Aspects of the mean residual life order for weighted distributions

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Aspects of the mean residual life order for weighted distributions

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In this paper, we first provide conditions for preservation of the mean residual life (mrl) order under weighting. Then we apply the obtained results to establish our results about preservation of the decreasing mrl class by weighted distributions. In addition, we present some results for comparing the original random variable to its weighted version in terms of the mrl order. Also, some examples are given to illustrate the results.

Keywords: weighted distributions; mean residual life order; preservation; generalized Pareto distribution; DMRL

AMS Subject Classification: 60E15; 62N05

1. Introduction

Stochastic comparisons of random variables (rv’s) are useful in many branches of statistics such as reliability and survival analysis. One of the well-known stochastic orders is the mean residual life (mrl) order which has been studied by several authors, for example, Gupta and Kirmani [1] and Alzaid.[2] In replacement and repair strategies, although the shape of the hazard rate (hr) function plays an important role, the mrl function is found to be more relevant than the hr function because the former summarizes the entire residual life function, whereas the latter involves only the risk of instantaneous failure at some time $t$. Therefore, in some situations, the mrl may be more appropriate than the hr in order to compare the lifetime of two devices. The mrl order is also of interest to analyse maintenance polices and renewal process (cf. [3]). Weighted distributions were first introduced by Fisher [4] and then Rao [5] studied them in a unified way. Many researchers such as Blumenthal,[6] Patil and Rao,[7] Mahfoud and Patil [8] and Gupta and Kirmani [9] discussed special cases of the weighted distributions. A survey on applications of the weighted distributions is available in Patil and Rao.[10] When an observation is recorded by nature according to a certain stochastic model, the recorded observation will not have the original distribution unless every observation is given an equal chance of being recorded. In this case, observations are recorded according to some weight functions. For example, in the analysis of data relating to human populations and ecology, the weighted distributions are applicable (see for instance [5]). Recently, the study of stochastic orders in the context of the weighted distributions has received more attention in the literature (see e.g. [11,12] with related references therein). Bartoszewicz and

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Skolimowska [13] investigated the problem of preservation of some well-known stochastic orders and aging classes by using a particular representation of the weighted distributions. Later, Misra et al. [11] discussed similar results under some different methods. The aim of the present work is to obtain more results on this topic. In this paper, we first concentrate on aspects of the mrl order of weighed distributions which mostly deal with the preservation property of this stochastic order under weighting. Various preservation properties of the decreasing mean residual life (DMRL) class by weighted distributions are then given. The obtained results for the preservation of the mrl order under weighting are utilized to study preservation property of the DMRL class by weighting. On the other hand, since the mrl function of the generalized Pareto (GP) distribution has a simple linear form and furthermore since two GP distributions fit the mrl order based on their ordered parameters, thus it is a useful model for our study. Throughout the paper, the term \((Z \mid A)\) stands for any rv that its distribution corresponds to the conditional distribution of \(Z\) given \(A\). The indicator function of a set \(A\) is denoted by \(I_A(t)\). All ratios are assumed to be well defined and all expectations are assumed to exist whenever they appeared. We also use the terms ‘increasing’ and ‘decreasing’ in place of ‘non-decreasing’ and ‘non-increasing’, respectively.

2. Preliminaries

Let \(X\) be a non-negative rv on the real line \((\mathcal{R})\) with distribution function (df) \(F(\cdot)\), survival function (sf) \(\bar{F}(\cdot) = 1 - F(\cdot)\), probability density function (pdf) \(f(\cdot)\), whenever it exists, and the support \(S_X\). The left and right endpoints of \(S_X\) are

\[
l_X = \inf \{x \in \mathcal{R} : F(x) > 0\} \quad \text{and} \quad u_X = \sup \{x \in \mathcal{R} : F(x) < 1\},
\]

respectively, where \(u_X = +\infty\) is possible to occur. Define \(S_X = \{x \in \mathcal{R} : f(x) > 0\}\), whenever \(F(\cdot)\) is absolutely continuous. Let \(X\) have a finite mean and let \(l_X = 0\). Then define an rv with the df

\[
F^e(x) = \frac{\int_0^x \bar{F}(u) \, du}{\int_0^\infty \bar{F}(u) \, du}, \quad x \geq 0,
\]

which is called the equilibrium distribution of \(X\) and it is denoted by \(X^e\). From Equation (1), the support of \(X^e\) is \(S_{X^e} = (0, u_X)\). The rv \(X_t = (X - t \mid X > t)\), \(t < u_X\) is called residual life rv which has the sf \(\bar{F}_t(x) = \bar{F}(t + x)/\bar{F}(t), x \geq 0\). The support of \(X_t\) is \(S_{X_t} = (0, u_X - t)\). Some reliability measures of \(X\) are \(r(t) = f(t)/\bar{F}(t), t < u_X, \bar{r}(t) = f(t)/F(t), t > l_X\) and

\[
m(t) = E(X_t) = \begin{cases} \int_t^{\infty} \frac{\bar{F}(x)}{\bar{F}(t)} \, dx, & t < u_X \\ 0, & t \geq u_X. \end{cases}
\]

which are known as, respectively, hr, reversed hr and mrl of the rv \(X\).

Let us give some definitions that are needed in the paper.

**Definition 1** A non-negative measurable function \(K(x, y)\) is said to be totally positive of order 2 (denoted by \(TP_2(x, y)\)), in \(x \in \mathcal{R}\) and \(y \in \mathcal{R}\), if

\[
\begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) \\ K(x_2, y_1) & K(x_2, y_2) \end{vmatrix} \geq 0 \quad \text{for every} \ x_1 \leq x_2 \ \text{and} \ y_1 \leq y_2.
\]
**Definition 2** A real valued non-negative function $h(\cdot)$ is said to be log-concave if and only if the function $K(x,y) = h(y - x)$ is $TP_2(x,y)$.

The rv $X$ is said to have a GP distribution, if for $a > 0$ and $b > 0$, its sf is given by

$$\bar{F}(x) = \left(\frac{b}{ax + b}\right)^{1/a+1}, \quad x \geq 0;$$

and for $a \in (-1,0)$ and $b > 0$, its sf is given by

$$\bar{F}(x) = \begin{cases} \left(\frac{b}{ax + b}\right)^{1/a+1}, & x \in \left[0, \frac{-b}{a}\right); \\ 0, & x \geq \frac{-b}{a}. \end{cases}$$

We shall denote it by $X \sim GP(a,b)$. It is to be mentioned here that the support of $X$ depends on the sign of the parameter $a$. Formally, when $-1 < a < 0$, then $S_X = (0, -b/a)$ and when $a > 0$, we have $S_X = (0, \infty)$. It is to be noted here also that for $a > 0$ and $-1 < a < 0$ this model is called, respectively, a Pareto (Lomax) distribution and a Power distribution. Moreover, its mrl is $m(x) = ax + b$.

### 2.1. Weighted distributions

Let $w(\cdot)$ be a non-negative real function for which $0 < E(w(X)) < \infty$. Define an rv $X^w$ with the df of the form

$$F^w(x) = \frac{1}{E(w(X))} \int_{-\infty}^x w(u) \, dF(u).$$

$X^w$ is called the weighted rv corresponding to $X$. In the absolutely continuous case, the pdf of $X^w$ is given by

$$f^w(x) = \frac{w(x)f(x)}{E(w(X))},$$

where $w(\cdot)$ is called the weight function. Denote by $m^w(\cdot)$ and $r^w(\cdot)$, the mrl function and the hr function of $X^w$, respectively. The support of $X^w$ is assumed to be $S_X^w = \{x \in S_X : w(x) > 0\} = (l_X^w, u_X^w)$. Clearly, $l_X \leq l_X^w < u_X^w \leq u_X^w$. In this paper, the supports of all weighted rv’s are assumed to be open intervals contained in $R$.

### 2.2. Stochastic orders and aging notions

The following stochastic orders that can be found in Shaked and Shanthikumar,[3] are defined for non-negative rv $X_i$, $i = 1,2$ with finite means and with the sf $\bar{F}_i(\cdot)$, the hr $r_i(\cdot)$ and the mrl $m_i(\cdot)$. Assume that $S_i = (l_i, u_i)$ is the support of $X_i$, $i = 1,2$, so that $l_1 \leq l_2$, $u_1 \leq u_2$ and $S_1 \cap S_2 = (l_2,u_1) \neq \emptyset$. We also use the convention that $a/0 = +\infty$, whenever $a > 0$. Note that when $X_1$ and $X_2$ have absolutely continuous df’s we take $S_i = \{x \in R : f_i(x) > 0\}$, where $f_i$ is the pdf of $X_i$ for $i = 1,2$.

(a) $X_1$ is said to be smaller than $X_2$ in hr ordering (denoted by $X_1 \preceq_{hr} X_2$) if $\bar{F}_2(x)/\bar{F}_1(x)$ is increasing in $x \in S_1 \cap S_2$, or equivalently $r_1(x) \geq r_2(x)$, for all $x \in S_1 \cap S_2$. 
(b) $X_1$ is said to be smaller than $X_2$ in mrl ordering (denoted by $X_1 \preceq_{\text{mrl}} X_2$) if $\int_{x}^{\infty} F_2(y) \, dy / \int_{x}^{\infty} \tilde{F}_1(y) \, dy$ is increasing in $x \in S_1 = (l_1, u_1)$, or equivalently $m_1(x) \leq m_2(x)$, for all $x \in S_1 = (l_1, u_1)$.

We know that $X_1 \preceq_{\text{hr}} X_2$ implies $X_1 \preceq_{\text{mrl}} X_2$.

(c) An rv $X$ is said to have DMRL (denoted by $X \in \text{DMRL}$) if $m(x)$ is decreasing in $x$, or equivalently if $\int_{x}^{\infty} \tilde{F}(y) \, dy$ is log-concave on $S_X$.

3. Comparing the original to its weighted version

Here, the mrl order is made between the original distribution and its weighted distribution when appropriate assumptions are assumed. First recall the following proposition which indicates that some reliability measures of $X^w$ are connected with $A(x) = E(w(X) \mid X > x)$.

**Proposition 1** (Jain et al. [14]) Let $A(x) = E(w(X) \mid X > x)$. Then,

$$1 - F^w(x) = \frac{A(x)\tilde{F}(x)}{E(w(X))}, \quad r^w(x) = \frac{w(x)r(x)}{A(x)} \quad \text{and} \quad m^w(x) = \int_{x}^{\infty} \frac{A(u)\tilde{F}(u)}{A(x)\tilde{F}(x)} \, du.$$ 

The next result presents an equivalent condition for comparing $X$ to $X^w$ based on the mrl order.

**Theorem 1** Suppose that the supports of $X^w$ and $X$ are the same, i.e., $S_X = S^*_X = (l_X^*, u_X^*)$. Then, $X \preceq_{\text{mrl}} (\geq_{\text{mrl}})X^w$ holds, if and only if, $\text{Cov}(X, w(X) \mid X > x) \geq (\leq)0$, for all $x \in S^*_X$.

**Proof** $X \preceq_{\text{mrl}} (\geq_{\text{mrl}})X^w$ means that, for all $x \in S^*_X$, $E(X \mid X > x) \leq (\geq)E(X^w \mid X^w > x)$. Now, suppose that $(X \mid X > x)$ and $(X^w \mid X^w > x)$ have respective pdf’s $f_{X \mid X > x}(\cdot)$ and $f_{X^w \mid X^w > x}(\cdot)$, which are given by

$$f_{X \mid X > x}(u) = \frac{f(u)}{F(x)}I_{(x,a_X)}(u) \quad \text{and} \quad f_{X^w \mid X^w > x}(u) = \frac{w(u)f(u)}{F(x)E(w(X) \mid X > x)}I_{(x,a^*_X)}(u).$$

Write,

$$E(X^w \mid X^w > x) = \int_{-\infty}^{+\infty} uf_{X^w \mid X^w > x}(u) \, du$$

$$= \int_{x}^{u_X^*} \frac{uw(u)f(u)}{F(x)E(w(X) \mid X > x)} \, du$$

$$= \int_{-\infty}^{+\infty} \frac{uw(u)f_{X \mid X > x}(u)}{E(w(X) \mid X > x)} \, du = \frac{E(Xw(X) \mid X > x)}{E(w(X) \mid X > x)}.$$ 

Therefore,

$$X \preceq_{\text{mrl}} (\geq_{\text{mrl}})X^w \quad \iff \quad E(X \mid X > x) \leq (\geq) \frac{E(Xw(X) \mid X > x)}{E(w(X) \mid X > x)},$$

$$\iff \quad \text{Cov}(X, w(X) \mid X > x) \geq (\leq)0.$$ 

**Remark 1** Assume that the df of $X$ is absolutely continuous. Note that $X_i$ is the weighted version of $X$ with the weight function $w(x) = f(t + x)f(x)$ (see Jain et al. [14]). Since $X$ is DMRL, if
and only if, $X \geq_{mrl} X_t$, for all $t > 0$ (see e.g. [3]) thus using Theorem 1, $X$ is DMRL, if and only if,
\[
\text{Cov} \left( X, \frac{f(t + X) - f(X)}{f(X)} \mid X > s \right) \leq 0, \quad \text{for all } t, s > 0.
\]

Take $g(t, s) = -\text{Cov}(X, (f(t + X)/f(X)) \mid X > s)$, $t, s > 0$. After some calculations, we arrive at
\[
g(t, s) = \frac{\tilde{F}(t + s)}{F(s)} (m(s) - m(t + s)).
\]

Now assume that $T$ and $S$ are i.i.d copies from $F$. Set $\triangle(F) = E(g(T, S))$. Then $\triangle(F)$ could be a potential index to test the hypothesis test $H_0: X$ is exponential versus $H_1: X$ is DMRL and is not exponential. We refer the reader to Hollander and Proschan [15] for some similar kind of the proposed deviation measure.

4. Preservation of the mrl order

We devote this section to the problem of preservation of the mrl order under weighting. Let $X_i^{wi}$ be the weighted version of $X_i$ with the corresponding weight function $w_i(\cdot)$, $i = 1, 2$. Write $F_i^{wi}(\cdot)$ and $F_i^{w2}(\cdot)$ for the df's of the rv's $X_i^{wi}$ and $X_i^{w2}$, respectively. Also, assume that
\[
S_i^* = \{ x \in S_i: w_i(x) > 0 \} = (l_i^*, u_i^*),
\]

is the support of $X_i^{wi}$, $i = 1, 2$, where $S_i = (l_i, u_i)$ is the support of $X_i$, $i = 1, 2$, with $l_1 \leq l_2$ and $u_1 \leq u_2$. We further assume that $l_1^* \leq l_2^*$ and $u_1^* \leq u_2^*$, such that $S_i^* \cap S_j^* = (l_i^*, u_i^*) \neq \emptyset$. Obviously, $l_i \leq l_i^*$ and $u_i \leq u_i^*$, since $S_i^* \subseteq S_i$, $i = 1, 2$. Take $A_i(x) = E(w_i(X_i) \mid X_i > x)$, $i = 1, 2$. From Proposition 1, we recall that
\[
1 - F_i^{wi}(x) = \frac{A_1(x)\tilde{F}_1(x)}{E(w_1(X_1))} \quad \text{and} \quad 1 - F_2^{w2}(x) = \frac{A_2(x)\tilde{F}_2(x)}{E(w_2(X_2))},
\]

are the sf of $X_i^{wi}$ and $X_i^{w2}$, respectively. Now, by definition, we get
\[
X_i^{wi} \leq_{mrl} X_i^{w2} \iff \frac{\int_x^\infty (1 - F_2^{w2}(u)) \, du}{\int_x^\infty (1 - F_1^{wi}(u)) \, du} \text{ is increasing in } x \in S_1^*,
\]
\[
\iff \frac{\int_x^\infty A_2(u)\tilde{F}_2(u) \, du}{\int_x^\infty A_1(u)\tilde{F}_1(u) \, du} \text{ is increasing in } x \in S_1^*.
\]

Now, we are ready to prove the following theorem.

THEOREM 2 Let $l_1 = l_2 = 0$ and $u_1 \leq u_2$. If $A_2(x)$ is increasing in $x \in (0, u_2)$ and $A_2(x)/A_1(x)$ is increasing in $x \in (0, u_1)$, then, $X_1 \leq_{mrl} X_2 \implies X_1^{wi} \leq_{mrl} X_2^{w2}$.

Proof Let $X_i^\nu$ be the equilibrium version of $X_i$ with $S_i^\nu = (0, u_i)$, $i = 1, 2$. Using Theorem 22 of Hu et al.,[16] $X_1 \leq_{mrl} X_2$ implies $X_1^\nu \leq_{hr} X_2^\nu$. By taking $(X_i^\nu)^\lambda_i$ as the weighted version of $X_i^\nu$ with
The last equality followed since the weight function \( S_i(x) = (l_i(x), u_i(x)) \), from Equation (3) we get

\[
S_i(x) = \{x \in \mathbb{X}_i : A_i(x) > 0\} = \{x \in (0, u_i) : \frac{1 - F_i^w(x)}{1 - F_i(x)} E(w_i(X_i)) > 0\} = \{x \in (0, u_i) : F_i^w(x) < 1\} = (0, u_i) \cap (0, u_i^*), \quad i = 1, 2.
\]

(5)

The last equality followed since \( u_i^* \leq u_i, \ i = 1, 2 \). From assumptions, \( A_2(x) \) is increasing in \( x \in S_{X_1} \cup S_{X_2} = (0, u_2) \) and \( A_2(x)/A_1(x) \) is increasing in \( x \in S_1 \cap S_2 = (0, u_1) \). Thus, Theorem 3.2. (b) of Misra et al. [11] is applicable and tells that \( X_1 \leq_{hr} X_2 \) implies \( (X_1)_{A_1} \leq_{hr} (X_2)_{A_2} \). Write \( g_i(\cdot) \) and \( G_i(\cdot) \) for the respective pdf and sf of \( (X_1)_{A_1}, i = 1, 2 \). Then,

\[
g_i(x) = \frac{A_i(x)F_i(x)}{E(A_i(X_1))E(X_i)}, \quad i = 1, 2,
\]

from which we get

\[
G_i(x) = \int_x^\infty A_i(u)F_i(u) E(A_i(X_1))E(X_i) \ du, \quad i = 1, 2.
\]

Finally, from \( (X_1)_{A_1} \leq_{hr} (X_2)_{A_2} \) the ratio

\[
\frac{G_2(x)}{G_1(x)} = \frac{\int_x^\infty A_2(u) F_2(u) E(X_1) E(A_1(X_1))}{\int_x^\infty A_1(u) F_1(u) E(X_2) E(A_2(X_2))},
\]

increases in \( x \). Hence, by (4), \( X_1^{w_1} \leq_{mrl} X_2^{w_2} \) holds. 

By Proposition 2.4 (ii) of Nanda and Jain, [17] the following is concluded.

**Corollary 1** Let \( l_1 = l_2 = 0 \). If \( A_1(x) \) or \( w_1(x) \) is decreasing and \( A_2(x) \) or \( w_2(x) \) is increasing, then, \( X_1 \leq_{mrl} X_2 \implies X_1^{w_1} \leq_{mrl} X_2^{w_2} \).

To present some examples, we need the following lemma. It states that the the mrl order is made between two rv’s with GP df’s.

**Lemma 1** Let \( X_1 \sim \text{GP}(a_1, b_1) \) and \( X_2 \sim \text{GP}(a_2, b_2) \). Then \( X_1 \leq_{mrl} X_2 \), if and only if, \( a_1 \leq a_2, \ b_1 \leq b_2 \).

The proof can be easily obtained from the definition of the mrl order and hence omitted. Let us see some examples for Theorem 2.

**Example 1** Let \( X_1 \sim \text{GP}(\frac{1}{3}, 1) \) and \( X_2 \sim \text{GP}(\frac{1}{7}, 1) \). Based on Lemma 1, \( X_1 \leq_{mrl} X_2 \) holds. Note that here, \( X_1 \not\leq_{hr} X_2 \). By taking \( w_1(x) = (3 + x)^2 \) and \( w_2(x) = (2 + x)^2 \), we have

\[
A_2(x) = 3(2 + x)^2, \quad \text{and} \quad \frac{A_2(x)}{A_1(x)} = \frac{3}{2} \left( \frac{2 + x}{3 + x} \right)^2,
\]

which are increasing in \( x \). By Theorem 2 then \( X_1^{w_1} \leq_{mrl} X_2^{w_2} \).
Example 2 Let $X_1 \sim \text{Exp}(1)$ and $X_2 \sim \text{Gp}(1, 1)$. It can be verified that $X_1 \leq_{mrl} X_2$ (but $X_1 \not\leq_{hr} X_2$). Take $w_1(x) = (1 + 2x)e^{-x}$ and $w_2(x) = 1 + x$. It can be seen that $A_1(x) = (1 + x)e^{-x}$ which is decreasing. In parallel, $w_2(x)$ is increasing. Then Corollary 1 gives $X_1^{w_1} \leq_{mrl} X_2^{w_2}$.

In the following, a dual result for Proposition 3.3 of Belzunce et al. [18] is given (see also, Theorem 3.2 of Ortega [19]).

Assume now that $F_1(\cdot)$ and $F_2(\cdot)$ are absolutely continuous.

Lemma 2 Let $l_1 = l_2 = 0$ and $u_1 \leq u_2 < 1$. Then,

$$X_1 \leq_{mrl} X_2 \iff \frac{1}{s}(1 - X_1) \leq_{hr} \frac{1}{s}(1 - X_2), \quad \text{for all } s > 0.$$ 

Applying Lemma 2, another result for preservation of the mrl order under weighting is derived under the condition that both $X_1$ and $X_2$ have bounded supports.

Theorem 3 Let $l_1 = l_2 = 0$, and $u_1 \leq u_2 < \infty$. If $w_2(x)$ is increasing on $(0, u_2)$ and $w_2(x)/w_1(x)$ is increasing on $(0, u_1)$, then $X_1 \leq_{mrl} X_2 \implies X_1^{w_1} \leq_{mrl} X_2^{w_2}$.

Proof Suppose $a = u_1 + u_2$. Then, $X_1 \leq_{mrl} X_2$ implies $0 \leq X_1/a \leq_{mrl} X_2/a < 1$. For fixed $s > 0$, set $V_s(x) = -(1/s) \ln(1 - x/a)$, $0 \leq x < a$, which is left continuous and strictly increasing in $x$. Taking $Z_1$ as the rv $V_s(X_i)$, $i = 1, 2$, Lemma 2 tells that $Z_1 \leq_{hr} Z_2$, where $Z_i$ has the support $S_{Z_i} = (0, -(1/s) \ln(1 - u_i/a))$, $i = 1, 2$. Let

$$\phi_1(x) = w_1(V_s^{-1}(x)) = w_1(a(1 - e^{-sx})), \quad \text{and} \quad \phi_2(x) = w_2(V_s^{-1}(x)) = w_2(a(1 - e^{-sx})),
$$

be two weights and $Z_1^{\phi_1}$ and $Z_2^{\phi_2}$ as weighted versions of $Z_1$ and $Z_2$, with corresponding weights $\phi_1(\cdot)$ and $\phi_2(\cdot)$, respectively. Let $S_{Z_i}^{\phi_i} = (l_{Z_i}^{\phi_i}, u_{Z_i}^{\phi_i})$ be the support of $Z_i^{\phi_i}, i = 1, 2$. By Equation (2), we obtain

$$S_{Z_i}^{\phi_i} = \{x \in S_{Z_i} : \phi_i(x) > 0\}
= \left\{x \in \left(0, -\frac{1}{s} \ln \left(1 - \frac{u_i}{a}\right) \right) : w_i(a(1 - e^{-sx})) > 0\right\}
= \left(0, -\frac{1}{s} \ln \left(1 - \frac{u_i}{a}\right) \right) \cap \left(-\frac{1}{s} \ln \left(1 - \frac{l_i}{a}\right), -\frac{1}{s} \ln \left(1 - \frac{u_i}{a}\right)\right)
= \left(-\frac{1}{s} \ln \left(1 - \frac{l_i}{a}\right), -\frac{1}{s} \ln \left(1 - \frac{u_i}{a}\right)\right), \quad i = 1, 2,
$$

where $u_i \leq u_i, \ i = 1, 2$, yielding the last equality. By elementary assumptions $l_1 \leq l_2$ and $u_1 \leq u_2$, we have $l_{Z_1}^{\phi_1} \leq l_{Z_2}^{\phi_2}$ and $u_{Z_1}^{\phi_1} \leq u_{Z_2}^{\phi_2}$. By assumptions, $\phi_2(x)$ is increasing on $S_{Z_2} = (0, -(1/s) \ln(1 - u_2/a))$ and $\phi_2(x)/\phi_1(x)$ is increasing in $x$ on $S_{Z_1} \cap S_{Z_2} = (0, -(1/s) \ln(1 - (u_1/a)))$. Now by Theorem 3.2, (b) of Misra et al.,[11] $Z_1 \leq_{hr} Z_2$ implies $Z_1^{\phi_1} \leq_{hr} Z_2^{\phi_2}$ and further from Theorem 1 of Bartoszewicz,[12] we have $Z_1^{\phi_1} = (V_s(X_1))^{\phi_1} \overset{sf}{=} V_s(X_1^{w_1})$ and also $Z_2^{\phi_2} = (V_s(X_2))^{\phi_1} \overset{sf}{=} V_s(X_2^{w_2})$, where $\overset{sf}{=} \text{means equality in law}$. Thus, it follows that

$$-\frac{1}{s} \ln \left(1 - \frac{X_1^{w_1}}{a}\right) \overset{sf}{\leq_{hr}} -\frac{1}{s} \ln \left(1 - \frac{X_2^{w_2}}{a}\right), \quad \text{for all } s > 0,$$

which by Lemma 2, it gives $X_1^{w_1}/a \leq_{mrl} X_2^{w_2}/a$ or equivalently $X_1^{w_1} \leq_{mrl} X_2^{w_2}$.
In this theorem, by taking \( w_1(x) = w_2(x) = w(x) \), we conclude the following.

**Corollary 2** If \( w(x) \) is increasing, then \( X_1 \leq_{\text{mrl}} X_2 \implies X_1^w \leq_{\text{mrl}} X_2^w \).

As mentioned before, for the GP distribution if we take \( -1 < a < 0 \), then it becomes a power distribution with the support \((0, -b/a)\). We, therefore, have a distribution with bounded support.

**Example 3** Let \( X_1 \sim \text{Gp}(-\frac{1}{2}, 1) \) and \( X_2 \sim \text{Gp}(-\frac{1}{3}, 2) \). Based upon Lemma 1, we have \( X_1 \leq_{\text{mrl}} X_2 \) (but \( X_1 \not\leq_{\text{hr}} X_2 \)). If we take \( w_1(x) = x^3 \) and \( w_2(x) = x^4 \), then the assumptions of Theorem 3 are satisfied and so, \( X_1^{w_1} \leq_{\text{mrl}} X_2^{w_2} \). Note that using Theorem 2 this result is not concluded, since \( A_2(x)/A_1(x) \) is not increasing.

The next result is a direct conclusion of Theorem 1 which is stated without proof.

**Theorem 4** Suppose \( S_i = S_i^* \), \( i = 1, 2 \), and \( X_1 \leq_{\text{mrl}} X_2 \). Let for all \( x \in S_i^* \), \( \text{Cov}(X_i, w_1(X_i) \mid X_1 > x) \leq 0 \), and for all \( x \in S_2^* \), \( \text{Cov}(X_2, w_2(X_2) \mid X_2 > x) \geq 0 \). Then, \( X_1^{w_1} \leq_{\text{mrl}} X_2^{w_2} \).

### 5. Preservation of DMRL class under weighting

In this section, we try to find appropriate conditions under which the DMRL class is preserved by weighted distributions. Some examples are also given to illustrate the concepts. Motivated by the following characterization of the DMRL class together with applying the results of Theorems 2 and 3, we establish our findings here.

**Lemma 3 (Shaked and Shanthikumar [3])** The rv \( X \) belongs to DMRL class, if and only if, \( X_t \leq_{\text{mrl}} X_{t'}, \) for all \( t < t' \leq t_1 < u_X \).

Below is a useful lemma to prove the main results of this section.

**Lemma 4** If \( w_1(x) = w(t + x) \), then

\[
(X_t)^{w_1} = (X^w - t \mid X^w > t), \quad t \in (0, u_X^w),
\]

where \( (X_t)^{w_1} \) is the weighted version of \( X_t \) with corresponding weight \( w_1(\cdot) \).

**Theorem 5** Let \( X \in \text{DMRL} \). If \( A(x) = E(w(X) \mid X > x) \) is increasing and log-concave on \( S_X \), then \( X^w \in \text{DMRL} \).

**Proof** For \( t_{iX}^w < t_2 \leq t_1 < u_{iX}^w \), by Lemma 3, \( X_{t_1} \leq_{\text{mrl}} X_{t_2} \). Assume \( w_j(x) = w(t_j + x), \ i = 1, 2 \). Denote by \( (X_{t_i})^{w_i} \) the weighted version of \( X_{t_i} \) with weight \( w_i(x), \ i = 1, 2 \). Note that \( S_{X_{t_i}} = (l_{X_{t_i}}, u_{X_{t_i}}) = (0, u_X - t_i) \), for which clearly, \( l_{X_{t_i}} = l_{X_{t_2}} = 0 \) and \( u_{X_{t_i}} \leq u_{X_{t_2}} \), and also \( S_{(X_{t_i})^{w_i}} = (l_{(X_{t_i})^{w_i}}, u_{(X_{t_i})^{w_i}}) = (0, u_{X_{t_i}} - t_i) \), with \( l_{(X_{t_i})^{w_i}} \leq l_{(X_{t_2})^{w_i}} \) and \( u_{(X_{t_i})^{w_i}} \leq u_{(X_{t_2})^{w_i}} \). By setting \( A_i^w(x) = E(w_i(X_{t_i}) \mid X_{t_i} > x), i = 1, 2 \), we derive

\[
A_i^w(x) = E(w(t_i + X_{t_i}) \mid X_{t_i} > x).
\]

From assumptions, \( A(x) \) is increasing on \( S_X = (l_X, u_X) \), thus readily \( A_2^w(x) \) is increasing on \( (l_X - t_2, u_X - t_2) \), which from \( l_X < t_2 \), it follows that \( A_2^w(x) \) is also increasing on \( S_{X_{t_2}} = (0, u_X - t_2) \).
Moreover, since \( A(x) \) is log-concave, from Definition 2, \( A(y^* - x^*) \) should be \( TP_2(x^*, y^*) \). Thus, for \( x_1 \leq x_2 \) if
\[
y_1^* = t_1 + t_2 + x_1 \leq t_1 + t_2 + x_2 = y_2^*, \quad x_1^* = t_2 \leq t_1 = x_2^*,
\]
then
\[
\begin{vmatrix}
A_1^*(x_1) & A_1^*(x_2) \\
A_2^*(x_1) & A_2^*(x_2)
\end{vmatrix}
\begin{vmatrix}
A(t_1 + x_1) & A(t_1 + x_2) \\
A(t_2 + x_1) & A(t_2 + x_2)
\end{vmatrix}
= \begin{vmatrix}
A(y_1^* - x_1^*) & A(y_2^* - x_1^*) \\
A(y_1^* - x_2^*) & A(y_2^* - x_2^*)
\end{vmatrix} \geq 0.
\]
So, from definition \( A_i^*(x) \) is \( TP_2(i, x) \). Now, on using Theorem 2, \( X_{t_1} \leq_{mrl} X_{t_2} \) implies that \( (X_{t_1})^{w_1} \leq_{mrl} (X_{t_2})^{w_2} \). Also based on Lemma 4, \( (X_{t_1})^{w_1} \overset{st}{\equiv} (X_{t_2} - t_1 \mid X_{t_2} > t_1) \). Consequently, it follows that \( (X_{t_2} - t_1 \mid X_{t_2} > t_1) \leq_{mrl} (X_{t_2} - t_2 \mid X_{t_2} > t_2) \), for every \( l_X \leq t_2 < t_1 < u_X^* \leq u_X \). The desired result now immediately follows from Lemma 3.

**THEOREM 6** Let \( uv_X < \infty \) and \( X \in DMRL \). If \( w(x) \) is increasing and log-concave on \( S_X \), then \( X^w \in DMRL \).

The proof is similar to that of Theorem 5 when Theorem 3 is applied in place of Theorem 2. The next example is stated in connection with Theorem 6.

**Example 4** Let \( X \) follow \( GP(-\frac{1}{2}, 1) \). Then \( u_X = 2 \). The sf is
\[
\bar{F}(x) = \begin{cases} 
1, & x \leq 0 \\
1 - \frac{x}{2}, & 0 \leq x \leq 2 \\
1, & x \geq 2 
\end{cases}
\]
and the mrl is \( m(x) = -\frac{1}{2}x + 1 \). So \( X \) is DMRL. Take \( w(x) = x^2 \), which is clearly increasing and log-concave. Theorem 6 now yields \( X^w \) is also DMRL. It is worthwhile to mention here that in the case of this example, the rv \( X^w \) is said to have a size biased distribution of order 2, which is of interest in some statistical applications (cf. [11]).

**LEMMA 5** (Bassan et al. [20]) Let \( S_X = (0, \infty) \) and let \( X \) have a finite mean. Then, \( X \in DMRL \), if and only if, \( X^c \leq_{hr} X \).

We now give a technical lemma. Its proof is simple and is, therefore, omitted.

**LEMMA 6** Let \( S_X = S_X^c = (0, \infty) \) and let \( X \) and \( X^w \) have finite means. If \( T_1 \equiv (X^c)^{w_1} \) is the weighted version of \( X^c \) with the weight \( w_1(t) = A(t) \) and if \( T_2 \equiv (X^w)^c \) is the equilibrium version of \( X^w \), then \( T_1 \overset{st}{\equiv} T_2 \).

On applying Theorem 3.2. (b) of Misra et al.[11] Lemmas 5 and 6, we establish the following theorem.

**THEOREM 7** Let \( S_X = S_X^c = (0, \infty) \) and let the rv’s \( X \) and \( X^w \) have finite means. If \( w(x) \) and \( w(x)/A(x) \) are increasing on \( (0, \infty) \), then \( X \in DMRL \) implies \( X^w \in DMRL \).
Proof Since $X \in \text{DMRL}$, from Lemma 5, $X^w \leq_{hr} X$. Consider the notations given in Lemma 6. Taking $w_1(x) = A(x) = E(w(X) \mid X > x)$ and $w_2(x) = w(x)$, the assumptions of Theorem 3.2. (b) of Misra et al. [11] are satisfied. Thus, it follows that $(X^w)^{w_1} \leq_{hr} X^{w_2}$, which by Lemma 6 it provides that $(X^w)^w \leq_{hr} X^w$. Therefore, using Lemma 5 we conclude that $X^w \in \text{DMRL}$. \[\blacksquare\]

The next example provides a situation where Theorem 7 is applicable.

The weight function $w(x) = x^t \int F(x) \delta(x) \, dx$ was introduced by Greenwood et al. [21] as a probability weighted moment via some potential applications. Also, Oluyede [22] presented some stochastic comparisons using it. As a special case, when $l = 0$, $j = 1$ and $c = k - 1$, this weight is applied in the following example.

Example 5 Suppose that $S_X = (0, \infty)$ and $F^{-1}$ is the right continuous inverse of $F$. Let $X \in \text{DMRL}$ and take $w(x) = F(x)(\bar{F}(x))^{k-1}$, $0 < k \leq 1$. Assume further that

$$E(X) = \int_0^1 F^{-1}(y) \, dy < \infty, \quad \text{and} \quad E(X^w) = (k+1) \int_0^1 yF^{-1}(y)(1-y)^{k-1} \, dy < \infty.$$ 

Easy calculations yield

$$A(x) = \frac{F(x)\bar{F}^{k-1}(x)}{k} + \frac{\bar{F}^k(x)}{k(k+1)}$$

and thus,

$$w(x) = \frac{1}{A(x)} \left( \frac{1}{k} + \frac{\bar{F}(x)}{k(k+1)F(x)} \right)^{-1}.$$ 

Clearly, $w(x)$ and $w(x)/A(x)$ are increasing and according to Theorem 7, $X^w \in \text{DMRL}$.

It is worthwhile to mention here that Example 5 can be applied in a case with the order statistics. Formally, let $X_1$ and $X_2$ be i.i.d from $F$, and consider the weight function as in Example 5 when $k = 1$. Then, obviously, $X^w$ is equal in distribution with $X_{2:2} = \max\{X_1, X_2\}$. Hence, the DMRL property passes from the parent distribution to the distribution of the largest order statistics when we have a random sample of size $n = 2$ from $F$.

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