A note on superlinear Hamiltonian elliptic systems

Fukun Zhao,1,a) Leiga Zhao,2 and Yanheng Ding3
1Department of Mathematics, Yunnan Normal University, Kunming 650092 Yunnan, People’s Republic of China
2Department of Mathematics, Beijing University of Chemical Technology, Beijing 100029, People’s Republic of China
3Institute of Mathematics, AMSS, CAS, Beijing 100080, People’s Republic of China

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This paper is concerned with the superlinear periodic elliptic systems of Hamiltonian type in the whole space. The existence of a ground state solution as well as an infinite number of geometrically distinct solutions is obtained. © 2009 American Institute of Physics. [doi:10.1063/1.3256120]

I. INTRODUCTION AND MAIN RESULTS

In this paper, we study nonlinear elliptic systems of the Hamiltonian form

\[
\begin{cases}
-\Delta u + V(x)u = g(x,u) & \text{in } \mathbb{R}^N \\
-\Delta v + V(x)v = f(x,u) & \text{in } \mathbb{R}^N \\
u(x) \to 0 \quad \text{and} \quad v(x) \to 0 \text{ as } |x| \to \infty,
\end{cases}
\]

where \( N \geq 3 \), \( V \in C(\mathbb{R}^N, \mathbb{R}) \), and \( f, g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \).

For the case of a bounded domain \( \Omega \subset \mathbb{R}^N \), the systems like or similar to (ES) were studied by a number of authors, for instance, Benci and Rabinowitz,8 Clement, de Figueiredo and Mitidieri,9 de Figueiredo and Ding,12 De Figueiredo and Felmer,13 De Figueiredo, do ª, and Ruf,14 Hulshof and Vandervorst,17 and Kryszewski and Szulkin.18 If \( f(t) = |t|^{p-1}t \), \( g(t) = |t|^{q-1}t \), \( N \geq 3 \), and \( \Omega \) is a star shaped domain, by a Pohozaev type identity (see Refs. 23 and 31), one sees that classical positive solutions can exist only if \( p, q \geq 1 \) and

\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}.
\]  

Condition (1.1) means that \( f \) and \( g \) have different growth rates at \( t \) at infinity. An elementary approach was introduced to deal with this case in Refs. 13 and 17. The main idea is to destroy the symmetry between \( u \) and \( v \), i.e., demand different regularity of \( u \) and \( v \). So they used suitable fractional Sobolev spaces to replace \( H^1_0(\Omega) \times H^1_0(\Omega) \).

The problem (ES) in the whole space \( \mathbb{R}^N \) was considered in some works. Most of them focused on the case \( V \equiv 1 \). In such a case, one can work on the radially symmetric function spaces, which possess compact embedding (see, for example, Refs. 15, 27, 5, and 20). To avoid the indefinite character of the original functional, some authors used a dual variational method (see Refs. 3, 4, and 24 and references therein for singularly perturbed problems).

In this paper, we consider superlinear elliptic systems with periodic potential \( V(x) \). As far as we know, there is currently no multiplicity result for such systems. Our purpose is to obtain an existence and, under stronger assumptions, a multiplicity result. The main difficulties originate in at least three facts. First, there is a lack of compactness due to the fact that we are working in \( \mathbb{R}^N \). Second, due to the type of growth of \( f \) and \( g \), we need to work with nonradially symmetric
fractional Sobolev spaces. Third, the associated variational functional is strongly indefinite and its gradient is not of the form “identity minus compact.” Therefore, the generalized mountain pass theorem (see Ref. 8) does not apply. However, one can follow the methods in Refs. 10, 11, and 26 to analyze (PS)-sequence. Recall that a sequence \( \{z_n\} \) is called a (PS)-sequence of a \( C^1 \)-functional \( \Phi \) defined on a Banach space \( E \), if \( \Phi(z_n) \) is bounded and \( \Phi'(z_n) \to 0 \). This suffices to obtain the multiple results via some techniques which were developed recently by Bartsch and Ding.\(^7,6\)

For the potential \( V \), we assume (V) \( V \in C(\mathbb{R}^N, \mathbb{R}) \) is 1-periodic in each \( x_i \) for \( i=1,\ldots,N \) and \( V_0 := \min V > 0 \).

Let \( S := -\Delta + V \) be the Schrödinger operator. Denote by \( \sigma(S) \) and \( \sigma_{\text{cont}}(S) \) the spectrum and the continuous spectrum of the operator \( S \). It is well known that, under (V), the spectrum \( \sigma(S) = \sigma_{\text{cont}}(S) \subseteq [V_0, \infty) \) is a union of closed intervals (see Ref. 25). Next, we denote by \( F(x,t) \) and \( G(x,t) \) the primitives of \( f(x,t) \) and \( g(x,t) \), respectively. Our assumptions for \( f \) and \( g \) are standard, roughly speaking “superlinear” at zero and infinity and “subcritical” at infinity. More precisely, they are the following.

(i) \( (H_0) \) \( f, g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \) are 1-periodic in each \( x_i \) for \( i=1,\ldots,N \).

(ii) \( (H_1) \) \( f(x,t) = o(|t|) \) and \( g(x,t) = o(|t|) \) as \( |t| \to 0 \) uniformly in \( x \).

(iii) \( (H_2) \) there is a constant \( C > 0 \) such that

\[
|f(x,t)| \leq C(1 + |t|^p) \quad \text{and} \quad |g(x,t)| \leq C(1 + |t|^q)
\]

for all \((x,t)\), where \( p, q > 1 \) satisfy (1.1).

(iv) \( (H_3) \) there exists constants \( \alpha, \beta > 2 \) with \( p < \alpha \leq p+1 \) and \( q < \beta \leq q+1 \) such that

\[
0 < \alpha F(x,t) \leq tf(x,t) \quad \text{and} \quad 0 < \beta G(x,t) \leq tg(x,t)
\]

for all \((x,t) \) with \( t \neq 0 \).

Observe that due to the periodicity of \( V, f, \) and \( g \), if \( (u,v) \) is a solution of (ES), then so is \( (a \ast u, a \ast v) \) for each \( a \in \mathbb{Z}^N \), where \( (a \ast u)(x) = u(x+a) \). Two solutions \( (u_1, v_1) \) and \( (u_2, v_2) \) are said to be geometrically distinct if

\[
\text{either } a \ast u_1 \neq u_2 \quad \text{or} \quad a \ast v_1 \neq v_2, \quad \forall a \in \mathbb{Z}^N.
\]

Our main result is as follows.

**Theorem 1.1:** Let (V) and \( (H_0)-(H_3) \) be satisfied. Then (ES) has a positive ground state solution. If additionally \( f(x,t) \) and \( g(x,t) \) are odd in \( t \), then (ES) has an infinite number of geometrically distinct solutions.

This paper is organized as follows. In Sec. II, we recall the functional setting developed in Ref. 13. In Sec. III, we discuss some properties of \( \Phi \) with the help of techniques developed in Ref. 1. The proof of Theorem 1.1 announced above will be given in Sec. IV.

**II. VARIATIONAL SETTING**

Below, by \( | \cdot |_p \) we denote the usual \( L^p \)-norm, \( c \) or \( c_i \) stands for different positive constants. Let \( X \) and \( Y \) be two Hilbert spaces with inner products \( \langle \cdot, \cdot \rangle_X \) and \( \langle \cdot, \cdot \rangle_Y \). We choose the inner product \( \langle (x,y), (w,z) \rangle_{X \times Y} = (x,w)_X + (y,z)_Y \) on the product space \( X \times Y \).

Under (V), \( S \geq V_0 \). Let \( \{F_\lambda\}_{\lambda \in \mathbb{R}} \) be the spectral family of the operator \( S \). Since \( S \) is a positive operator, it has a square root

\[
S^{1/2} = \int_{V_0}^\infty \lambda^{1/2} dF(\lambda); D(S^{1/2}) \to L^2(\mathbb{R}^N, \mathbb{R})
\]

with

\[
D(S^{1/2}) = \left\{ u \in L^2(\mathbb{R}^N, \mathbb{R}); \int_{V_0}^\infty \lambda^{1/2} d(F(\lambda)u, u)_{L^2} < \infty \right\}.
\]
We know that $S^{1/2}$ is a positive self-adjoint operator. For each positive real $s$, define

$$S^{s/2} = \int_{V_0}^{\infty} \lambda^{s/2} dF(\lambda).$$

Set $A := S^{1/2}$, $A^t = S^{t/2}$, and define the space $E^t$ as

$$E^t := \mathcal{D}(A^t) = \left\{ u \in L^2(\mathbb{R}^N, \mathbb{R}): \int_{V_0}^{\infty} \lambda^{t/2} d(F(\lambda)u,u)_{L^2} < \infty \right\}.$$

Each $E^t$ is a Hilbert space endowed with inner product

$$\langle u, v \rangle_{E^t} := \langle A^tu, A^tv \rangle_{L^2}.$$

In fact, by the complex interpolation theory (see Refs. 21 and 28), we have for all $0 \leq s \leq 2$,

$$E^s = [H^2(\mathbb{R}^N), L^2(\mathbb{R}^N)]_{s/2} = H^0(\mathbb{R}^N).$$

It is clear that

$$|A^tu|_2 \geq V_0^{s/2}|u|_2. \quad (2.1)$$

In view of (2.1), $A^s: E^s \rightarrow L^2(\mathbb{R}^N, \mathbb{R})$ is an isomorphism. We denote by $A^{-s}$ the inverse of $A^s$.

Next, we recall the following embedding theorem (see Ref. 28).

Proposition 2.1: For $r > 0$, the embedding $E^r \hookrightarrow L^k(\mathbb{R}^N)$ is continuous for $2 \leq k \leq 2N/(N-2r)$ and is locally compact for $2 \leq k < 2N/(N-2r)$.

Now, we choose $s$, $t > 0$ with $s+t=2$ such that

$$1 - \frac{1}{p+1} < \frac{1}{2} + \frac{s}{N},$$

$$1 - \frac{1}{q+1} < \frac{1}{2} + \frac{t}{N}.$$

Set $E := E^s \times E^t$. Then, by Proposition 2.1, the embedding $E \hookrightarrow L^{p+1}(\mathbb{R}^N) \times L^{q+1}(\mathbb{R}^N)$ is continuous and is locally compact.

The system (ES) is naturally variational and its energy functional is

$$\Phi(z) = \Phi(u, v) = \langle A^t, A^t v \rangle_{L^2} - \Psi(z)$$

for $z = (u,v) \in E$, where $\Psi(z) = \Psi(u, v) = \int_{\mathbb{R}^N} \left[ F(x, u) + G(x, v) \right]$. Our hypotheses imply that $\Phi \in C^1(E, \mathbb{R})$. Moreover, a standard argument shows that the critical points of $\Phi$ are weak solutions of (ES) (see, e.g., Ref. 17). For the regularity of weak solutions, we refer the readers to Refs. 17 and 13.

Since

$$\Phi'(z) \eta = \Phi'(u, v)(\varphi, \psi) = \int_{\mathbb{R}^N} (A^t u A^t \varphi + A^t v A^t \varphi) - \int_{\mathbb{R}^N} (f(x, u) \varphi + g(x, v) \psi),$$

following Refs. 13 and 17 (see also Ref. 15), we introduce a bilinear form

$$B[(u,v),(\varphi, \psi)] = \langle A^t u, A^t \varphi \rangle_{L^2} + \langle A^t v, A^t \psi \rangle_{L^2}.$$

It is easy to check that $B$ is continuous and symmetric. Hence, $B$ induces a self-adjoint bounded linear operator $L: E \rightarrow E$ such that
It is obvious that studies differential equations via critical point theory the references therein.

By an easy computation, we have

\[ L_z = (A^{-\lambda}A'u, A^{-\lambda}A'v) \quad \text{for} \quad z = (u, v) \in E. \]

We can then prove that \( L \) has only two eigenvalues \(-1\) and \(1\), whose corresponding eigenspaces are

\[ E^-=\{(u,-A^{-\lambda}A'u) : u \in E^+\} \quad \text{for} \quad \lambda = -1, \]

\[ E^+=\{(u,A^{-\lambda}A'u) : u \in E^+\} \quad \text{for} \quad \lambda = 1. \]

Clearly, \( E = E^- \oplus E^+ \). For each \( z = (u, v) \in E \), set

\[ z^+ = \left( \frac{u + A^{-\lambda}A'u}{2}, \frac{u + A^{-\lambda}A'v}{2} \right), \]

\[ z^- = \left( \frac{u - A^{-\lambda}A'u}{2}, \frac{u - A^{-\lambda}A'v}{2} \right). \]

It is obvious that \( z = z^+ + z^- \),

\[ \|z^+\|^2_E = \frac{1}{2} \|A'u + A'v\|^2 = 2 \left\| \frac{u + A^{-\lambda}A'u}{2} \right\|^2_E = 2 \left\| \frac{A^{-\lambda}A'u + v}{2} \right\|^2_E, \]

\[ \|z^-\|^2_E = \frac{1}{2} \|A'v - A'u\|^2 = 2 \left\| \frac{u - A^{-\lambda}A'v}{2} \right\|^2_E = 2 \left\| \frac{A^{-\lambda}A'u - v}{2} \right\|^2_E, \]

and

\[ \frac{1}{2} (\|z^+\|^2_E - \|z^-\|^2_E) = \langle A'u, A'v \rangle_{L^2}. \]

Therefore, the functional \( \Phi \) can be rewritten in a standard way

\[ \Phi(z) = \frac{1}{2} (\|z^+\|^2_E - \|z^-\|^2_E) - \Psi(z) \quad \text{for} \quad z = (u, v) \in E. \quad (2.2) \]

The functional \( \Phi \) is strongly indefinite, this type of functional appears extensively when one studies differential equations via critical point theory (see, for example, Refs. 19, 29, and 30 and the references therein).

**III. LINKING STRUCTURE AND (PS)-SEQUENCE**

We assume all assumptions in Theorem 1.1 are satisfied in the sequel. With only minor modifications, one can verify the conditions on \( \Psi \) that are used in the abstract part of Ackermann.1

First, we have the following.

(i) \((P_1)\) There is a \( \rho > 0 \) such that \( \kappa = \inf \Phi(\partial B_\rho \cap E^+) > 0 \).

(ii) \((P_2)\) \( \Psi \) is weakly sequentially lower semicontinuous, and \( \Psi' \) is weakly sequentially continuous.

(iii) \((P_3)\) Let \( Z \) be a finite dimensional subspace of \( E^+ \). Then, \( \Phi(u) \to -\infty \) as \( \|u\| \to \infty \) in \( E^- \oplus Z \).

**Remark 3.1:** \((P_1)\) and \((P_3)\) imply that the functional \( \Phi \) has a linking structure in the sense as in Ref. 19.

Next, we discuss the properties of \((PS)\)-sequence.

**Lemma 3.1:** Suppose that \((H_0)-(H_3)\) are satisfied. Then, any \((PS)\)-sequence of \( \Phi \) is bounded.
Proof: One can obtain the conclusion in a standard way (see, e.g., Refs. 1 and 19). Let \( \mathcal{K} = \{ z \in E : \Phi'(z) = 0, \; z \neq 0 \} \) be the set of nontrivial critical points of \( \Phi \).

Lemma 3.2: The following two conclusions hold:
(1) \( \nu := \inf \{|z| : z \in \mathcal{K} \} > 0 \) and
(2) \( \theta := \inf \{ \Phi(z) | z \in \mathcal{K} \} > 0. \)

Proof:
(1) For any \( z = (u, v) \in \mathcal{K} \), there holds
\[
0 = \Phi'(z)(z^* - z) = \|z\|^2 - \int_{\mathbb{R}^N} \left[ f(x, u)A^{-1}A'v + g(x, v)A^{-1}A'u \right],
\]
which implies that
\[
\|z\|^2 = \int_{\mathbb{R}^N} \left[ f(x, u)A^{-1}A'v + g(x, v)A^{-1}A'u \right] \leq \int_{\mathbb{R}^N} \left[ (e|u| + C_1|u|^p)|A^{-1}A'v| + (e|u| + C_1|u|^q) \right.
\]
\[
\times |A^{-1}A'u| \leq e|u|_2|A^{-1}A'v|_2 + C_1|u|_p|A^{-1}A'v|_{p+1} + |u|_2|A^{-1}A'u|_2 + C_1|u|_q|A^{-1}A'u|_{q+1}
\leq 2eC_1|u|_E|v|_E + C_1C_2(|u|_p^p|v|_E + |u|_q^q|v|_E) \leq 2eC_1|u|_E|v|_E + C_1C_2(|u|_E^p + |v|_E^q)
\]
\[
+ |u|_E^{p+1} + |v|_E^{q+1} \leq eC_1\|z\|^2 + C_1C_2(|z|^{p+1} + |z|^{q+1}).
\]

Choose \( \varepsilon \) small enough, hence there exists \( c_\varepsilon > 0 \) such that
\[
c_\varepsilon \leq \|z|^{p-1} + \|z|^{q-1}
\]
for each \( z \in \mathcal{K} \). Hence, (1) holds.

(2) For the details of the proof, we refer the reader to Lemma 4.3 in Ref. 1.

Let \( [l] \) denote the integer part of \( l \in \mathbb{R} \). The following lemma is essentially standard (see Refs. 2, 10, 11, 19, and 26). It was proved in Ref. 1 under our weak regularity assumptions on \( f \) and \( g \).

Lemma 3.3: Let \( \{ z_j \} \subset E \) be a (PS)-sequence of \( \Phi \). Then, either

(i) \( z_j \rightarrow 0 \) (and hence \( c=0 \)) or
(ii) \( c \geq \theta \) and there exists a positive integer \( l \leq \lfloor c/\theta \rfloor \), \( y_1, \ldots, y_l \in \mathcal{K} \) and sequences \( \{a_j^i \} \subset \mathbb{Z}^N \), \( i=1, 2, \ldots, l \), such that after extraction of a subsequence of \( \{ z_j \} \),
\[
\| z_j - \sum_{i=1}^l a_j^i \cdot y_i \| \rightarrow 0,
\]
\[
\sum_{i=1}^k \Phi(y_i) = c
\]
and for \( i \neq k \),
\[
|a_j^i - a_j^j| \rightarrow \infty
\]
as \( j \rightarrow \infty \).

IV. PROOF OF THEOREM 1.1

Proof of Theorem 1.1:

(1) Existence of a least energy solution. By \((P_1)-(P_3)\) and the abstract critical point theorem in Refs. 7 and 6 (see also Ref. 16), there exists a sequence \( \{ z_j \} \subset E \) such that \( \Phi(z_j) \rightarrow c \geq \kappa \) and
\( \Phi \gamma(z_i) \to 0 \). Lemma 3.1 implies that \( \{z_i\} \) is bounded. By the vanishing lemma (see Ref. 22), one can rule out the case of vanishing. So nonvanishing occurs. Using a standard translation argument, we can obtain a nontrivial critical point \( \bar{z} = (\bar{u}, \bar{v}) \), which implies that \( K \) is nonempty, and hence \( \theta := \inf \{ \Phi(z) | z \in K \} \) is finite.

Next, we claim that \( \theta \) is achieved.

Let \( \{z_i\} \subset K \) be a minimizing sequence for \( \theta \). Clearly, \( \{z_i\} \) is a \((PS)_\theta\)-sequence of \( \Phi \), hence is bounded by Lemma 3.1. From part (1) of Lemma 3.2, \( \|z_i\| \to \nu > 0 \), one can rule out case (i) of Lemma 3.3. Hence, (ii) applies with \( \lambda = 1 \), that is, there exists a ground state.

(2) Multiplicity. \( \Phi \) is even provided that \( H(x, z) \) is even in \( z \). Lemma 3.2 says that \( \Phi \) satisfies \( (\Phi_\lambda) \). Next, we assume

\[
K/\mathbb{Z}^N \text{ is a finite set. (4.1)}
\]

Let \( F \) be a set consisting of arbitrarily chosen representatives of the \( \mathbb{Z}^N \)-orbits of \( K \). Given a compact interval \( I \subset [0, \infty) \) with \( d = \max I \) we set \( I = [d/\theta] \) and \( A = [F, I] \). Then as in Ref. 16, one can check that \( A \) is a bounded discrete \((PS)_\theta\)-attractor. Therefore, by multiple critical point theorem due to Bartsch and Ding, \( \Phi \) has an unbounded sequence of critical values which contradicts with the assumption (4.1), and hence \( \Phi \) has infinite many geometrically distinct nontrivial critical points. Thus, our multiplicity results follow.

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