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COMMUTATIVITY, DIRECT AND STRONG CONVERSE RESULTS FOR PHILIPS OPERATORS

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Dedicated to the memory of Borislav Bojanov

We study the so-called Phillips operators which can be considered as genuine Szász-Mirakjan-Durrmeyer operators. As main results we prove the commutativity of the operators as well as their commutativity with an appropriate differential operator and establish a strong converse inequality of type A for the approximation of real valued continuous bounded functions f on $[0, \infty)$. Together with the corresponding direct theorem we derive an equivalence result for the error of approximation and an appropriate K-functional and modulus of smoothness.

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1. Introduction

We consider a sequence $\widetilde{S}_n$ of positive linear operators (see [15, 13]) which are known in the literature as Phillips operators. These operators can also be considered as genuine Szász-Mirakjan-Durrmeyer operators in the same meaning as the genuine Bernstein-Durrmeyer operators (see e. g. [14]) and the genuine Baskakov-Durrmeyer operators (see e. g. [11]), i. e., they commute, preserve linear functions and commute with an appropriate differential operator.

In [6, Theorem 2] Z. Finta and V. Gupta proved a strong converse inequality of type B in the terminology of K. G. Ivanov and Z. Ditzian [4] for the Phillips operators. They use a general theorem developed by V. Totik [17, Theorem 1] where a direct and a strong converse result of type B is proved for positive linear operators satisfying certain conditions. In [7] Z. Finta also proved a general converse result of type B under certain conditions for the considered operators involving some general weight functions and applied his results (among others) to the Phillips operators. In [6, 7] estimates for the difference of Phillips operators and the classical Szász-Mirakjan operators are used extensively.

The aim of our paper is to establish a direct and a strong converse result of type A with explicit constants for the Phillips operators. To do so, we proceed in a similar way as H.-B. Knoop and X. L. Zhou in [12] for the Bernstein operators. A crucial step in this method is an appropriate strong Voronovskaja type theorem similar to [4, Lemma 8.3] and good estimates for the norms of weighted derivatives. In the proof of the strong converse inequality of type A we take advantage of a nice representation for the iterates of the operators (see Theorem 3.1) which is specific for the Phillips and the Szász-Mirakjan-Durrmeyer operators.

Let us mention that we first used a different approach to establish a strong converse inequality of type A based on the strong converse result of type B in [6] and especially using Theorem 3.1 and (16) to estimate $\|\widetilde{S}_n f - f\|$ by $K\|\widetilde{S}_{Kn} f - f\|$, K being a constant greater than 1. However, the constants appearing in the result of the present paper are much better.

Let $f \in C[0,\infty)$ be a real valued continuous function on $[0,\infty)$ satisfying an exponential growth condition, i. e.,

$$f \in C_\alpha[0,\infty) := \{f \in C[0,\infty) : |f(t)| \le Me^{\alpha t}, t \in [0,\infty)\}$$

with some constants $M > 0$ and $\alpha > 0$. Then the Phillips operators $\widetilde{S}_n$, $n > \alpha$, are defined by

$$\widetilde{S}_n(f,x) = n\sum_{k=1}^{\infty} s_{n,k}(x)\int_0^\infty s_{n,k-1}(t)f(t)dt + e^{-nx}f(0), \tag{1}$$

where

$$s_{n,k}(x) = \frac{(nx)^k}{k!}e^{-nx}, \quad k \in \mathbb{N}_0,\ x \in [0,\infty).$$

Throughout this paper f will mostly be considered as a function in $C_B[0,\infty)$, the space of real valued continuous bounded functions on $[0,\infty)$ endowed with the norm $\|f\| = \sup_{x\ge 0}|f(x)|$. We also need the space

$$W^2_\infty(\varphi) = \{g \in C_B[0,\infty) : g' \in AC_{loc}[0,\infty),\ \varphi^2 g'' \in C_B[0,\infty)\},$$

where here and in what follows $\varphi(x) = \sqrt{x}$, $x \in [0,\infty)$.

We point out that n is considered as a natural number in [6, 7], but in our statements here n may be considered also as an arbitrary positive number.

The Phillips operators are closely related to the Szász-Mirakjan operators [16] defined by

$$(2) \qquad S_n(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right),$$

to its Kantorovitch variants

$$\widehat{S}_n(f,x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt$$

and the Durrmeyer version

$$\overline{S}_n(f,x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k}(t) f(t)dt.$$

All these operators preserve constants and $\widetilde{S}_n$ and S_n also preserve linear functions and interpolate f at 0.

The operators $\widetilde{S}_n$ and $\overline{S}_n$ are connected in the same way as the operators S_n and $\widehat{S}_n$, i. e.,

$$(3) \qquad (S_n f)' = \widehat{S}_n f' \text{ and } (\widetilde{S}_n f)' = \overline{S}_n f'$$

if $f \in C^1_\alpha[0,\infty) = \{f \in C^1[0,\infty) : f, f' \in C_\alpha[0,\infty)\}$. For the proof of $(\widetilde{S}_n f)' = \overline{S}_n f'$ see (19).

The paper is organized as follows. In Section 2 we give some basic and auxiliary results such as the moments, the image of the Phillips operators for monomials and some identities which will be used throughout the paper. Section 3 is devoted to the proof of the commutativity results. It turns out that the commutativity of the Phillips operators can be derived as a corollary from a nice representation of $\widetilde{S}_n(\widetilde{S}_m)$ as a Phillips operator. The strong Voronovkaja type result is proved in Section 4. In Section 5 we prove a direct and a strong converse result of type A for the Phillips operators with explicit constants.

2. Basic results

In this section we collect some elementary and basic results which will be used throughout this paper. First we list some identities for the basis functions $s_{n,k}$ which follow directly from their definition. For the sake of simplicity in the notation we set $s_{n,k}(x) = 0$ for $k < 0$. The identities we shall need are:

$$\sum_{k=0}^{\infty} s_{n,k}(x) = 1, \tag{4}$$

$$\int_0^{\infty} s_{n,k}(t)dt = \frac{1}{n}, \tag{5}$$

$$s'_{n,k}(x) = n[s_{n,k-1}(x) - s_{n,k}(x)], \tag{6}$$

$$\varphi(x)^2 s'_{n,k}(x) = (k - nx)s_{n,k}(x), \tag{7}$$

$$\varphi(x)^4 s''_{n,k} = [(k - nx)^2 - k]s_{n,k}(x), \tag{8}$$

$$\varphi(x)^2 s_{n,k}(x)s_{n,k+1}(t) = s_{n,k+1}(x)\varphi(t)^2 s_{n,k}(t). \tag{9}$$

We will also need the second moment of the original Szász-Mirakjan operators S_n given by

$$S_n((t - x)^2, x) = \frac{x}{n}. \tag{10}$$

In our first lemma we state an explicit formula for the moments and the images of the Phillips operators for monomials. Note that there exists a formula for the image of the monomials by the Phillips operators given in [2, p. 1504], where the coefficients are given in terms of a recursion formula. By comparing the moments of the Phillips operators with the functions $H_{n,m}$ in [9, Lemma 4.10], case $c = 0$, one can see that apart from a factor $(-1)^m$ the functions $H_{n,m}$ are exactly the moments of the Phillips operators. In [9] a recursion formula and other representations are proved but now we are able to give an explicit formula. The same connection holds true for the functions $H_{n-1,m}$ in [3, Lemma 6.4] and the moments of the genuine Bernstein-Durrmeyer operators as well as for the functions $H_{n+1,m}$ in [9, Lemma 4.10], case $c = 1$ for the genuine Baskakov-Durrmeyer operators.

Throughout this paper we denote by $e_\nu(t) = t^\nu$, $\nu \in \mathbb{N}_0$, the ν-th monomial and define $f_{\mu,x}(t) = (t - x)^\mu$, $\mu \in \mathbb{N}_0$.

Lemma 2.1. *For the images of the operators $\widetilde{S}_n$ for the monomials we have*

$$\widetilde{S}_n(e_0, x) = 1,$$

$$\widetilde{S}_n(e_\nu, x) = \sum_{j=1}^{\nu} \binom{\nu - 1}{j - 1} \frac{\nu!}{j!} n^{j-\nu} x^j, \quad \nu \in \mathbb{N}.$$

The moments of the operators $\widetilde{S}_n$ are given by

$$\begin{aligned}
\widetilde{S}_n(f_{0,x},x) &= 1,\\
\widetilde{S}_n(f_{1,x},x) &= 0,\\
\widetilde{S}_n(f_{\mu,x},x) &= \sum_{j=1}^{\left[\frac{\mu}{2}\right]} \binom{\mu-j-1}{j-1}\frac{\mu!}{j!}n^{j-\mu}x^j,\ \mu\geq 2,
\end{aligned}$$

where $\left[\frac{\mu}{2}\right]$ denotes the integer part of $\frac{\mu}{2}$.

Proof. The identity $\widetilde{S}_n(e_0,x)=1$ follows directly from (4) and (5). As $\widetilde{S}_n$ interpolates at 0 we can use (3) and the image of $\overline{S}_n$ for the monomials given in [10, Satz 4.1] to calculate for $\nu\in\mathbb{N}$

$$\begin{aligned}
\int_0^x \overline{S}_n(\nu e_{\nu-1},u)du &= \nu\sum_{j=1}^{\nu}\binom{\nu-1}{j-1}\frac{(\nu-1)!}{(j-1)!}n^{j-\nu}\cdot\frac{1}{j}x^j\\
&= \int_0^x (\widetilde{S}_n(e_\nu,u)'du\\
&= \widetilde{S}_n(e_\nu,x).
\end{aligned}$$

For calculating the moments we follow the lines of [10, (4.6) in Korollar 4.4]. By using the binomial formula, the image of the monomials for $\widetilde{S}_n$, appropriate transformations for the summation indices and interchanging the order of summation we get

$$\begin{aligned}
\widetilde{S}_n(f_{\mu,x}(t),x) &= \sum_{\nu=0}^{\mu}\binom{\mu}{\nu}(-x)^{\mu-\nu}\widetilde{S}_n(e_\nu,x)\\
&= (-x)^\mu+\sum_{\nu=1}^{\mu}\binom{\mu}{\nu}(-x)^{\mu-\nu}\sum_{j=1}^{\nu}\binom{\nu-1}{j-1}\frac{\nu!}{j!}n^{j-\nu}x^j
\end{aligned}$$

$$
\begin{aligned}
&= (-x)^\mu + \sum_{\nu=1}^{\mu} \binom{\mu}{\nu} (-1)^{\mu-\nu} \sum_{j=\mu-\nu+1}^{\mu} \binom{\nu-1}{j+\nu-\mu-1} \frac{\nu!}{(j+\nu-\mu)!} n^{j-\mu} x^j \\
&= (-x)^\mu + \sum_{j=1}^{\mu} x^j n^{j-\mu} \sum_{\nu=\mu-j+1}^{\mu} \binom{\mu}{\nu} (-1)^{\mu-\nu} \binom{\nu-1}{j+\nu-\mu-1} \frac{\nu!}{(j+\nu-\mu)!} \\
&= (-x)^\mu + \sum_{j=1}^{\mu-1} x^j n^{j-\mu} \sum_{\nu=0}^{j-1} \binom{\mu}{\nu+\mu-j+1} \\
&\qquad\qquad \times (-1)^{j-1-\nu} \binom{\nu+\mu-j}{\nu} \frac{(\nu+\mu-j+1)!}{(\nu+1)!} \\
&= (-x)^\mu + \sum_{j=1}^{\mu-1} x^j n^{j-\mu} (-1)^{j-1} \frac{\mu!}{(j-1)!} \\
&\qquad\qquad \times \frac{1}{\mu-j} \sum_{\nu=0}^{j-1} (-1)^\nu \binom{j-1}{\nu} \binom{\nu+\mu-j}{\nu+1} \\
&= \sum_{j=1}^{\left[\frac{\mu}{2}\right]} \binom{\mu-j-1}{j-1} \frac{\mu!}{j!} n^{j-\mu} x^j .
\end{aligned}
$$

For the last equality we have used [8, (3.48)]. □

3. Commutativity results

In contrast with the Szász-Mirakjan operators, the Phillips operators have the very nice property of commutativity $\widetilde{S}_m(\widetilde{S}_n f) = \widetilde{S}_n(\widetilde{S}_m f)$. Another important property is the commutativity with an appropriate differential operator. This is a crucial step to establish a strong converse result of type A in the terminology of [4].

In our first theorem we prove a nice identity for $\widetilde{S}_m(\widetilde{S}_n f)$. An analogous result was proved by Abel and Ivan in [1] for the operators $\overline{S}_n$. From Theorem 3.1 the commutativity then follows as a corollary as well as a nice representation for iterates of $\widetilde{S}_n$.

Theorem 3.1. *For all $f \in C_\alpha[0,\infty)$, $m > \alpha$, $n > \alpha$, $\frac{mn}{m+n} > \alpha$ we have*

$$
\widetilde{S}_m(\widetilde{S}_n f) = \widetilde{S}_{\frac{mn}{m+n}}. \tag{11}
$$

Proof. As $\widetilde{S}_n(f,0)=f(0)$, the proposition is equivalent to

$$
\begin{aligned}
& m\sum_{j=1}^{\infty} s_{m,j}(x)\int_0^{\infty}\left\{s_{m,j-1}(t)n\sum_{k=1}^{\infty}s_{n,k}(t)\int_0^{\infty}s_{n,k-1}(y)f(y)dy\right\}dt \\
& \qquad +f(0)\left\{m\sum_{j=1}^{\infty}s_{m,j}(x)\int_0^{\infty}s_{m,j-1}(t)e^{-nt}dt+e^{-mx}\right\} \\
& = \frac{mn}{m+n}\sum_{l=1}^{\infty}s_{\frac{mn}{n+m},l}(x)\int_0^{\infty}s_{\frac{mn}{n+m},l-1}(y)f(y)dy+e^{-\frac{mn}{n+m}x}f(0).
\end{aligned}
\tag{12}
$$

Since

$$
\begin{aligned}
m\sum_{j=1}^{\infty}s_{m,j}(x)\int_0^{\infty}s_{m,j-1}(t)e^{-nt}dt &= m\sum_{j=1}^{\infty}s_{m,j}(x)\frac{m^{j-1}}{(n+m)^j} \\
&= e^{-mx}\sum_{j=1}^{\infty}\frac{\left(\frac{m^2}{n+m}x\right)^j}{j!} \\
&= e^{-mx}\left(e^{\frac{m^2}{n+m}x}-1\right),
\end{aligned}
$$

we get that (12) is equivalent to

$$
\begin{aligned}
& mn\int_0^{\infty}\left\{f(y)\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}s_{m,j}(x)s_{n,k-1}(y)\int_0^{\infty}s_{m,j-1}(t)s_{n,k}(t)dt\right\}dy \\
& \qquad = \frac{mn}{m+n}\sum_{l=1}^{\infty}s_{\frac{mn}{n+m},l}(x)\int_0^{\infty}s_{\frac{mn}{n+m},l-1}(y)f(y)dy.
\end{aligned}
\tag{13}
$$

We denote the left-hand side of (13) by $\int_0^{\infty}f(y)T_{m,n}(x,y)dy$. From

$$
\int_0^{\infty}s_{m,j-1}(t)s_{n,k}(t)dt=\frac{m^{j-1}n^k}{(m+n)^{j+k}}\cdot\frac{(k+j-1)!}{(j-1)!k!}
$$

we get

$$
\begin{aligned}
T_{m,n}(x,y) &= mn\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}s_{m,j}(x)s_{n,k-1}(y)\frac{m^{j-1}n^k}{(m+n)^{j+k}}\cdot\frac{(k+j-1)!}{(j-1)!k!} \\
&= mn\,e^{-mx-ny}\sum_{j=1}^{\infty}\frac{m^{2j-1}x^j}{(m+n)^j j!(j-1)!} \\
&\qquad \times\sum_{k=1}^{\infty}\frac{n^{2k-1}y^{k-1}}{(m+n)^k k!(k-1)!}\cdot(k+j-1)!.
\end{aligned}
\tag{14}
$$

For the inner sum of the right-hand side of (14) we can write

$$\sum_{k=1}^{\infty}\frac{n^{2k-1}y^{k-1}}{(m+n)^k k!(k-1)!}\cdot(k+j-1)! \;=\; \frac{n}{m+n}\sum_{k=0}^{\infty}\frac{(\frac{n^2}{m+n}y)^k}{k!(k+1)!}\cdot(k+j)! \;=:\; T(y).$$

From

$$z^j e^{az}=\sum_{k=0}^{\infty}\frac{a^k z^{k+j}}{k!}$$

and the Leibniz formula we obtain

$$\begin{aligned}\frac{d^{j-1}}{dz^{j-1}}\left(z^j e^{az}\right) &= \sum_{k=0}^{\infty}\frac{(j+k)!a^k z^{k+1}}{k!(k+1)!}\\ &= \sum_{l=0}^{j-1}\binom{j-1}{l}\frac{j!}{(l+1)!}z^{l+1}a^l e^{az}.\end{aligned}$$

Substituting $z=1$ and $a=\frac{n^2}{m+n}y$ in the above equation we derive

$$T(y)=\frac{n}{m+n}e^{\frac{n^2}{m+n}y}\sum_{l=0}^{j-1}\binom{j-1}{l}\frac{j!}{(l+1)!}\left(\frac{n^2}{m+n}y\right)^l.$$

Inserting this expression in (14) we get

$$\begin{aligned}T_{m,n}(x,y) &= \frac{n^2 m}{m+n}e^{-mx}e^{-ny}e^{\frac{n^2}{m+n}y}\sum_{j=1}^{\infty}\frac{m^{2j-1}x^j}{(m+n)^j j!(j-1)!}\\ &\qquad\times\sum_{l=0}^{j-1}\binom{j-1}{l}\frac{j!}{(l+1)!}\left(\frac{n^2}{m+n}y\right)^l\\ &= \frac{n^2 m}{m+n}e^{-mx}e^{-ny}e^{\frac{n^2}{m+n}y}\sum_{l=0}^{\infty}\frac{1}{(l+1)!l!}\left(\frac{n^2}{m+n}y\right)^l\\ &\qquad\times\sum_{j=l+1}^{\infty}\frac{m^{2j-1}x^j}{(m+n)^j}\cdot\frac{1}{(j-1-l)!}.\end{aligned}$$

Since

$$\begin{aligned}\sum_{j=l+1}^{\infty}\frac{m^{2j-1}x^j}{(m+n)^j}\cdot\frac{1}{(j-1-l)!} &= \frac{m^{2l+1}x^{l+1}}{(m+n)^{l+1}}\sum_{j=0}^{\infty}\frac{\left(\frac{m^2 x}{m+n}\right)^j}{j!}\\ &= \frac{m^{2l+1}x^{l+1}}{(m+n)^{l+1}}e^{\frac{m^2}{m+n}x},\end{aligned}$$

we get

$$\begin{aligned} T_{m,n}(x,y) &= \frac{mn}{m+n} e^{-\frac{mn}{m+n}x} e^{-\frac{mn}{m+n}y} \sum_{l=0}^{\infty} \frac{\left(\frac{mn}{m+n}x\right)^{l+1}}{(l+1)!} \frac{\left(\frac{mn}{m+n}y\right)^{l}}{(l)!} \\ &= \frac{mn}{m+n} \sum_{l=1}^{\infty} s_{\frac{mn}{n+m},l}(x) s_{\frac{mn}{n+m},l-1}(y). \end{aligned}$$

So, in view of (13), we have proved our proposition. □

Corollary 3.1. *For all $f \in C_\alpha[0,\infty)$, $m > \alpha$, $n > \alpha$, $\frac{mn}{m+n} > \alpha$ we have*

$$\widetilde{S}_m(\widetilde{S}_n f) = \widetilde{S}_n(\widetilde{S}_m f) \tag{15}$$

and for $l \in \mathbb{N}$, $\frac{m}{l} > \alpha$

$$\widetilde{S}_m^l = \widetilde{S}_{\frac{m}{l}}. \tag{16}$$

We point out that (16) is essential for the derivation of a strong converse result of type A.

As an appropriate differential operator we will use

$$\widetilde{D}^2 f := \varphi^2 D^2 f \tag{17}$$

where D denotes the ordinary differentiation of a function with respect to its variable.

Our second commutativity result is

Theorem 3.2. *Let $f \in C_\alpha[0,\infty)$ and $f', f'' \in C_\alpha[0,\infty)$. Then the operators $\widetilde{S}_n$, $n > \alpha$, and $\widetilde{D}^2$ commute, namely*

$$(\widetilde{D}^2 \circ \widetilde{S}_n) f = (\widetilde{S}_n \circ \widetilde{D}^2) f. \tag{18}$$

Proof. On using (6) we compute the first derivative of $\widetilde{S}_n f$.

$$\begin{aligned} D(\widetilde{S}_n(f,x)) &= -ne^{-nx} f(0) + n \sum_{k=1}^{\infty} \underbrace{n\left[s_{n,k-1}(x) - s_{n,k}(x)\right]}_{s'_{n,k}(x)} \int_0^\infty s_{n,k-1}(t) f(t) dt \\ &= -ne^{-nx} f(0) + n \sum_{k=0}^{\infty} n s_{n,k}(x) \int_0^\infty s_{n,k}(t) f(t) dt \\ &\quad - n \sum_{k=1}^{\infty} n s_{n,k}(x) \int_0^\infty s_{n,k-1}(t) f(t) dt \end{aligned}$$

$$= -ne^{-nx} f(0) + n^2 e^{-nx} \int_0^\infty e^{-nt} f(t)dt$$
$$- n \sum_{k=1}^\infty s_{n,k}(x) \int_0^\infty \underbrace{n\left[s_{n,k-1}(t) - s_{n,k}(t)\right]}_{s'_{n,k}(t)} f(t)dt$$
$$= ne^{-nx} \left\{ -f(0) + \left(\underbrace{-e^{-nt} f(t)dt|_0^\infty}_{=f(0)} + \int_0^\infty e^{-nt} f'(t)dt \right) \right\}$$
$$- n \sum_{k=1}^\infty s_{n,k}(x) \left\{ \underbrace{s_{n,k}(t)f(t)|_0^\infty}_{=0 \text{ as } k\geq 1} - \int_0^\infty s_{n,k}(t)f'(t)dt \right\}$$
$$= n \sum_{k=0}^\infty s_{n,k}(x) \int_0^\infty s_{n,k}(t)f'(t)dt$$
$$= \overline{S}_n(f', x).$$

Thus we proved that

$$D(\widetilde{S}_n(f, x)) = \overline{S}_n(f', x). \tag{19}$$

Now from [10, (3.1)] we get for the second derivative

$$\begin{aligned} D^2(\widetilde{S}_n(f, x)) &= D(\overline{S}_n(f', x)) \\ &= n \sum_{k=0}^\infty s_{n,k}(x) \int_0^\infty s_{n,k+1}(t)f''(t)dt. \end{aligned}$$

Finally, on using (9) and observing that $(\widetilde{D}^2 f)(0) = 0$ we get

$$\begin{aligned} \widetilde{D}^2(\widetilde{S}_n(f, x) &= \varphi(x)^2 D(\overline{S}_n(f', x)) \\ &= n \sum_{k=1}^\infty s_{n,k}(x) \int_0^\infty s_{n,k-1}(t)(\widetilde{D}^2 f)(t)dt \\ &= \widetilde{S}_n(\widetilde{D}^2 f, x). \end{aligned}$$

Theorem 3.2 is proved. □

As an immediate consequence of Theorem 3.2 we get the following corollary which improves significantly the constant in the result of [6, Lemma 5].

Corollary 3.2. *For $f \in W_\infty^2(\varphi)$ there holds*

$$\|\widetilde{D}^2(\widetilde{S}_n f)\| \leq \|\widetilde{D}^2 f\|.$$

Remark 3.1. *For sufficiently smooth functions we get from Theorem 3.2 via induction*

$$(\widetilde{D}^2)^l \circ \widetilde{S}_n = \widetilde{S}_n \circ (\widetilde{D}^2)^l.$$

It is easy to verify that

$$(\widetilde{D}^2)^l = \widetilde{D}^{2l} := D^{l-1}\varphi^{2l}D^{l+1}.$$

Such an identity is not true in the case of the corresponding appropriate differential operators for the genuine Bernstein-Durrmeyer and Baskakov-Durrmeyer operators.

4. Strong Voronovskaja type theorem

In order to derive a strong converse inequality of type A we need an appropriate strong Voronovskaja type result. We formulate our results in terms of a positive constant c which will be choosen later to get a good constant in the strong converse inequality.

Theorem 4.1. *Let $g \in C_B[0,\infty)$, $\varphi^2 g''', \varphi^3 g''' \in C_B[0,\infty)$ and $n > 0$. Then*

$$\left\|\widetilde{S}_n g - g - \frac{1}{n}\,\varphi^2 g''\right\| \le \frac{\sqrt{6}}{2}\cdot\frac{1}{n}\max\left\{\frac{4}{3}\sqrt{1+2c}\cdot\frac{1}{\sqrt{n}}\,\|\varphi^3 g'''\|, \sqrt{\frac{1+2c}{c}}\cdot\frac{1}{n}\,\|\varphi^2 g'''\|\right\},$$

where c denotes an arbitrary positive constant.

Proof. We apply the operators $\widetilde{S}_n$ to the Taylor expansion of g

$$g(t) = g(x) + g'(x)(t-x) + \frac{1}{2}g''(x)(t-x)^2 + \frac{1}{2}\int_x^t g'''(u)(t-u)^2 du$$

and use Lemma 2.1 to derive

$$\left|\widetilde{S}_n(g,x) - g(x) - \frac{1}{n}\,(\widetilde{D}^2 g)(x)\right| \le \frac{1}{2}\widetilde{S}_n\left(\left|\int_x^t g'''(u)(t-u)^2 du\right|, x\right). \tag{20}$$

Case 1: $x \ge \frac{1}{cn}$. We have

$$\widetilde{S}_n\left(\left|\int_x^t g'''(u)(t-u)^2 du\right|, x\right) \le \|\varphi^3 g'''\|\widetilde{S}_n\left(\left|\int_x^t \frac{(t-u)^2}{\varphi(u)^3} du\right|, x\right). \tag{21}$$

As $\frac{|t-u|}{u} \leq \frac{|t-x|}{x}$ we now observe that

$$
\begin{aligned}
&\widetilde{S}_n\left(\left|\int_x^t \frac{(t-u)^2}{\varphi(u)^3}du\right|, x\right) \\
&\quad= \sum_{k=1}^{\infty} s_{n,k}(x) n \int_0^{\infty} s_{n,k-1}(t)\left|\int_x^t \frac{(t-u)^2}{u^{\frac{3}{2}}}du\right| dt + e^{-nx}\left|\int_0^x u^{\frac{1}{2}}du\right| \\
&\quad\leq \sum_{k=1}^{\infty} s_{n,k}(x) n \int_0^{\infty} s_{n,k-1}(t)\frac{|t-x|^{\frac{3}{2}}}{x^{\frac{3}{2}}}\left|\int_x^t |t-u|^{\frac{1}{2}}du\right| dt + \frac{2}{3}e^{-nx}x^{\frac{3}{2}} \\
&\quad\leq \frac{2}{3}\cdot\frac{1}{x^{\frac{3}{2}}}\widetilde{S}_n\left(|t-x|^3, x\right).
\end{aligned} \tag{22}
$$

Now using the Cauchy-Schwarz inequality we get the estimate

$$
\begin{aligned}
\widetilde{S}_n\left(|t-x|^3, x\right) &\leq \sqrt{\widetilde{S}_n\left((t-x)^2, x\right)}\cdot\sqrt{\widetilde{S}_n\left((t-x)^4, x\right)} \\
&= \sqrt{\frac{2x}{n}}\cdot\sqrt{\frac{12x}{n^2}\left(x+\frac{2}{n}\right)},
\end{aligned} \tag{23}
$$

where we have used Lemma 2.1 for the moments. For $x \geq \frac{1}{cn}$ (22) and (23) imply

$$
\widetilde{S}_n\left(\left|\int_x^t \frac{(t-u)^2}{\varphi(u)^3}du\right|, x\right) \leq \frac{4\sqrt{6}}{3}\sqrt{1+2c}\, n^{-3/2}.
$$

Thus for the case $x \geq \frac{1}{cn}$ we have

$$
\widetilde{S}_n\left(\left|\int_x^t g'''(u)(t-u)^2du\right|, x\right) \leq \frac{4\sqrt{6}}{3}\sqrt{1+2c}n^{-3/2}\|\varphi^3 g'''\|. \tag{24}
$$

Case 2: $x \leq \frac{1}{cn}$. We have

$$
\begin{aligned}
&\varphi(x)^2\widetilde{S}_n\left(\left|\int_x^t g'''(u)(t-u)^2du\right|, x\right) \\
&\qquad\qquad\leq \|\varphi^2 g'''\|\varphi(x)^2\widetilde{S}_n\left(\left|\int_x^t \frac{(t-u)^2}{\varphi(u)^2}du\right|, x\right).
\end{aligned} \tag{25}
$$

Proceeding in a similar way as in *Case 1* we get

$$
\begin{aligned}
\varphi(x)^2\widetilde{S}_n\left(\left|\int_x^t \frac{(t-u)^2}{\varphi(u)^2}du\right|, x\right) &\leq \widetilde{S}_n\left(\left||t-x|\int_x^t |t-u|du\right|, x\right) \\
&\leq \frac{1}{2}\widetilde{S}_n\left(|t-x|^3, x\right) \\
&\leq \varphi(x)^2\sqrt{6}\sqrt{\frac{1+2c}{c}}\,n^{-2}.
\end{aligned}
$$

Thus we obtained for the case $x \le \frac{1}{cn}$

$$\widetilde{S}_n\left(\left|\int_x^t g'''(u)(t-u)^2 du\right|, x\right) \le \sqrt{6}\sqrt{\frac{1+2c}{c}}\, n^{-2}\|\varphi^2 g'''\|. \tag{26}$$

Substitution of (24) and (26) into (20) proves the proposition. □

5. Direct and strong converse result

Our direct and converse results will be formulated in terms of the following K-functional

$$K_\varphi^2(f,\delta^2) := \inf\{\|f-g\| + \delta^2\|\widetilde{D}^2 g\| : g \in W_\infty^2(\varphi)\}. \tag{27}$$

The corresponding second order Ditzian-Totik modulus of smoothness is given by

$$\omega_\varphi^2(f,\delta) = \sup_{0<h\le\delta} \|\Delta_{h\varphi}^2 f\|, \tag{28}$$

where

$$\Delta_{h\varphi(x)}^2 f(x) = \begin{cases} f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x)), & \text{if } x \pm h\varphi(x) \in [0,\infty), \\ 0, & \text{otherwise.} \end{cases}$$

It is known that $K_\varphi^2(f,\delta^2)$ and $\omega_\varphi^2(f,\delta)$ are equivalent (see [5, Theorem 2.1.1]), i.e., there exists an absolute constant $C>0$ and δ_0 such that

$$C^{-1}\omega_\varphi^2(f,\delta) \le K_\varphi^2(f,\delta^2) \le C\omega_\varphi^2(f,\delta), \quad 0<\delta\le\delta_0.$$

Our main results are

Theorem 5.1. *For every $f \in C_B[0,\infty)$ and $n>0$ there holds*

$$\|\widetilde{S}_n f - f\| \le 2\, K_\varphi^2\left(f, \frac{1}{n}\right).$$

Theorem 5.2. *For every $f \in C_B[0,\infty)$ and $n>0$ the following inequality holds true*

$$K_\varphi^2\left(f, \frac{1}{n}\right) \le 92.16\, \|\widetilde{S}_n f - f\|. \tag{29}$$

As a consequence of Theorems 5.1 and 5.2 we get the following equivalence result.

Corollary 5.1. *For $f \in C_B[0,\infty)$, $n > 0$ we have the following equivalences*

$$\begin{aligned} \frac{1}{2}\,\|\widetilde{S}_n f - f\| &\leq K_\varphi^2\left(f, \frac{1}{n}\right) \leq 92.16\,\|\widetilde{S}_n f - f\|, \\ C_1\,\|\widetilde{S}_n f - f\| &\leq \omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) \leq C_2\,\|\widetilde{S}_n f - f\|, \end{aligned}$$

where $C_1, C_2 > 0$ are absolute constants.

Proof of Theorem 5.1. Let $g \in W_\infty^2(\varphi)$ be arbitrary and x be fixed. From the Taylor expansion

$$g(t) - g(x) = g'(x)(t-x) + \int_x^t g''(s)(t-s)ds, \quad t \in [0,\infty),$$

we see that

$$|\tilde{S}_n(g,x) - g(x)| \leq \tilde{S}_n\left\{\left|\int_x^t g''(u)(t-u)du\right|, x\right) . \tag{30}$$

It is known (see [5, (9.6.1)]) that

$$\begin{aligned} \varphi(x)^2\left|\int_x^t g''(u)(t-u)du\right| &\leq |t-x|\left|\int_x^t \varphi(u)^2 g''(u)du\right| \\ &\leq \|\varphi^2 g''\|(t-x)^2. \end{aligned}$$

From (30) and the second moment $\widetilde{S}_n((t-x)^2, x) = \frac{2x}{n}$ (see Lemma 2.1) we get

$$|\tilde{S}_n(g,x) - g(x)| \leq \frac{2}{n}\|\varphi^2 g''\|.$$

This gives

$$\begin{aligned} |\tilde{S}_n(f,x) - f(x)| &\leq |\tilde{S}_n(f-g)(x)| + |g(x) - f(x)| + |\tilde{S}_n(g,x) - g(x)| \\ &\leq 2(\|f-g\| + \frac{1}{n}\|\varphi^2 g''\|). \end{aligned}$$

Taking the infimum of the right-hand side term over all $g \in W_\infty^2(\varphi)$ we obtain the statement of the theorem. □

For the proof of Theorem 5.2 we will need three further estimates which are proved in the next lemmas. Note that Lemma 5.1 improves upon the constant in [6, Lemma 6] and Lemma 5.2 gives an explicit value for the constant in [6, Lemma 7].

Lemma 5.1. *For $f \in C_B[0,\infty)$, $n > 0$ we have*

$$\|\widetilde{D}^2(\widetilde{S}_n f)\| \leq 2n\,\|f\|.$$

Proof. From (8) we have

$$\varphi(x)^2 \left|\widetilde{D}^2 \widetilde{S}_n(f,x)\right| \le \|f\| \left\{ \sum_{k=0}^{\infty} (k-nx)^2 s_{n,k}(x) + \sum_{k=1}^{\infty} k s_{n,k}(x) \right\}. \tag{31}$$

For the second moment of the Szász-Mirakjan operator S_n we make use of (10). As S_n preserves linear functions we derive from (31)

$$\left|\widetilde{D}^2 \widetilde{S}_n(f,x)\right| \le 2n \, \|f\|. \qquad \square$$

Lemma 5.2. *For every $g \in W^2_\infty(\varphi)$ and $n > 0$ we have*

$$\|\varphi^3 D^3(\widetilde{S}_n g)\| \le 1.47\sqrt{n}\, \|\widetilde{D}^2 g\|.$$

Proof. From (19) and [10, (3.1)] we have

$$\begin{aligned} \varphi(x)^4 D^3(\widetilde{S}_n(g,x)) &= \varphi(x)^4 D^2(\overline{S}_n(g',x)) \\ &= \varphi(x)^4 n \sum_{k=0}^{\infty} s'_{n,k}(x) \int_0^\infty s_{n,k+1}(t) D^2 g(t) dt \\ &= n \sum_{k=0}^{\infty} (k-nx) s_{n,k+1}(x) \int_0^\infty s_{n,k}(t) (\widetilde{D}^2 g(t)) dt \,, \end{aligned} \tag{32}$$

where we have used (7) and (9). On using (5) we get

$$\begin{aligned} \left|\varphi(x)^4 D^3(\widetilde{S}_n(g,x))\right| &\le \|\widetilde{D}^2 g\| \sum_{k=0}^{\infty} (|k+1-nx|+1) s_{n,k+1}(x) \\ &= \|\widetilde{D}^2 g\| \left\{ \sum_{k=0}^{\infty} (|k-nx|+1) s_{n,k}(x) - e^{-nx}(1+nx) \right\} \\ &\le \|\widetilde{D}^2 g\| \left\{ n\sqrt{S_n((t-x)^2,x)} + 1 - e^{-nx}(1+nx) \right\} \\ &= \|\widetilde{D}^2 g\| \left\{ \varphi(x)\sqrt{n} + 1 - e^{-nx}(1+nx) \right\}. \end{aligned} \tag{33}$$

The maximum of $\frac{1-e^{-nx}(1+nx)}{\sqrt{x}}$ is attained for $nx \in [3.2, 3.25]$, hence we have the estimate

$$\frac{1-e^{-nx}(1+nx)}{\sqrt{nx}} \le \frac{1-e^{-3.25}(1+3.2)}{\sqrt{3.2}} < 0.47.$$

Inserting this bound into (33) we get

$$\left|\varphi(x)^3 D^3(\widetilde{S}_n(g,x))\right| \le 1.47\sqrt{n}\, \|\widetilde{D}^2 g\|. \qquad \square$$

Lemma 5.3. *For every $g \in W^2_\infty(\varphi)$ we have*

$$\|\varphi^2 D^3(\widetilde{S}_n g)\| \le 2n\, \|\widetilde{D}^2 g\|.$$

Proof. Making again use of (19) and [10, (3.1)], combined with (6), we get

$$\begin{aligned}\varphi(x)^2 D^3(\widetilde{S}_n(g,x)) &= \varphi(x)^2 D^2(\overline{S}_n(g',x))\\ &= \varphi(x)^2 n \sum_{k=0}^{\infty} s'_{n,k}(x) \int_0^\infty s_{n,k+1}(t) g''(t)dt\\ &= \varphi(x)^2 n \sum_{k=0}^{\infty} n[s_{n,k-1}(x) - s_{n,k}(x)] \int_0^\infty \frac{n}{k+1} s_{n,k}(t) \widetilde{D}^2 g(t)dt.\end{aligned}$$

Applying (5) we deduce the result as follows:

$$\begin{aligned}|\varphi(x)^2 D^3((\widetilde{S}_n(g,x))| &\le \|\widetilde{D}^2 g\| n \left\{ \sum_{k=1}^{\infty} \frac{nx s_{n,k-1}(x)}{k+1} + \sum_{k=0}^{\infty} \frac{nx s_{n,k}(x)}{k+1} \right\}\\ &\le \|\widetilde{D}^2 g\| 2n \sum_{k=0}^{\infty} s_{n,k}(x)\\ &= 2n\, \|\widetilde{D}^2 g\|.\end{aligned}$$

□

Proof of Theorem 5.2. Since

$$K^2_\varphi\left(f, \frac{1}{n}\right) \le \|\widetilde{S}_n f - f\| + \frac{1}{n} \|\widetilde{D}^2(\widetilde{S}_n f)\|\,, \tag{34}$$

we have to estimate $\frac{1}{n}\|\widetilde{D}^2(\widetilde{S}_n f)\|$.

Let $N \in \mathbb{N}$. From Lemma 5.1 we obtain the estimate

$$\begin{aligned}\frac{1}{n}\|\widetilde{D}^2(\widetilde{S}_n f)\| &\le \frac{1}{n}\|\widetilde{D}^2[\widetilde{S}_n(f - \widetilde{S}^N_n f)]\| + \frac{1}{n}\|\widetilde{D}^2[\widetilde{S}_n(\widetilde{S}^N_n f)]\|\\ &\le 2N\|f - \widetilde{S}_n f\| + \frac{1}{n}\|\widetilde{D}^2(\widetilde{S}^{N+1}_n f)\|.\end{aligned} \tag{35}$$

We now apply the strong Voronovskaja–type result of Theorem 4.1 to the function $g = \widetilde{S}^{N+1}_n f$ to obtain

$$\begin{aligned}&\frac{1}{n}\|\widetilde{D}^2(\widetilde{S}^{N+1}_n f)\|\\ &\le \|\widetilde{S}^{N+2}_n f - \widetilde{S}^{N+1}_n f\| + \|\widetilde{S}^{N+2}_n f - \widetilde{S}^{N+1}_n f - \frac{1}{n}\widetilde{D}^2(\widetilde{S}^{N+1}_n f)\|\\ &\le \|f - \widetilde{S}_n f\|\\ &\quad + \frac{\sqrt{6}}{2n} \max\left\{ \frac{4\sqrt{1+2c}}{3\sqrt{n}} \|\varphi^3(\widetilde{S}^{N+1}_n f)'''\|, \sqrt{\frac{1+2c}{c}} \cdot \frac{1}{n}\|\varphi^2(\widetilde{S}^{N+1}_n f)'''\| \right\}.\end{aligned} \tag{36}$$

From Corollary 3.1, Lemma 5.2 and Lemma 5.3 we obtain the estimates

$$\begin{aligned} \|\varphi^3(\widetilde{S}_n^{N+1}f)'''\| &= \|\varphi^3(\widetilde{S}_{\frac{n}{N}}\widetilde{S}_nf)'''\| \le 1.47\sqrt{\frac{n}{N}}\|\widetilde{D}^2(\widetilde{S}_nf)\| \\ \|\varphi^2(\widetilde{S}_n^{N+1}f)'''\| &= \|\varphi^2(\widetilde{S}_{\frac{n}{N}}\widetilde{S}_nf)'''\| \le 2\frac{n}{N}\|\widetilde{D}^2(\widetilde{S}_nf)\|. \end{aligned}$$

The latter estimates together with (35) and (36) imply

$$\begin{aligned} \frac{1}{n}\|\widetilde{D}^2(\widetilde{S}_nf)\| &\le (2N+1)\|f-\widetilde{S}_nf\| \\ &\quad + \frac{\sqrt{6}}{2n}\|\widetilde{D}^2(\widetilde{S}_nf)\|\max\left\{\frac{4\cdot 1.47}{3}\cdot\frac{\sqrt{1+2c}}{\sqrt{N}}, 2\sqrt{\frac{1+2c}{c}}\cdot\frac{1}{N}\right\}. \end{aligned} \tag{37}$$

In order to get a good constant we now choose $N=16$ and $c=\frac{625}{9604}$. With this choice we have

$$\max\left\{\frac{4\cdot 1.47}{3}\cdot\frac{\sqrt{1+2c}}{\sqrt{N}},\ 2\sqrt{\frac{1+2c}{c}}\cdot\frac{1}{N}\right\} = \frac{9\sqrt{2\cdot 67}}{200}.$$

Substituting this into (37) leads to

$$\frac{1}{n}\|\widetilde{D}^2(\widetilde{S}_nf)\| \le 33\|f-\widetilde{S}_nf\| + \frac{1}{n}\frac{9\sqrt{2\cdot 6\cdot 67}}{400}\|\widetilde{D}^2(\widetilde{S}_nf)\|. \tag{38}$$

Note that $\frac{9\sqrt{2\cdot 6\cdot 67}}{400} < 1$, so (38) is equivalent to

$$\frac{1}{n}\|\widetilde{D}^2(\widetilde{S}_nf)\| \le \frac{6600}{200-9\sqrt{3\cdot 67}}\|f-\widetilde{S}_nf\|.$$

Together with (34) we end up with the estimate

$$K_\varphi^2\left(f,\frac{1}{n}\right) \le \frac{6800-9\sqrt{3\cdot 67}}{200-9\sqrt{3\cdot 67}}\|\widetilde{S}_nf-f\| \le 92.16\|\widetilde{S}_nf-f\|.$$

Theorem 5.2 is proved. □

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