Some relations between variational-like inequality problems and vectorial optimization problems in Banach spaces

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Abstract

In this work, we will establish some relations between variational-like inequality problems and vectorial optimization problems in Banach spaces under invexity hypotheses. This paper extends the earlier work of Ruiz-Garzón et al. [G. Ruiz-Garzón, R. Osuna-Gómez, A. Rufián-Lizana, Relationships between vector variational-like inequality and optimization problems, European J. Oper. Res. 157 (2004) 113–119].

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1. Introduction

It is well-known that variational inequalities are appearing naturally in problems from Physics, Economics, Optimization and Control, Elasticity and the Applied Sciences (see for instance, [1–3]).

In the scalar case, Mancino and Stampacchia [4] obtained the following result: if $F : S \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the gradient of a convex function $\theta : S \rightarrow \mathbb{R}^n$ and $S$ is an open and convex set, then the variational inequality problem (VIP) is equivalent to the optimization problem (MP), where (VIP) is:

(VIP): Find $\bar{x} \in S$ such that

$$(y - \bar{x})^T F(\bar{x}) \geq 0, \forall y \in S,$$

and (MP) is:

\[ \text{Find } x \in S \text{ such that } F(x) \cdot y \geq \lambda, \forall y \in S, \]

where $\lambda$ is a real number.

\[ \text{Find } x \in S \text{ such that } F(x) \cdot y \geq \lambda, \forall y \in S, \]

This work extends the earlier work of Ruiz-Garzón et al. to the vectorial case.

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We say that the variational-like inequality problem (VVLIP) is:

\[ \text{Find } \eta \in S \text{ such that } \eta(y, \eta) F(\eta) \geq 0, \forall y \in S. \]

Parida et al. in [5] studied the existence of the solution of (VVLIP) and the convex mathematical programming. Ruiz-Garzón et al. in [6] proved that the solution of (VVLIP) is coincident with the solution of a certain mathematical programming problem under certain hypotheses of the generalized invexity and monotonicity. In our work, we shall extend the results of [7] in another direction to that proposed by [9], extending these results to the vectorial optimization problems in Banach spaces, when the domination structure is defined by convex cones.

Throughout the paper unless otherwise stated, let \( E_1, E_2 \) be two Banach spaces, \( \mathcal{L}(E_1, E_2) \) denote the space of all continuous linear operators from \( E_1 \) to \( E_2 \), and let \( f : S \to E_2 \) be a given function where \( S \) is a nonempty subset of \( E_1 \). Let \( Q \subset E_2 \), be a pointed closed, convex cone with nonempty interior and different from \( E_2 \).

The following concepts are used in the following:

**Definition 1.1.** (a) We say that \( \eta \in S \) is efficient of \( f \) if there exists no \( y \in S \) such that

\[ f(y) - f(\eta) \in -Q \setminus \{0\}; \]

(b) We say that \( \eta \in S \) is weakly efficient of \( f \) if there exists no \( y \in S \) such that

\[ f(y) - f(\eta) \in -\text{int } Q, \]

where \( \text{int } Q \) denotes the interior set of \( Q \).

We denote by \( E(f; S) \) the set of all efficient points of \( f \) and \( WE(f; S) \) the set of all weakly efficient points of \( f \). Obviously, \( E(f; S) \subset WE(f; S) \).

Now we consider the following vectorial optimization problem:

\[ \text{(VOP): V-min } f(x) \text{ subject to } x \in S, \]

whose resolution consists of the determination of the set \( E(f; S) \) and the weak vectorial optimization problem:

\[ \text{(WVOP): W-min } f(x) \text{ subject to } x \in S, \]

whose resolution consists of the determination of the set \( WE(f; S) \).

Next, \( \eta : S \times S \to E_1 \) and \( F : S \to \mathcal{L}(E_1, E_2) \) be two given functions, we consider the following Vectorial variational-like inequality problem (VVLIP):

\[ \text{(VVLIP): Find a point } \eta \in S \text{ such that } F(\eta) \eta(y, \eta) \not\in -Q \setminus \{0\}, \forall y \in S, \]

where, we denote by \( F(\eta) \eta(y, \eta) \) the value of the operator \( F(\eta) \) applied on vector \( \eta(y, \eta) \), and the Weak vectorial variational-like inequality problem (WVVLIP):

\[ \text{(WVVLIP): Find a point } \eta \in S \text{ such that } F(\eta) \eta(y, \eta) \not\in -\text{int } Q, \forall y \in S. \]
We remark that in finite-dimensional case, i.e., $E_1 = \mathbb{R}^n$, $E_2 = \mathbb{R}^m$ and $Q = \mathbb{R}^m_+$, the above problems were studied by Ruiz-Garzón et al. [7]. Here we generalize the results due to [5,7] for the infinite-dimensional case.

In [10], Yang and Goh proved that, under the hypotheses of convexity, if $F = \nabla f$, the resolution of (WVOP) is equivalent to the resolution of (WVVLIP). Similar results can be found in [11]. Further, we remark that Ansari and Siddiqi [12], Kazmi [13] and Yang [14] characterize the weakly efficient points for (VOP) through the solutions of weak variational-like inequalities under the hypothesis of pre-invexity.

In this paper, we shall prove that the solutions of vectorial problems (VOP) and (WVOP) can be characterized through the solutions of vectorial variational-like inequality problems (VVLIP) and (WVVLIP), respectively, under some pseudoinvexity hypotheses which are weaker than pre-invexity hypothesis. Our results generalize and unify the results due to Yang and Goh [10] and Lee and Kum [11].

Finally, we recall the following concept:

**Definition 1.2.** A function $f : S \rightarrow E_2$, is called Fréchet differentiable (or, differentiable) at $\bar{x} \in \text{int} S$ iff there exists a bounded operator $A \in L(E_1, E_2)$ such that

$$f(\bar{x} + h) - f(\bar{x}) = Ah + \|h\|\epsilon(h)$$

for all $h \in E_1$ in an open neighborhood of $h = 0$, where $\epsilon(h) \rightarrow 0$ as $h \rightarrow 0$. We denote $A := Df(\bar{x})$ (see [15]).

2. Relations between variational-like inequality problems and vectorial optimization problems

The notions of the generalized invexity introduced by Osuna-Gómez et al. [16] in finite-dimensional context, can be generalized as follows:

**Definition 2.1.** Let $S$ be a nonempty subset of $E_1$ and let $f : S \rightarrow E_2$ be Fréchet differentiable (or, differentiable) function at $x \in \text{int} S$.

(a) We say that $f$ is **invex (IX)** at $x \in S$ iff there exists a vectorial function $\eta : S \times S \rightarrow E_1$ such that

$$f(y) - f(x) - Df(x)\eta(y, x) \in Q, \forall y \in S.$$

(b) The function $f$ is called **strictly invex (SIX)** at $x \in S$ iff, there exists a vectorial function $\eta : S \times S \rightarrow E_1$ such that

$$f(y) - f(x) - Df(x)\eta(y, x) \in \text{int} Q, \forall y \in S, \ y \neq x;$$

(c) The function $f$ is called **pseudoinvex (PIX)** at $x \in S$ iff, there exists a vectorial function $\eta : S \times S \rightarrow E_1$ such that

$$f(y) - f(x) \in -\text{int} Q \Rightarrow Df(x)\eta(y, x) \in -\text{int} Q, \forall y \in S.$$

We observe that

(SIX) $\Rightarrow$ (IX) $\Rightarrow$ (PIX).

It is well-known that in the case $E_2 = \mathbb{R}$ and $Q = \mathbb{R}^+$, the class of invex functions is exactly equal to pseudoinvex functions, but it is not a true vectorial case (see [7]).

**Theorem 2.2.** Let $f : S \subset E_1 \rightarrow E_2$ be a differentiable function and invex at $\bar{x} \in \text{int} S$, with respect to $\eta$. If $F \equiv Df$ and $\bar{x}$ is a solution of (VVLIP), then $\bar{x}$ is an efficient solution of (VOP).

**Proof.** Assume that $\bar{x}$ is a solution of (VVLIP). If possible, let $\bar{x}$ not be an efficient solution of (VOP). Then, there exists $y \in S$ such that

$$f(y) - f(\bar{x}) \in -Q \setminus \{0\}. \quad (3)$$

From the invexity hypothesis on $f$ we obtain

$$Df(\bar{x})\eta(y, \bar{x}) \in f(y) - f(\bar{x}) - Q. \quad (4)$$

From (3) and (4) we have
\[Df(\overline{x})\eta(y, \overline{x}) \in -Q \setminus \{0\}.\]  

Which contradicts our assumption. Hence \(\overline{x}\) is an efficient solution of (VOP). \(\square\)

Consequently, under invexity hypothesis, the solutions of (VVLIP) are efficient solutions of (VOP). To show the converse of the preceding theorem, we set some more strong conditions. More precisely, we have:

**Theorem 2.3.** Let \(f : S \subset E_1 \rightarrow E_2\) be a differentiable function at \(\overline{x} \in \text{int} S\). Assume that \(F \equiv Df\) and that \(-f\) is strictly invex. If \(\overline{x}\) is a solution of (WVOP), then \(\overline{x}\) also is a solution of (VVLIP).

**Proof.** We shall prove this Theorem by an indirect method. Assume that \(\overline{x}\) is solution of (WVOP). If possible, let \(\overline{x}\) not be a solution of (VVLIP). Then, there exists \(y \in S\) such that

\[Df(\overline{x})\eta(y, \overline{x}) \in -Q \setminus \{0\}.\]  

On the other hand, \(-f\) is strictly invex, consequently

\[f(y) - f(\overline{x}) \in Df(\overline{x})\eta(y, \overline{x}) - \text{int} Q \subset -Q \subset -\text{int} Q,\]  

which is a contradiction to the assumption that \(\overline{x}\) is a weakly efficient solution of (WVOP). Hence, \(\overline{x}\) is a solution of (VVLIP). \(\square\)

**Theorem 2.4.** Let \(f : S \subset E_1 \rightarrow E_2\) be a differentiable function at \(\overline{x} \in \text{int} S\) and \(F \equiv Df\).

(i) If \(\overline{x}\) is a weakly efficient solution of (WVOP) then \(\overline{x}\) is a solution of (WVVLIP).

(ii) If \(f\) is a pseudoinvex function at \(\overline{x}\) and if \(\overline{x}\) is a solution of (WVVLIP) then \(\overline{x}\) is a weakly efficient solution of (WVOP).

**Proof.** (i) Assume that \(\overline{x}\) is a weakly efficient solution of (WVOP). Let \(y \in S\). Since \(\overline{x} \in \text{int} S\), then, for each \(t > 0\) sufficiently small, the point \(\overline{x} + t\eta(y, \overline{x})\) belongs to \(S\). Further, it follows that

\[f(\overline{x} + t\eta(y, \overline{x})) - f(\overline{x}) \not\in -\text{int} Q\]  

and since \(-\text{int} Q\) is a cone, for such \(t\), we have

\[\frac{1}{t}[f(\overline{x} + t\eta(y, \overline{x})) - f(\overline{x})] \not\in -\text{int} Q.\]  

Letting \(t \downarrow 0\) and recalling the fact that \((-\text{int} Q)\) is closed, it follows that

\[Df(\overline{x})\eta(y, \overline{x}) \not\in -\text{int} Q, \quad \forall y \in S,\]  

i.e \(\overline{x}\) is a solution of (WVVLIP).

(ii) Assume that \(\overline{x} \in S\) is a solution of (WVVLIP). If possible, let \(\overline{x}\) not be a weakly efficient solution of (WVOP). Then, there exists \(y \in S\) such that

\[f(y) - f(\overline{x}) \in -\text{int} Q\]  

and, since \(f\) is pseudoinvex at \(\overline{x}\), we have

\[Df(\overline{x})\eta(y, \overline{x}) \in -\text{int} Q,\]  

which contradicts our assumption that \(\overline{x}\) is a solution of (WVVLIP). \(\square\)

**Theorem 2.5.** Let \(f : S \subset E_1 \rightarrow E_2\) be a differentiable function at point \(\overline{x}\). Assume that \(F \equiv Df\) and that \(f\) is strictly invex at \(\overline{x}\). If \(\overline{x}\) is a solution of (WVOP), then it is also a solution of (VOP).

**Proof.** Assume that \(\overline{x}\) is a weakly efficient solution of (WOP). If possible, let \(\overline{x}\) not be a solution of (VOP), then there exists \(y \in S\) such that

\[f(y) - f(\overline{x}) \in -Q \setminus \{0\}.\]  

Since \(f\) is strictly invex at \(\overline{x}\), we have

\[f(y) - f(\overline{x}) - Df(\overline{x})\eta(y, \overline{x}) \in \text{int} Q.\]  

Then, from (13) and (14), we obtain
\[
Df(\bar{x})\eta(y, \bar{x}) \in f(y) - f(\bar{x}) - \text{int } Q \subset -Q \setminus \{0\} - \text{int } Q \subset -\text{int } Q,
\]
which implies that $\bar{x}$ is not a solution of (WVVLIP) and, by using Theorem 2.4, we obtain that $\bar{x}$ is not a weakly efficient solution, which contradicts the hypothesis. □

Let $C \subset E_2$ be a cone, we define the dual cone of $C$ as follows
\[
C^* := \{\xi \in E_2^* : \langle \xi, x \rangle \geq 0, \forall x \in C\},
\]
(16)
where $E_2^*$ denotes the topological dual of $E_2$ and $\langle \cdot, \cdot \rangle$ is the canonical duality pairing between $E_2^*$ and $E_2$.

**Definition 2.6.** We say that $x \in S$ is a vectorial critical point (VCP) of $f$ if there exists a functional $\lambda^* \in C^* \setminus \{0\}$ such that $\lambda^* \circ Df(x) = 0$.

In [17], Craven proved that every vectorial critical point is a necessary condition for the weak efficiency of (WVOP). Next, we will prove, under some hypotheses, the inverse affirmation. First, we recall some necessary results.

**Lemma 2.7.** Let $F$ be a Banach space and let $C \subset F$ be a closed, convex cone with $C \neq F$ and $\text{int } C \neq \emptyset$. If $x \in \text{int } C$ and $\xi \in C^* \setminus \{0\}$, then $\langle \xi, x \rangle > 0$.

The following result is a generalization of the classical alternative Farkas’ theorem for the infinite-dimensional spaces, see [18].

**Lemma 2.8.** Let $X, Y, V$ be three normed spaces; let $T \subset V$, $Q \subset Y$ be convex cones and let $A \in \mathcal{L}(X, V)$, $M \in \mathcal{L}(X, Y)$, $b \in -T$, $s \in -Q$. Assume that the set $[Ab]^T$ $(T^*)$ is w$^*$-closed. Then, the following system
\[
\begin{align*}
Ax + b &\in -T \\
Mx + s &\in -\text{int } Q
\end{align*}
\]
(17)
has no solution $x \in X$ iff there exist $\tau \in Q^* \setminus \{0\}$ and $\lambda \in T^*$ such that
\[
\begin{align*}
\tau M + \lambda A &= 0 \\
\langle \lambda, b \rangle &= 0 \\
\langle \tau, s \rangle &= 0.
\end{align*}
\]
(18)

**Proposition 2.9.** All the vectorial points critical are solutions of (WVOP) iff the function $f$ is pseudoinvex.

**Proof.** Let $f$ be a pseudoinvex function and let $\bar{x} \in S$ be a vectorial critical point. We assume that $\bar{x}$ is not a weakly efficient solution of (WVOP) and exhibits a contradiction. Then, there exists $x \in S$ such that
\[
f(x) - f(\bar{x}) \in -\text{int } Q.
\]
(19)
On the other hand, there exists $\lambda^* \in Q^* \setminus \{0\}$ such that
\[
\lambda^* \circ Df(\bar{x}) = 0.
\]
(20)
Since $f$ is pseudoinvex, it follows from (19)
\[
Df(\bar{x})\eta(x, \bar{x}) \in -\text{int } Q
\]
(21)
and, using Lemma 2.7,
\[
\lambda^* \circ Df(\bar{x})\eta(x, \bar{x}) < 0.
\]
(22)
which contradicts (20).

Now, we will prove the other implication. We assume that all vectorial critical points are weakly efficient solutions of (WVOP). We fix $\bar{x} \in S$ and we consider the systems:
\[
f(x) - f(\bar{x}) \in -\text{int } Q
\]
(23)
and

\[ Df(\bar{x})u \in -\text{int } Q. \]  

(24)

Next, we claim that the system (24) has a solution \( u \in E_1 \) when the system (23) has a solution \( x \in S \).

In fact, if the system (23) has a solution \( x \in S \), then \( \bar{x} \) is not a weakly efficient solution of (WVOP) and, by hypothesis, is not a vectorial critical point, i.e., there does not exist \( \lambda^* \in Q^* \setminus \{0\} \) such that \( \lambda^* \circ Df(\bar{x}) = 0 \).

Since \( \bar{x} \) is not a vectorial critical point, there does not exist \( \tau \in Q^* \setminus \{0\}, \lambda \in Q^* \) such that

\[
\begin{align*}
\tau M + \lambda A &= 0 \\
\langle \lambda, b \rangle &= 0 \\
\langle \tau, s \rangle &= 0,
\end{align*}
\]  

(25)

where:

\[
\begin{align*}
A &:= 0 \in \mathcal{L}(E_1, E_2) \\
M &:= Df(\bar{x}) \in \mathcal{L}(E_1, E_2) \\
b &:= 0 \in E_1 \\
s &:= 0 \in E_2.
\end{align*}
\]  

(26)

From Lemma 2.8, there exists \( u \in E_1 \) such that

\[
\begin{align*}
Au + b &= 0 \in -T \\
Mu + s &= Df(\bar{x})u \in -\text{int } Q
\end{align*}
\]  

(27)

and, in particular, the system (24) has a solution \( u \in E_1 \).

It is sufficient to put \( \eta(x, \bar{x}) := u \) and we obtain that \( f \) is pseudoinvex.  

From Theorem 2.4 and Proposition 2.9, we can relate the vectorial critical points, the weakly efficient solutions of (WVOP) and the solutions of (WVVLIP). More precisely, we have:

**Corollary 2.10.** Assume that \( S \) is an open subset and \( F \equiv Df \). If \( f \) is pseudoinvex, then the vectorial critical points, the weakly efficient points of (WVOP) and the solutions of (WVVLIP) are coincident.

The results obtained in this paper can be described in the following diagram:

<table>
<thead>
<tr>
<th>VVLIP</th>
<th>( f(I \times X), F \equiv Df )</th>
<th>VOP</th>
<th>( f(SI \times X), F \equiv Df )</th>
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<td>( f(PI \times X), F \equiv Df )</td>
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3. Conclusions

In Ruiz-Garzón et al. [6], it is proved that the solutions of the variational-like inequality problem (VLIP) in the scalar case are equivalent to the minimum of the mathematical programming problem in invex environments. In [7], it is proved that these results can be generalized to the vectorial problem between Euclidean spaces and in [9] these results are extended under non-smooth invexity. In this work, we have extended these results to the vectorial optimization problems in Banach spaces, when the domination structure is defined by convex cones. Under the condition of pseudoinvexity, we have seen the relationship between vector variational-like problems and vector optimization problems and managed to identify the weakly efficient points, the solutions of the weak vector variational-like inequality problems (WVVLIP) and the vector critical points.
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References