences $\{\tau_k^q\}$ be uniformly almost periodic. Then for each $p > 0$ there exists a positive integer $N$ such that on each interval of length $p$ no more than $N$ elements of the sequence $\{\tau_k\}$, i.e.,

$$i(s, t) \leq N(t - s) + N,$$

where $i(s, t)$ is the number of points $\tau_k$ in the interval $(s, t)$.

**3. Main results**

**Theorem 1.** Let the following conditions be fulfilled:

1. **Conditions H1–H7** are fulfilled.
2. **For the Cauchy matrix** $W(t, s)$ **of the system (3)** there exist positive constants $K$ and $\lambda$ such that

$$|W(t, s)| \leq Ke^{-\lambda(t-s)}, \quad t \geq s, \quad t, s \in \mathbb{R}$$
and the number

\[ r = K \left\{ \max_i \lambda^{-1} L_1 \sum_{j=1}^n \mathbb{L}_{ij} + \frac{L_2}{1 - e^{-\lambda}} \right\} < 1. \]

Then:

1. There exists unique almost periodic solution \( x(t) \) of (1).
2. If the following inequalities hold:

\[ 1 + KL_2 < e, \quad \lambda - N \ln(1 + KL_2) - L_1 \max_i \sum_{j=1}^n \mathbb{L}_{ij} e^{2t} > 0 \]

then the solution \( x(t) \) is exponentially stable.

**Proof of assertion 1.** We denote with \( D \) the set of all almost periodic functions \( \varphi(t) \) satisfying the inequality \( ||\varphi|| < \mathcal{K}, ||\varphi|| = \sup_{t \in \mathbb{R}} |\varphi(t)|, \mathcal{K} = K C_0 \left( \frac{1}{1 - e^{-\lambda}} \right) \).

Set \( F(t, x(t-h)) = \text{col} \{ F_1(t, x(t-h)), F_2(t, x(t-h)), \ldots, F_n(t, x(t-h)) \} \), where

\[ F_i(t, x(t-h)) = \sum_{j=1}^n \alpha_{ij}(t)f_j(x_j(t-h)), \quad i = 1, 2, \ldots, n \]

and \( \gamma(t) = \text{col} \{ \gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t) \} \).

Define in \( D \) an operator \( S \),

\[ S\varphi = \int_{-\infty}^t W(t, s) [F(s, \varphi(s-h)) + \gamma(s)] ds + \sum_{\tau_k < t} W(t, \tau_k) [I_k(\varphi(\tau_k)), + \gamma_k] \quad (4) \]

and subset \( D^* \), \( D^* \subset D \),

\[ D^* = \left\{ \varphi \in D : ||\varphi| - \varphi_0|| \leq \frac{r\mathcal{K}}{1 - r} \right\}, \]

where

\[ \varphi_0 = \int_{-\infty}^t W(t, s) \gamma(s) ds + \sum_{\tau_k < t} W(t, \tau_k) \gamma_k. \]

We have

\[ ||\varphi_0|| = \sup_{t \in \mathbb{R}} \left\{ \max_i \int_{-\infty}^t |W(t, s)||\gamma_i(s)| ds \right\} + \sum_{\tau_k < t} |W(t, \tau_k)||\gamma_k| \leq \sup_{t \in \mathbb{R}} \left\{ \max_i \int_{-\infty}^t Ke^{-\lambda(s-t)}|\gamma_i(s)| ds + \sum_{\tau_k < t} Ke^{-\lambda(t-\tau_k)}|\gamma_k| \right\} \leq K \left( \frac{C_0}{\lambda} + \frac{C_0}{1 - e^{-\lambda}} \right) = \mathcal{K}. \quad (5) \]

Then for arbitrary \( \varphi \in D^* \) from (4) and (5) we have

\[ ||\varphi|| \leq ||\varphi - \varphi_0|| + ||\varphi_0|| \leq \frac{r\mathcal{K}}{1 - r} + \mathcal{K} = \frac{\mathcal{K}}{1 - r}. \]

Now we prove that \( S \) is self-mapping from \( D^* \) to \( D^* \).
For arbitrary \( \varphi \in D^* \) it follows:

\[
\|S\varphi - \varphi_0\| = \sup_{t \in \mathbb{R}} \left\{ \max_i \left( \int_{-\infty}^{t} |W(t, s) \sum_{j=1}^{n} \alpha_j(s) |f_j(\varphi_j(s - h))| \, ds \right) + \sum_{\tau_k < t} |W(t, \tau_k)| \| I_k(\varphi(\tau_k)) \| \right\}
\]

\[
\leq \left\{ \max_i \left( \int_{-\infty}^{t} Ke^{-i(t-s)} \left[ \sum_{j=1}^{n} \alpha_j L_1 \right] \, ds \right) + \sum_{\tau_k < t} Ke^{-i(t-\tau_k)} L_2 \right\} \| \varphi \|
\]

\[
\leq K \left\{ \max_i \lambda^{-1} L_1 \left( \sum_{j=1}^{n} \alpha_j \right) + \frac{L_2}{1 - e^{-t}} \right\} \| \varphi \| = r \| \varphi \| \leq \frac{rK}{1 - r}.
\]

(6)

Let \( \tau \in T, q \in Q \), where the sets \( T \) and \( Q \) are determined in Lemma 2. Then

\[
\|S\varphi(t + \tau) - S\varphi(t)\| \leq \sup_{t \in \mathbb{R}} \left\{ \max_i \left( \int_{-\infty}^{t} |W(t, s + \tau + t) - W(t, s) \sum_{j=1}^{n} \alpha_j(s + \tau) f_j(\varphi_j(s + \tau - h)) + \gamma_j(s + \tau) \right) \, ds \right. 
\]

\[
+ \left. \int_{-\infty}^{t} |W(t, s) \sum_{j=1}^{n} \alpha_j(s + \tau) f_j(\varphi_j(s + \tau - h)) + \gamma_j(s + \tau) - \sum_{j=1}^{n} \alpha_j(s) f_j \times (\varphi_j(s - h) - \gamma_j(s)) \, ds \right) + \sum_{\tau_k < t} |W(t + \tau, \tau_k + q) - W(t, \tau_k)| |I_k(q(\tau_{k+q})) + \gamma_{k+q}| 
\]

\[
+ \sum_{\tau_k < t} |W(t, \tau_k)| |I_k(q(\tau_{k+q})) - I_k(\varphi(\tau_k)) + \gamma_{k+q} - \gamma_k \right\} \leq 2C_1,
\]

(7)

where

\[
C_1 = \frac{1}{\lambda} \left( \max_i \left( \sum_{j=1}^{n} (2\Gamma L_1 + K) \alpha_j \right) + KL_1 \right) + 2\Gamma C_0 + K + \frac{(L_2 + C_0)C_0}{1 - e^{-t}}.
\]

From (6) and (7) we obtain that \( S\varphi \in D^* \).

Let \( \varphi \in D^*, \psi \in D^* \). We get

\[
\|S\varphi - S\psi\| \leq \sup_{t \in \mathbb{R}} \left\{ \max_i \left( \int_{-\infty}^{t} |W(t, s) \sum_{j=1}^{n} \alpha_j(s) |f_j(\varphi_j(s - h)) - f_j(\psi_j(s - h))| \, ds \right) + \sum_{\tau_k < t} |W(t, \tau_k) \times |I_k(\varphi(\tau_k)) - I_k(\psi(\tau_k))| \right\} \leq K \left( \max_i \lambda^{-1} L_1 \sum_{j=1}^{n} \alpha_j \right) \frac{L_2}{1 - e^{-t}} \| \varphi - \psi \| = r \| \varphi - \psi \|.
\]

(8)

Then from (8) it follows that \( S \) is contracting operator in \( D^* \). So there exists unique almost periodic solution of (1). □

**Proof of assertion 2.** Let \( y(t) \) be arbitrary solution of (1) with initial condition \( y(t_0 + 0, t_0, \varpi_0) = \varpi_0, \varpi_0 \in PC(t_0) \). Then from (4) we obtain

\[
y(t) - x(t) = W(t, t_0)(\varpi_0 - \varphi_0) + \int_{t_0}^{t} W(t, s)[F(x, y(s - h)) - F(s, x(s - h))] \, ds + \sum_{t_0 < \tau_k < t} W(t, \tau_k) \times [I_k(y(\tau_k)) - I_k(x(\tau_k))].
\]

Then

\[
|y(t) - x(t)| \leq Ke^{-i(t-t_0)}|\varpi_0 - \varphi_0| + \max_i \left( \int_{t_0}^{t} Ke^{-i(t-h-i)} L_1 \sum_{j=1}^{n} \alpha_j |y_j(v) - x_j(v)| \, ds \right)
\]

\[
+ \max_i \left( \int_{t_0}^{t-h} Ke^{-i(t-h-i)} L_1 \sum_{j=1}^{n} \alpha_j |y_j(v) - x_j(v)| \, ds \right) + \sum_{t_0 < \tau_k < t} Ke^{-i(t-\tau_k)} L_2 |y(\tau_k) - x(\tau_k)|.
\]
Theorem 2. \( W \) is in the form and from Gronwall–Bellman’s lemma [3] we have

\[
|y(t) - x(t)| \leq K \left( 1 + L_1 \max_i \sum_{j=1}^n T_{ij} e^{\lambda h} \right) |\mu_0 - \phi_0| e^{N \ln(1 + L_2 N)}
\]

\[
\times \exp \left\{ -\lambda + N \ln(1 + KL_2) + L_1 \max_i \sum_{j=1}^n \bar{T}_{ij} e^{\lambda h/2} \right\} (t - t_0).
\]

Thus Theorem 1 is complete. \( \square \)

We note that the main inequalities which are used in proof of Theorem 1 are connect with the properties of matrix \( W(t, s) \) for a system (3). Now we will consider some special case in which these properties are accomplished.


\[
\begin{cases}
C_i \dot{x}_i(t) = -\frac{1}{K} x_i(t) + \sum_{j=1}^n T_{ij} f_j(x_j(t-h)) + \gamma_i(t), & t \neq \tau_k, \ i = 1, 2, \ldots, n, \\
\Delta x(t) = Gx(t) + I_k(x(t)) + \gamma_k, & t = \tau_k, \ k \in \mathbb{Z},
\end{cases}
\]

where

(i) \( t \in \mathbb{R}, \ C_i > 0, \ R_i > 0, \ T_{ij} \in \mathbb{R}, \ h > 0, \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, n; \)

(ii) \( \gamma_i(t) \in C(\mathbb{R}, \mathbb{R}), \ i = 1, 2, \ldots, n, \ I_k(x) \in C(\Omega, \mathbb{R}^n), \ \{ \tau_k \} \in \mathbb{B}, \ k \in \mathbb{Z}; \)

(iii) \( G = \text{diag}(g_i), \ g_i \in \mathbb{R}, \ i = 1, 2, \ldots, n, \ \gamma_k \in \mathbb{R}^n, \ \Delta x(t) = x(t+0) - x(t-0). \)

Theorem 2. Let the following conditions be fulfilled:

1. Conditions H3, H4, H6, H7 are fulfilled.
2. The following inequalities hold:

\[
\lambda = \min_i \frac{1}{C_i R_i} - N \max_i \ln(1 + |g_i|) > 0,
\]

\[
r = \exp\left\{ N \max_i \ln(1 + |g_i|) \right\} \left\{ \max_i \lambda^{-1} L_1 \sum_{j=1}^n \frac{T_{ij}}{C_i} + \frac{L_2}{1 - e^{-\lambda}} \right\} < 1.
\]

Then there exists unique almost periodic solution \( x(t) \) of (9).

If the following inequalities hold:

\[
1 + \exp\left\{ N \max_i \ln(1 + |g_i|) \right\} L_2 < e,
\]

\[
\lambda - \exp\left\{ N \max_i \ln(1 + |g_i|) L_1 \right\} \sum_{j=1}^n \frac{T_{ij}}{C_i} - N \ln(1 + \{ N \max_i \ln(1 + |g_i|) \} L_2) > 0
\]

then the solution \( x(t) \) is exponentially stable.

Proof. Recall [3] the matrix \( W(t, s) \) for the linear system of system (9)

\[
\begin{cases}
\dot{x}_i(t) = -\frac{1}{C_i R_i} x_i(t), & t \neq \tau_k, \\
\Delta x(t) = Gx(t), & t = \tau_k, \ k \in \mathbb{Z}.
\end{cases}
\]

is in the form \( W(t, s) = e^{K(t-s)}(E + G)^{\delta_{s,t}} \), \( A = \text{diag}\left( -\frac{1}{C_i R_i} \right), \ i = 1, 2, \ldots, n. \)
Then
\[ |W(t, s)| \leq e^{N \max_{i} \ln(1+|g_i|)} e^{-\lambda(t-s)}, \quad t > s, \quad s \in \mathbb{R}. \]
and the proof follows from Theorem 1. □

4. Conclusions

In this paper, we have formulated and studied a new type of neural networks: the impulsive cellular neural networks. This general neural network model is useful for describing evolutionary processes that have sequential abrupt change that cannot be appropriately described by pure continuous or pure discrete neural network model. The efficient conditions for the existence and uniqueness of almost periodic solutions for such model were obtained. The obtained results can be used to develop and analyze the mathematical models of neural networks without impulsive perturbations in order to design tools that can distinguish among network configurations.

References