Enumeration of perfect matchings of a type of Cartesian products of graphs*

Weigen Yan$^{a,b}$ 1 and Fuji Zhang$^b$ 2

$^a$School of Sciences, Jimei University, Xiamen 361021, China

$^b$Department of Mathematics, Xiamen University, Xiamen 361005, China

Abstract

Let $G$ be a graph and let $\text{Pm}(G)$ denote the number of perfect matchings of $G$. We denote the path with $m$ vertices by $P_m$ and the Cartesian product of graphs $G$ and $H$ by $G \times H$. In this paper, as the continuance of our paper [19], we enumerate perfect matchings in a type of Cartesian products of graphs by the Pfaffian method, which was discovered by Kasteleyn. Here are some of our results:

1. Let $T$ be a tree and let $C_n$ denote the cycle with $n$ vertices. Then $\text{Pm}(C_4 \times T) = \prod (2 + \alpha^2)$, where the product ranges over all eigenvalues $\alpha$ of $T$. Moreover, we prove that $\text{Pm}(C_4 \times T)$ is always a square or double a square.

2. Let $T$ be a tree. Then $\text{Pm}(P_4 \times T) = \prod (1 + 3 \alpha^2 + \alpha^4)$, where the product ranges over all non-negative eigenvalues $\alpha$ of $T$.

3. Let $T$ be a tree with a perfect matching. Then $\text{Pm}(P_3 \times T) = \prod (2 + \alpha^2)$, where the product ranges over all positive eigenvalues $\alpha$ of $T$. Moreover, we prove that $\text{Pm}(C_4 \times T) = [\text{Pm}(P_3 \times T)]^2$.

Keywords: Perfect matchings, Pfaffian orientation, Skew adjacency matrix, Cartesian product, Bipartite graph, Nice cycle.

1. Introduction

A perfect matching of a graph $G$ is a set of independent edges of $G$ covering all vertices of $G$. Problems involving enumeration of perfect matchings of a graph were first examined by chemists and physicists in the 1930s (for history see [4,16]), for two different (and

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1This work is supported by FMSTF(2004J024) and FJCEF(JA03131)

2Partially supported by NSFC(10371102).

Email address: weigenyan@263.net (W. Yan), fjzhang@jingxian.xmu.edu.cn (F. Zhang)
unrelated) purposes: the study of aromatic hydrocarbons and the attempt to create a theory of the liquid state.

Shortly after the advent of quantum chemistry, chemists turned their attention to molecules like benzene composed of carbon rings with attached hydrogen atoms. For these researchers, perfect matchings of a polyhex graph corresponded to "Kekulé structures", i.e., assigning single and double bonds in the associated hydrocarbon (with carbon atoms at the vertices and tacit hydrogen atoms attached to carbon atoms with only two neighboring carbon atoms). There are strong connections between combinatorial and chemical properties for such molecules; for instance, those edges which are present in comparatively few of the perfect matchings of a graph turn out to correspond to the bonds that are least stable, and the more perfect matchings a polyhex graph possesses the more stable is the corresponding benzenoid molecule. The number of perfect matchings is an important topological index which had been applied for estimation of the resonant energy and total $\pi$-electron energy and calculation of pauling bond order (see [6,15,18]). So far, many mathematicians, physicists and chemists have given most of their attention to counting perfect matchings of graphs. See for example papers [2,5,7,16,17,19–23].

By a simple graph $G = (V(G), E(G))$ we mean a finite undirected graph, that is, one with no loops or parallel edges, with the vertex-set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge-set $E(G) = \{e_1, e_2, \ldots, e_m\}$, if not specified. We denote by $\text{Pm}(G)$ the number of perfect matchings of $G$. If $M$ is a perfect matching of $G$, an $M$–alternating cycle in $G$ is a cycle whose edges are alternately in $E(G) \setminus M$ and $M$. Let $G$ be a graph. A cycle $C$ of $G$ is called to be nice if $G - C$ contains a perfect matching, where $G - C$ denotes the induced subgraph of $G$ obtained from $G$ by deleting the vertices of $C$. Throughout this paper, we denote a tree by $T$ and a path with $n$ vertices by $P_n$. For two graphs $G$ and $H$, let $G \times H$ denote the Cartesian product of graphs $G$ and $H$.

Let $G = (V(G), E(G))$ be a simple graph and let $G^e$ be an arbitrary orientation of $G$. The skew adjacency matrix of $G^e$, denoted by $A(G^e)$, is defined as follows:

$$A(G^e) = (a_{ij})_{n \times n},$$

$$a_{ij} = \begin{cases} 
1 & \text{if } (v_i, v_j) \in E(G^e), \\
-1 & \text{if } (v_j, v_i) \in E(G^e), \\
0 & \text{otherwise}. 
\end{cases}$$

It is clear that $A(G^e)$ is a skew symmetric matrix, that is, $A(G^e)^T = -A(G^e)$. 
Let $G$ be a simple graph. We say $G$ has **reflective symmetry** if it is invariant under the reflection across some straight line or plane $l$ (the symmetry plane or axis) (see Ciucu’s paper [2]). Ciucu [2] gave a matching factorization theorem of the number of perfect matchings of a symmetric plane bipartite graph in which there are some vertices lying on the symmetry axis $l$ but no edges crossing $l$. Ciucu’s theorem expresses the number of perfect matchings of $G$ in terms of the product of the number of perfect matchings of two subgraphs of $G$ each one of which has nearly half the number of vertices of $G$. On the other hand, in [21] Zhang and Yan proved that if a bipartite graph $G$ without nice cycles of length $4s, s \in \{1, 2, \ldots\}$ was invariant under the reflection across some plane (or straight line) and there are no vertices lying on the symmetry plane (or axis) then $Pm(G) = |\det A(G^+)|$, where $G^+$ is a graph having loops with half the number of vertices of $G$, and $A(G^+)$ is the adjacency matrix of $G^+$. Furthermore, in [19] Yan and Zhang obtained the following results on the symmetric graphs:

1. If $G$ is a reflective symmetric plane graph (which does not need to be bipartite) without vertices on the symmetry axis, then the number of perfect matchings of $G$ can be expressed by a determinant of order $\frac{1}{2}|G|$, where $|G|$ denotes the number of vertices of $G$. 
2. Let $G$ be a bipartite graph without cycles of length $4s, s \in \{1, 2, \ldots\}$. Then the number of perfect matchings of $G \times P_2$ equals $\prod (1 + \alpha^2)$, where the product ranges over all non-negative eigenvalues $\alpha$ of $G$. Particularly, if $T$ is a tree then $Pm(T \times P_2)$ equals $\prod (1 + \theta^2)$, where the product ranges over all non-negative eigenvalues $\theta$ of $T$.

As the continuance of our paper [19], in this paper we obtain the following results:

1. Let $T$ be a tree and let $C_n$ denote the cycle with $n$ vertices. Then $Pm(C_4 \times T) = \prod (2 + \alpha^2)$, where the product ranges over all of eigenvalues $\alpha$ of $T$. This makes it possible to obtain a formula for the number of perfect matchings for the linear $2 \times 2 \times n$ cubic lattice, which was previously obtained by H. Narumi and H. Hosoya [14]. Moreover, we prove that $Pm(C_4 \times T)$ is always a square or double a square (such a number was called squarish in [7]).
2. Let $T$ be a tree. Then $Pm(P_4 \times T) = \prod (1 + 3\alpha^2 + \alpha^4)$, where the product ranges over all non-negative eigenvalues $\alpha$ of $T$.
3. Let $T$ be a tree with a perfect matching. Then $Pm(P_3 \times T) = \prod (2 + \alpha^2)$, where the product ranges over all positive eigenvalues $\alpha$ of $T$. Moreover, we prove that $Pm(C_4 \times T) = [Pm(P_3 \times T)]^2$. 
The start point of this paper is the fact that we can use the Pfaffian method, which was discovered by Kasteleyn [8,9,11], to enumerate perfect matchings of some graphs. In order to formulate lemmas we need to introduce some terminology and notation as follows.

If $D$ is an orientation of a simple graph $G$ and $C$ is a cycle of even length, we say that $C$ is **oddly oriented** in $D$ if $C$ contains odd number of edges that are directed in $D$ in the direction of each orientation of $C$ (see [5,11]). We say that $D$ is a **Pfaffian orientation** of $G$ if every nice cycle of even length of $G$ is oddly oriented in $D$. It is well known that if a graph $G$ contains no subdivision of $K_{3,3}$ then $G$ has a Pfaffian orientation (see Little [10]). McCuaig [12], and McCuaig, Robertson et al [13], and Robertson, Seymour et al [17] found a polynomial-time algorithm to determine whether a bipartite graph has a Pfaffian orientation.

**Lemma 1** [8,9,11] Let $G^e$ be a Pfaffian orientation of a graph $G$. Then

$$[\text{Pm}(G)]^2 = \det A(G^e),$$

where $A(G^e)$ is the skew adjacency matrix of $G^e$.

**Lemma 2** [11] Let $G$ be any simple graph with even number of vertices, and $G^e$ an orientation of $G$. Then the following three properties are equivalent:

1. $G^e$ is a Pfaffian orientation.
2. Every nice cycle of even length in $G$ is oddly oriented in $G^e$.
3. If $G$ contains a perfect matching, then for some perfect matching $F$, every $F$-alternating cycle is oddly oriented in $G^e$.

![Figure 1](image.png)

Figure 1: (a) A graph $G$. (b) An orientation $G^e$ of $G$. (c) The orientation $(P_2 \times G)^e$ of $P_2 \times G$. 

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2. Enumeration of perfect matchings of $C_4 \times T$

First, we introduce a method to orient a type of symmetric graphs. Let $G$ be a simple graph, and $G^e$ an orientation of $G$. We take a copy of $G^e$, denoted by $G_1^e$. If we reverse the orientation of each arc of $G_1^e$, then we obtain another orientation of $G$, denoted by $G_2^e$. Hence $G_2^e$ is the converse of $G_1^e$. Note that $P_2 \times G$ can be obtained as follows: Take two copies of $G$, denoted by $G_1$ with vertex-set $V(G_1) = \{v'_1, v'_2, \ldots, v'_n\}$ and $G_2$ with vertex-set $V(G_2) = \{v''_1, v''_2, \ldots, v''_n\}$ (we consider that $G_1$ and $G_2$ are the left half and right half of $P_2 \times G$, respectively), and add an edge $v'_i v''_i$ between every pair of corresponding vertices $v'_i$ and $v''_i$ of $G_1$ and $G_2$, respectively. It is obvious that the resulting graph is $P_2 \times G$ and all edges $v'_i v''_i$ (for $1 \leq i \leq n$) added between the left half and the right half of $P_2 \times G$ form a perfect matching of $P_2 \times G$, denoted by $M$, and $G_1^e$ (or $G_2^e$) is an orientation of $G_1$ (or $G_2$). If we define the direction of every edge in $M$ is from the left to the right, then an orientation of $P_2 \times G$ is obtained, denoted by $(P_2 \times G)^e$. Figure 1 illustrates this procedure. By using a result from Fischer and Little [5], the following lemma was proved by Yan and Zhang in [19].

**Lemma 3** [19] Let $G$ be a simple graph. If $G^e$ is an orientation of $G$ under which every cycle of even length is oddly oriented in $G^e$, then the orientation $(P_2 \times G)^e$ defined as above is a Pfaffian orientation of $P_2 \times G$.

Now we prove the following lemma.

**Lemma 4** Let $T$ be a tree. Then every cycle of $P_2 \times T$ is a nice cycle.

**Proof** Let $T_1$ and $T_2$ denote the left half and the right half of $P_2 \times T$, respectively. Suppose that $C$ is a cycle of $P_2 \times T$. We claim that $C$ has the following form:

$$v'_{i_1} - v'_{i_2} - \ldots - v'_{i_m} - v''_{i_{m-1}} - \ldots - v''_{i_2} - v''_{i_1},$$

where $v'_{i_1}, v'_{i_2}, \ldots, v'_{i_m} \in V(T_1)$ and $v''_{i_1}, v''_{i_2}, \ldots, v''_{i_m} \in V(T_2)$.

Note that there is a unique path between two vertices of tree $T$. Hence we only need to prove that $|E(C) \cap M| = 2$, where $M$ denotes the edge set of $P_2 \times T$ between the left half and the right half of $P_2 \times T$. Suppose $|E(C) \cap M| = k > 2$. It is obvious that $k$ even. We suppose that $k = 2t$ ($t > 1$). Then $C$ has the following form:

$P(i_0 \rightarrow i_1) \cup v'_{i_1} v''_{i_1} \cup P(i_1 \rightarrow i_2) \cup v''_{i_2} v'_{i_2} \cup P(i_2 \rightarrow i_3) \cup v'_{i_3} v''_{i_3} \cup \ldots \cup P(i_{2t-1} \rightarrow i_{2t}) \cup v''_{i_{2t}} v'_{i_{2t}} \cup P(i_{2t} \rightarrow i_{2t+1}),$

where $P(i_j \rightarrow i_{j+1})$ are paths from vertex $v'_{i_j}$ to vertex $v'_{i_{j+1}}$ in $T_1$ when $j$ ($0 \leq j \leq 2t$)
are even, and paths from vertex \(v_{ij}'\) to vertex \(v_{ij+1}'\) in \(T_2\) when \(j\) (\(0 \leq j \leq 2t - 1\)) are odd, and \(v'_{i2t+1} = v'_{i0}\). Hence there exists a cycle of the form \(v'_{i0} \rightarrow v'_{i1} \rightarrow v'_{i2} \rightarrow \ldots \rightarrow v'_{i2t-1} \rightarrow v'_{i2t} \rightarrow v'_{i0}\) in \(T_1\). This is a contradiction. Hence the claim holds.

The lemma is immediate from the claim.

**Corollary 5** Let \(T\) be a tree and \(T^e\) be an arbitrary orientation of \(T\). Then the orientation \((P_2 \times T)^e\) defined as above is a Pfaffian orientation of \(P_2 \times T\) under which every cycle of even length of \(P_2 \times T\) is oddly oriented in \((P_2 \times T)^e\).

**Proof** By Lemma 3, \((P_2 \times T)^e\) is a Pfaffian orientation of \(P_2 \times T\). Hence, by Lemma 2, every nice cycle in \((P_2 \times T)^e\) is oddly oriented. Then Corollary 5 is immediate from Lemma 4.

![Figure 2: (a) An orientation \((P_2 \times T)^e\) (=\(G^e\)). (b) The corresponding orientation \((C_4 \times T)^e\).](image)

Let \(T\) be a tree and \(T^e\) an arbitrary orientation of \(T\). Then, by Corollary 5, the orientation \((P_2 \times T)^e\) defined as above is a Pfaffian orientation of \(P_2 \times T\) under which every cycle of even length of \(P_2 \times T\) is oddly oriented in \((P_2 \times T)^e\). Let \(G = P_2 \times T\) and \(G^e = (P_2 \times T)^e\) and let \((P_2 \times P_2 \times T)^e = (P_2 \times P_2 \times T)^e\) be the orientation defined as above. Figure 2 illustrates this procedure, where both of \(T_1^e\) and \(T_3^e\) are \(T^e\), and both of \(T_2^e\) and \(T_4^e\) are the converse of \(T^e\). Note that \(P_2 \times P_2 = C_4\). Hence we have \((C_4 \times T)^e = (P_2 \times P_2 \times T)^e\). From Lemma 3 and Corollary 5, the following theorem is immediate.

**Theorem 6** Let \(T\) be a tree and let \(T^e\) be an arbitrary orientation of \(T\). Then the orientation \((C_4 \times T)^e\) of \(C_4 \times T\) defined as above is a Pfaffian orientation.

**Lemma 7** Let \(T\) be a tree, and \(T^e\) an arbitrary orientation. Then \(\theta\) is an eigenvalue of \(A(T)\) with multiplicity \(m_\theta\) if and only if \(i\theta\) is an eigenvalue of \(A(T^e)\) with multiplicity \(m_{i\theta}\), where \(A(T)\) and \(A(T^e)\) are the adjacency matrix of \(T\) and the skew adjacency matrix of \(T^e\), respectively, and \(i^2 = -1\).
Proof Let $\phi(T,x) = \det(xI - A(T))$. Since $T$ is a bipartite graph, we may assume that
\[
\phi(T,x) = x^n - a_1x^{n-2} + a_2x^{n-4} + \ldots + (-1)^i a_i x^{n-2i} + \ldots + (-1)^r a_r x^{n-2r},
\] (1)
where $n$ and $r$ are the number of vertices of $T$ and the maximum number of edges in a matching of $T$ (see Biggs [1]). Note that $(-1)^i a_i$ equals the sum of all principal minors of $A(T)$ of order $2i$. Hence $(-1)^i a_i$ equals the sum of $\det A(H)$ over all induced subgraphs $H$ of $T$ with $2i$ vertices, where $A(H)$ is the adjacency matrix of subgraph $H$. Note that every induced subgraph $H$ of $T$ with $2i$ vertices is either a subtree of $T$ or some subtrees of $T$. Hence $\det A(H)$ equals $(-1)^i$ if $H$ has a perfect matching and 0 otherwise. Thus we have proved the following claim.

Claim 1 Every $a_i$ equals the number of the induced subgraphs of $T$ with $2i$ vertices that have a perfect matching.

Note that the spectrum of a bipartite graph is symmetric with respect to 0 (see Coulson and Rushbrooke [3] or Biggs [1]). Hence, by Claims 1 and 2, the lemma follows.

Theorem 8 Let $T$ be a tree with $n$ vertices. Then
\[
\text{Pm}(C_4 \times T) = \prod_{j=1}^{n} (2 + \theta_j^2),
\]
where the eigenvalues of $T$ are $\theta_1, \theta_2, \ldots, \theta_n$.

Proof Suppose that $(C_4 \times T)^e$ is the Pfaffian orientation of $C_4 \times T$ defined as that in Theorem 6. Let $A(T^e)$ be the skew adjacency matrix of $T^e$. By a suitable labelling of
vertices of \((C_4 \times T)^e\), the skew adjacency matrix of \((C_4 \times T)^e\) has the following form:

\[
A((C_4 \times T)^e) = \begin{bmatrix}
A(T^e) & I & 0 \\
-I & -A(T^e) & I \\
-I & 0 & -A(T^e) \\
0 & -I & I
\end{bmatrix}.
\]

where \(I\) is the identity matrix. Hence, by Lemma 1, we have

\[
[Pm(C_4 \times T)]^2 = \det A((P_2 \times P_2 \times T)^e)
\]

\[
= \det \begin{bmatrix}
A(T^e) & I & 0 \\
-I & -A(T^e) & I \\
-I & 0 & -A(T^e) \\
0 & -I & I
\end{bmatrix}
\]

\[
= \det \left\{ -\begin{bmatrix}
A(T^e) & I \\
-I & -A(T^e)
\end{bmatrix}^2 + \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix} \right\}
\]

\[
= \det \begin{bmatrix}
2I - (A(T^e))^2 & 0 \\
0 & 2I - (A(T^e))^2
\end{bmatrix}.
\]

Hence we have proved

\[
Pm(C_4 \times T) = |\det(2I - (A(T^e))^2)|.
\]

Hence, by Lemma 7, we have

\[
Pm(C_4 \times T) = |\det(2I - (A(T^e))^2)| = \prod_{j=1}^{n}(2 + \theta_j^2),
\]

where the eigenvalues of \(T\) are \(\theta_1, \theta_2, \ldots, \theta_n\). The theorem is thus proved.

**Remark 9** Note that if \(T\) is a path with \(n\) vertices, then the set of eigenvalues of \(T\) is \(\{2 \cos \frac{k\pi}{n+1} | 1 \leq k \leq n\}\). Hence, by Theorem 8, the number of perfect matchings of \(C_4 \times P_n\) (the linear \(2 \times 2 \times n\) cubic lattice) equals \(\prod_{k=1}^{n} \left[2 + 4 \cos^2 \frac{k\pi}{n+1}\right]\). This formula was previously obtained by H. Narumi and H. Hosoya in [14].

**Corollary 10** Suppose \(T\) is a tree with \(n\) vertices. Then \(Pm(C_4 \times T)\) is always a square or double a square. Moreover, if \(T\) is a tree with a perfect matching, then \(Pm(C_4 \times T)\) is always a square.
**Proof** Suppose that \( \phi(T, x) \) is the characteristic polynomial of \( T \). Since \( T \) is a bipartite graph, the zeroes of \( \phi(T, x) \) are symmetric with respect to zero (a result obtained by Coulson and Rushbrooke [3], see also Biggs [1]). Without loss of generality, we may suppose that

\[
\phi(T, x) = x^n - a_1 x^{n-2} + a_2 x^{n-4} + \ldots + (-1)^j a_j x^{n-2j} + \ldots + (-1)^r a_r x^{n-2r},
\]

where \( r \) is the number of edges in a maximum matching of \( T \). Let \( s = n - 2r \). Thus, we have

\[
\phi(T, x) = x^s \prod_{j=1}^r (x - \theta_j)(x + \theta_j),
\]

where \( \pm \theta_j \) for \( 1 \leq j \leq r \) are all of non-zero eigenvalues of \( T \). Hence, by Theorem 8, we have

\[
\text{Pm}(C_4 \times T) = 2^s \prod_{j=1}^r (2 + \theta_j^2)^2.
\]  

(5)

Note that \( \phi(T, i\sqrt{2}) = (i\sqrt{2})^s \prod_{j=1}^r (i\sqrt{2} - \theta_j)(i\sqrt{2} + \theta_j) = (-1)^r (i\sqrt{2})^s \prod_{j=1}^r (2 + \theta_j^2), \) where \( i^2 = -1 \). Hence we have

\[
\phi^2(T, i\sqrt{2}) = (-1)^s 2^s \prod_{j=1}^r (2 + \theta_j^2)^2.
\]  

(6)

By equations (5) and (6), we have

\[
\text{Pm}(C_4 \times T) = 2^s \prod_{j=1}^r (2 + \theta_j^2)^2 = (-1)^s \phi^2(T, i\sqrt{2}).
\]  

(7)

Note that \( a_j \), for \( 1 \leq j \leq r \), is a non-negative integer, then \((-1)^s \phi^2(T, i\sqrt{2})\) equals

\[
(-1)^s \{(i\sqrt{2})^s [(i\sqrt{2})^{2r} - a_1 (i\sqrt{2})^{2r-2} + a_2 (i\sqrt{2})^{2r-4} + \ldots + (-1)^j a_j (i\sqrt{2})^{2r-2j} + \ldots + (-1)^r a_r]\}^2,
\]

which is a square or double a square. This implies that \( \text{Pm}(C_4 \times T) \) is a square or double a square. Hence the first assertion in Corollary 10 holds. If \( T \) is a tree with a perfect matching, then \( s = 0 \) and hence \((-1)^s \phi^2(T, i\sqrt{2})\) is a square. Thus the second assertion in Corollary 10 holds. The corollary is thus proved.
3. Enumeration of perfect matchings of $P_3 \times T$ and $P_4 \times T$

Suppose that $T$ is a tree and $P_m$ is a path with $m$ vertices. Let $T^e$ be any orientation of $T$ and let $T^e_\ast$ be the converse of $T^e$ which is the digraph obtained from $T^e$ by reversing the orientation of each arc. We define an orientation of $P_m \times T$ (denoted $(P_m \times T)^e$) as follows.

Let $V(T) = \{v_1, v_2, \ldots, v_n\}$ be the vertex-set of $T$. Take $m$ copies of $T$, denoted by $T_1, T_2, \ldots, T_m$, where $V(T_i) = \{v_1^{(i)}, v_2^{(i)}, \ldots, v_n^{(i)}\}$ is the vertex-set of $T_i$, $i = 1, 2, \ldots, m$. Clearly, the mapping $\phi_i$ (from $T$ to $T_i$): $v_j \mapsto v_j^{(i)} (1 \leq j \leq n)$ is an isomorphism between $T$ and $T_i$. If we add the set of edges $\{v_j^{(i)}v_j^{(i+1)} | 1 \leq j \leq n\}$ between every pair of trees $T_i$ and $T_{i+1}$ for $1 \leq i \leq m-1$, then the resulting graph is $P_m \times T$. We define the orientation of $T_i$ in $P_m \times T$ to be $T^e$ if $i$ is odd, denoted by $T_i^e$, and the converse $T^e_\ast$ otherwise, denoted also by $T_i^e$, and the direction of edges of the form $v_j^{(i)}v_j^{(i+1)} (1 \leq j \leq n, 1 \leq i \leq m-1)$ in $P_m \times T$ is from $v_j^{(i)}$ to $v_j^{(i+1)}$. Hence we obtain an orientation of $P_m \times T$, denoted by $(P_m \times T)^e$ (see Figure 3).

![Figure 3: An orientation $(P_m \times T)^e$ of $P_m \times T$.](image)

For the sake of convenience, we need introduce some notations. Let $T$ be a tree with $n$ vertices. For the graph $P_{2k} \times T$, let $M = M_1 \cup M_2 \cup \ldots \cup M_k$, where $M_i = \{v_j^{(2i-1)}, v_j^{(2i)} | 1 \leq j \leq n\}$ for $1 \leq i \leq k$. Clearly $M$ is a perfect matching of $P_{2k} \times T$. Suppose $T$ is a tree with $n$ vertices containing a perfect matching. For the graph $P_{2k+1} \times T$, let $M^* = M_1 \cup M_2 \cup \ldots \cup M_k \cup M'$, where $M_i = \{v_j^{(2i-1)}, v_j^{(2i)} | 1 \leq j \leq n\}$ for $1 \leq i \leq k$ and $M'$ is the unique perfect matching of $T_{2k+1}$. Then $M^*$ is a perfect matching of $P_{2k+1} \times T$.

**Lemma 11** Let $T$ be a tree. Then $(P_4 \times T)^e$ defined as above is a Pfaffian orientations of $P_4 \times T$.

**Proof** Let $M, M_1$ and $M_2$ be defined as above and let $C$ be an $M$–alternating cycle in $P_4 \times T$. By Lemma 2, we only need to prove that $C$ is oddly oriented in $(P_4 \times T)^e$. Noting the definitions of $C_4 \times T$ and $P_4 \times T$, every nice cycle in $P_4 \times T$ is also a nice cycle in $C_4 \times T$. Hence $C$ is a nice cycle in $C_4 \times T$. By the definitions of $(C_4 \times T)^e$ and $(P_4 \times T)^e$,
is a subdigraph of \((C_4 \times T)^e\). Since \((C_4 \times T)^e\) is a Pfaffian orientation of \(C_4 \times T\),
every nice cycle in \(C_4 \times T\) is oddly oriented in \((C_4 \times T)^e\). Thus \(C\) is oddly oriented in
\((P_4 \times T)^e\). The lemma thus follows.

**Lemma 12** Let \(T\) be a tree with a perfect matching. Then \((P_3 \times T)^e\) defined as above is
a Pfaffian orientations of \(P_3 \times T\).

**Proof** In Lemma 11 we proved that \((P_4 \times T)^e\) is a Pfaffian orientation of \(P_4 \times T\). Note
that \(T\) contains a perfect matching. Hence every nice cycle in \(P_3 \times T\) is also a nice cycle
in \(P_4 \times T\). By using the same method as in Lemma 11, we may prove that \((P_3 \times T)^e\) is a
Pfaffian orientation of \(P_3 \times T\). The lemma is thus proved.

**Theorem 13** Suppose \(T\) is a tree with \(n\) vertices. Then

\[
Pm(P_4 \times T) = \prod (1 + 3\alpha^2 + \alpha^4),
\]

where the product ranges over all non-negative eigenvalues \(\alpha\) of \(T\).

**Proof** By Lemma 11, \((P_4 \times T)^e\) defined as above is a Pfaffian orientation of \(P_4 \times T\). Hence,
by Lemma 1, we have

\[
[Pm(P_4 \times T)]^2 = \det[A((P_4 \times T)^e)],
\]

where \(A((P_4 \times T)^e)\) is the skew adjacency matrix of \((P_4 \times T)^e\). By a suitable labelling of
vertices of \((P_4 \times T)^e\), the skew adjacency matrix of \((P_4 \times T)^e\) has the following form:

\[
A((P_4 \times T)^e) = \begin{bmatrix}
A & I & 0 & 0 \\
-I & -A & I & 0 \\
0 & -I & A & I \\
0 & 0 & -I & -A
\end{bmatrix},
\]

where \(A\) denotes the skew adjacency matrix \(A(T^e)\) of \(T^e\).

Now multiplying the first column, then the third and fourth row, then the fourth
column of the partitioned matrix \(A((P_4 \times T)^e)\) by \(-1\), we do not change the absolute
value of the determinant and we obtain matrix \(Q\), where

\[
Q = \begin{bmatrix}
-A & I & 0 & 0 \\
I & -A & I & 0 \\
0 & I & -A & I \\
0 & 0 & I & -A
\end{bmatrix}.
\]
Denote by $B$ the adjacency matrix of the path with four vertices, that is,

$$B = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}.$$  

Then we may write

$$Q = -I_4 \otimes A + B \otimes I_n,$$

where $\otimes$ denotes the Kronecker product of matrices.

Note that, if $A$ has the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $B$ has the eigenvalues $\mu_1, \mu_2, \mu_3$ and $\mu_4$, then the eigenvalues of $-I_4 \otimes A + B \otimes I_n$ are as follows:

$$\mu_i - \lambda_j, \quad \text{where} \quad 1 \leq i \leq 4, \quad 1 \leq j \leq n.$$

Suppose that $T$ has the eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$. By Lemma 7, $A$ has the eigenvalues $i\alpha_j$ ($1 \leq j \leq n$), where $i^2 = -1$. Note that the eigenvalues of $B$ are as follows:

$$\pm \sqrt{\frac{3 + \sqrt{5}}{2}}, \quad \pm \sqrt{\frac{3 - \sqrt{5}}{2}}.$$

Thus the eigenvalues of $Q$ are as follows:

$$\pm \sqrt{\frac{3 + \sqrt{5}}{2}} - i\alpha_s, \pm \sqrt{\frac{3 - \sqrt{5}}{2}} - i\alpha_s, (s = 1, 2, \ldots, n).$$

Hence the determinant of matrix $Q$ is the product of these numbers. Since we are interested in the absolute value of this determinant, we may replace these $4n$ factors by their absolute values, and so the absolute value of the determinant of matrix $A((P_4 \times T^c))$ is

$$\prod_{s=1}^{n} \left( \sqrt{\frac{3 + \sqrt{5}}{2}} - i\alpha_s \right) \left( -\sqrt{\frac{3 + \sqrt{5}}{2}} - i\alpha_s \right) \left( \sqrt{\frac{3 - \sqrt{5}}{2}} - i\alpha_s \right) \left( -\sqrt{\frac{3 - \sqrt{5}}{2}} - i\alpha_s \right)$$

$$= \prod_{s=1}^{n} (1 + 3\alpha_s^2 + \alpha_s^4).$$

Hence

$$\text{Pm}(P_4 \times T) = \prod_{s=1}^{n} (1 + 3\alpha_s^2 + \alpha_s^4)^{\frac{1}{2}}.$$
Note that the spectrum of a tree is symmetric with respect to zero (see Coulson and Rushbrooke [3] or Biggs [1]). Hence we have

\[
Pm(P_4 \times T) = \prod (1 + 3\alpha^2 + \alpha^4),
\]

where the product ranges over all non-negative eigenvalues of \( T \). The theorem is thus proved.

Similarly, by using Lemma 12, we may prove the following theorem.

**Theorem 14** Suppose \( T \) is a tree with a perfect matching. Then

\[
Pm(P_3 \times T) = \prod (2 + \alpha^2),
\]

where the product ranges over all positive eigenvalues \( \alpha \) of \( T \).

**Corollary 15** Suppose \( T \) is a tree with a perfect matching. Then \([Pm(P_3 \times T)]^2 = Pm(C_4 \times T)\).

Corollary 15 is immediate from Theorems 8 and 14.

Although a tree \( T \) with even number of vertices has no perfect matching, \( P_3 \times T \) may contain perfect matchings. See for example the tree \( T \) in Figure 4, which has no perfect matching but \( P_3 \times T \) contains a perfect matching (the set of the bold edges). Hence we

\[\text{Figure 4: (a) A tree } T \text{ having no perfect matching. (b) } P_3 \times T.\]

pose naturally the following problems.

**Problem 1** Suppose that \( T \) is a tree with even number of vertices containing no perfect matching. Enumerate perfect matchings of \( P_3 \times T \).

**Problem 2** Suppose that \( T \) is a tree and \( m > 4 \). Enumerate perfect matchings of \( P_m \times T \).

**Remark 17** If the tree in Problem 2 is a path \( P_n \), then the number of perfect matchings
of $P_m \times P_n$ equals
\[ 2^{mn} \prod_{k=1}^{m} \prod_{l=1}^{n} \left( \cos^2 \left( \frac{\pi k}{m + 1} \right) + \cos^2 \left( \frac{\pi l}{n + 1} \right) \right)^{\frac{1}{4}}, \]
which was obtained by a physicist, Kasteleyn (see [8,9,11]). It is well known as the dimer problem, which has applications in statistical mechanics.

Acknowledgements

We wish to thank Professor Richard Kenyon for some useful discussions.

References


