A goodness of fit test for copulas based on Rosenblatt’s transformation

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Abstract
A goodness of fit test for copulas based on Rosenblatt’s transformation is investigated. This test performs well if the marginal distribution functions are known and are used in the test statistic. If the marginal distribution functions are unknown and are replaced by their empirical estimates, then the test’s properties change significantly. This is shown in detail by simulation for special cases. A bootstrap version of the test is suggested and it is shown by simulation that it performs well. An empirical application of this test to daily returns of German assets reveals that a Gaussian copula is unsuitable to describe their dependence structure. A $t_v$-copula with low degrees of freedom such as $v = 4$ or $5$ fits the data in some cases.

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1. Introduction

Copulas have become a popular tool for modelling the dependence structure of financial data, such as returns from assets or currencies. A fundamental paper on copulas is Sklar (1959). Joe (1997), Mari and Kotz (2001) and Nelsen (2006) give comprehensive expositions on copulas. Whether or not a certain copula or a parametric family of copulas is suitable for the description of the dependencies in the historical data under study can be investigated by applying specialized goodness of fit tests for copulas. There is a growing number of contributions to this field, see e.g. Mashal and Zeevi (2002), Malevergne and Sornette (2003), Breymann et al. (2003), Savu and Trede (2004), Xiaohong et al. (2004), Dobrić and Schmid (2005), Junker and May (2005), Berg and Bakken (2005), Fermanian (2005) and Genest et al. (2006).

This note addresses a test for parametric families of bivariate copulas based on the Rosenblatt transformation (see Rosenblatt, 1952). This test was suggested and applied in Breymann et al. (2003) and also applied in Dias and Embrechts (2004). In the following it will be called RTT.

This test works by definition in the case where the marginal distribution functions $F_{X_i}$ of the random variables $X_i$, $i = 1, \ldots, d$, are known. We will show in this note, however, that its properties change significantly in the relevant
case where the $F_{X_i}$ are not known, but are replaced by the empirical distribution functions $\hat{F}_{X_i}$ which depend on the observations. Thus the assumption of Breymann et al. (2003) that the test “will not be significantly affected by the use of the empirical distribution functions” is not true.

The structure of this note is as follows. Section 2 introduces the notation and gives a sketch of the RTT for the copulas under consideration. Section 3 presents a Monte Carlo (MC) study which for two different distributional settings shows that these test’s properties change significantly when the marginal distribution functions are unknown. Section 4 introduces a parametric bootstrap version of the RTT for the Gaussian copula. It is shown by simulation that the bootstrap version works well, i.e. it keeps the prescribed level and has power to detect a wrong null hypothesis. The test is applied to German asset returns in Section 5. Section 6 concludes and gives an outlook to extending the procedure from the bivariate to the multivariate case.

2. A goodness of fit test for copulas

Let $X$ and $Y$ denote two random variables with a joint distribution function $F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$ for $(x, y) \in \mathbb{R}^2$ and the marginal distribution functions $F_X(x) = P(X \leq x)$ and $F_Y(y) = P(Y \leq y)$ for $x, y \in \mathbb{R}$. We assume that $F_X$ and $F_Y$ are continuous functions. Therefore, there exists a unique copula $C : [0, 1]^2 \rightarrow [0, 1]$ such that

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)).$$

Here, $C$ is the joint distribution function of the variables $U = F_X(X)$ and $V = F_Y(Y)$, i.e. $C(u, v) = P(U \leq u, V \leq v)$ for $(u, v) \in [0, 1]^2$. The conditional distribution function of $V$ given $U = u$ is defined by

$$C(v \mid u) = \lim_{\Delta u \to 0} \frac{C(u + \Delta u, v) - C(u, v)}{\Delta u} = D_1 C(u, v),$$

where $D_1$ indicates the partial derivative with respect to the first argument, which we assume to exist. According to Rosenblatt (1952) the random variables

$$Z_1 = U = F_X(X)$$

and

$$Z_2 = C(V \mid U) = C(F_Y(Y) \mid F_X(X))$$

are independent and uniformly distributed on $[0, 1]$. Therefore, the random variable

$$S(X, Y) = \left[ \Phi^{-1}(F_X(X)) \right]^2 + \left[ \Phi^{-1}(C(F_Y(Y) \mid F_X(X))) \right]^2$$

has a $\chi^2_2$-distribution. Further, if $(X_1, Y_1), \ldots, (X_n, Y_n)$ is a random sample from $(X, Y)$ then $S(X_1, Y_1), \ldots, S(X_n, Y_n)$ is a random sample from a $\chi^2_2$-distributed random variable. These preliminaries can be used to perform a test for the null hypothesis of interest,

$$H_0: (X, Y) has copula C(u, v),$$

in case where the marginal distribution functions $F_X$ and $F_Y$ are known. In that case the values of $S(X_1, Y_1), \ldots, S(X_n, Y_n)$ can be computed and can be used to test the auxiliary null hypothesis

$$H_0^*: S(X, Y) is \chi^2_2-distributed.$$ 

As $H_0$ implies $H_0^*$ we reject $H_0$ if $H_0^*$ is rejected.
The null hypothesis $H_0^*$ can be tested e.g. with the Kolmogorov test, the Cramér von Mises test or the Anderson Darling (AD) test. We decide to use the AD test because of its excellent power properties against a variety of alternatives (see D’Agostino and Stephens, 1986, Chapter 4). The test statistic is (see Anderson and Darling, 1952, 1954)

$$AD = -n - \frac{1}{n} \sum_{j=1}^{n} (2j - 1) \cdot [\ln(F_0(S(j))) + \ln(1 - F_0(S(n-j+1)))],$$

where $S_j = S(X_j, Y_j)$, $j = 1, \ldots, n$, and $S(1) \leq \cdots \leq S(n)$ are in increasing order. $F_0$ is the distribution function of a $\chi^2$-distributed variable.

At least two problems occur in the empirical application of this approach. First, the hypothesis of interest to be tested is a composite hypothesis in the majority of cases, i.e. we are testing goodness of fit for a parametric family of copulas $C_\theta$ where $\theta \in \Theta \subset \mathbb{R}^d$ and $\theta$ denotes the $d$-dimensional parameter. $\theta$ has to be estimated from the observations which may have an effect on the distribution and the independence of the $S(X_i, Y_i)$, $i = 1, \ldots, n$. Second, the marginal distribution functions $F_X$ and $F_Y$ are usually unknown in applications and are estimated by their empirical versions, $\hat{F}_X(x) = \frac{1}{n} \sum_{k=1}^{n} I\{X_k \leq x\}$ and $\hat{F}_Y(y) = \frac{1}{n} \sum_{k=1}^{n} I\{Y_k \leq y\}$.

Therefore, we cannot compute the $S(X_i, Y_i)$, $i = 1, \ldots, n$. Instead we compute

$$\hat{S}(X_i, Y_i) = [\Phi^{-1}(\hat{F}_X(X_i))]^2 + [\Phi^{-1}(C(\hat{F}_Y(Y_i) | \hat{F}_X(X_i)))^2$$

for $i = 1, \ldots, n$ and the test of the auxiliary null hypothesis $H_0^*$ is based on $\hat{S}(X_i, Y_i)$ instead of $S(X_i, Y_i)$. Note that

$$\hat{F}_X(X_i) = \frac{\text{rank of } X_i \text{ in } X_1, \ldots, X_n}{n}$$

and a similar formula holds for $\hat{F}_Y(Y_i)$. Therefore, $\hat{S}(X_i, Y_i)$ is based on the ranks of $X_i$ and $Y_i$. Note that the two problems mentioned above will occur simultaneously in applications.

### 3. Performance of the test

In this section we will investigate some properties of the RTT described in Section 2 by means of an MC simulation in various settings. The focus is on the error probability of the first kind $\alpha$ and on the power of the test for selected alternatives. In particular, we are interested in whether or not the true error probability of the test (as determined by simulation) corresponds well to the prescribed level (such as $\alpha = 0.1, 0.05$ or 0.01).

It is well known that Gauss copulas and $t$-copulas are possible candidates for the description of the dependence structure of asset returns (see Cherubini et al., 2004). We therefore decide to investigate the properties of the RTT for these copulas (see Section 5 for an empirical application).

**Setting 1: Gauss copula**: The family of Gauss copulas is defined by

$$C_\rho(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left\{ \frac{-(s^2 - 2\rho st + t^2)}{2(1-\rho^2)} \right\} ds \, dr,$$

where $\Phi$ denotes the distribution function of the standard normal distribution and $\Phi_\rho(\ldots)$ denotes the distribution function of the bivariate standard normal distribution with parameter $-1 < \rho < 1$.

We are testing

$$H_0 : (X, Y) \text{ has Gaussian copula } C_\rho,$$

by testing the auxiliary hypothesis $H_0^*$ using the AD test as suggested in Breymann et al. (2003).

We are going to deal with the following cases:

**Case A**: The marginal distribution functions are known and $\rho$ is known.
Table 1  
Error probabilities of the first kind for the RTT for setting I in cases A–C (Results are rounded to two places behind the decimal point.)

<table>
<thead>
<tr>
<th>ρ \ x</th>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x = 0.1</td>
<td>x = 0.05</td>
<td>x = 0.01</td>
</tr>
<tr>
<td>ρ = 0</td>
<td>0.10</td>
<td>0.05</td>
<td>0.01</td>
</tr>
<tr>
<td>ρ = 0.2</td>
<td>0.11</td>
<td>0.06</td>
<td>0.01</td>
</tr>
<tr>
<td>ρ = 0.4</td>
<td>0.10</td>
<td>0.05</td>
<td>0.01</td>
</tr>
<tr>
<td>ρ = 0.6</td>
<td>0.10</td>
<td>0.05</td>
<td>0.01</td>
</tr>
<tr>
<td>ρ = 0.8</td>
<td>0.10</td>
<td>0.05</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Case B: The marginal distribution functions are unknown and are replaced by the corresponding empirical distribution functions \( \hat{F}_X \) and \( \hat{F}_Y \). Also \( \rho \) is unknown and has to be estimated using \( \hat{F}_X(X_i) \) and \( \hat{F}_Y(Y_i) \). Estimation is done in two steps. First the Spearman coefficient of correlation is estimated from \( \hat{F}_X(X_i) \) and \( \hat{F}_Y(Y_i) \) giving \( \hat{\rho}_{Sp} \). Then \( \rho \) is estimated (see Embrechts et al., 2002) by

\[
\hat{\rho} = 2 \cdot \sin \left( \frac{\hat{\rho}_{Sp} \cdot \pi}{6} \right).
\]

Case C: The marginal distribution functions are unknown and are replaced by the corresponding empirical distribution functions \( \hat{F}_X \) and \( \hat{F}_Y \). Further \( \rho \) is assumed to be known.

Obviously only case B is relevant for applications. Cases A and C are considered for comparison and to check the correctness of our programming which was done in MATLAB® (matrix laboratory from MathWorks, Inc.).

The sample size selected is \( n = 2500 \). Note that 2500 is roughly the number of daily returns in 10 years (see Section 5). The number of MC replications is \( M = 5000 \). The critical values of the AD test are 1.9330 for \( \alpha = 0.1 \), 2.4924 for \( \alpha = 0.05 \) and 3.8781 for \( \alpha = 0.01 \) (see Marsaglia and Marsaglia, 2004).

Table 1 contains MC simulations for the true error probabilities of the first kind in cases A–C; the results are easy to interpret. There is excellent agreement of the prescribed and true error probability of the first kind in case A. This is to be expected, because in this case the test works by definition.

In case B, however, the effect of replacing the true marginal distribution functions \( F_X \) and \( F_Y \) by their empirical counterparts \( \hat{F}_X \) and \( \hat{F}_Y \) and estimating \( \rho \) is strong. The true level of the tests is 0.00 regardless of the prescribed level.

Note that the results in case B are essentially due to the use of \( \hat{F}_X \) and \( \hat{F}_Y \) for \( F_X \) and \( F_Y \) and not due to estimation of \( \rho \). This can be seen from the results for case C where \( \rho \) is assumed to be known. The results from cases B and C are nearly identical.

In order to shed some light on the power of the RTT we have to choose special alternatives. The family of \( t_v \)-copulas is of special interest (see Section 5). It is defined by

\[
C_{\rho,v}(u, v) = \int_{-\infty}^{F_{t_v}^{-1}(u)} \int_{-\infty}^{F_{t_v}^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left( 1 + \frac{(s^2 - 2\rho st + t^2)}{v(1-\rho^2)} \right)^{-(v+2)/2} \, ds \, dt,
\]

where \( F_{t_v} \) denotes the univariate distribution function of a \( t_v \)-distribution with \( v \) degrees of freedom (see Demarta and McNeil, 2005). The parameters are \( v \in \mathbb{N} \) and \( -1 < \rho < 1 \). For \( v \to \infty \) the \( t_v \)-copula tends to the Gauss copula.

The power, i.e. the probability of rejecting \( H_0 : (X, Y) \) has a Gauss copula \( C_\rho \), is displayed in Fig. 1 (for \( \rho = 0.4 \)) and Fig. 2 (for \( \rho = 0.8 \)). Here, the line — refers to case A and the line - - - refers to case B. The degrees of freedom \( v \) of the alternative \( t_v \)-copula are graphed on the abscissa. The power is particularly high for small values of \( v \) such as \( v = 1, 2, 3 \) and it becomes smaller for increasing \( v \). In case A power converges to \( \alpha \) which is \( \alpha = 0.1, 0.05 \) and 0.01 for \( v \to \infty \). In case B power converges to 0.00. The difference in power between cases A and B is somewhat astonishing. The power is smaller in case B than in case A for \( v \geq 10 \), and it is substantially higher in case B than in case A for \( v \leq 7 \).

Another alternative of interest is the family of Clayton copulas (see Clayton, 1978),

\[
C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta},
\]
Fig. 1. Probability of rejection of $H_0 : (X, Y)$ has a Gauss copula when the true copula is a $t_{\nu}$-copula as a function of the degrees of freedom $\nu$. The sample size is $n = 2500$, the number of MC replications is $M = 5000$. --- refers to case A and - - - refers to case B. $\rho = 0.4$ was selected for the simulation. The prescribed rejection probabilities are $\alpha = 0.01, 0.05$ and 0.1.

Fig. 2. Probability of rejection of $H_0 : (X, Y)$ has a Gauss copula when the true copula is a $t_{\nu}$-copula as a function of the degrees of freedom $\nu$. The sample size is $n = 2500$, the number of MC replications is $M = 5000$. --- refers to case A and - - - refers to case B. $\rho = 0.8$ was selected for the simulation. The prescribed rejection probabilities are $\alpha = 0.01, 0.05$ and 0.1.

where $\theta \in \Theta = [0, \infty[$. It belongs to the class of Archimedean copulas and has some attractive features for applications. It interpolates the independence copula $\Pi(u, v) = uv$ and the copula of maximal dependence $M(u, v) = \min\{u, v\}$ and can have positive lower tail dependence. Further, generation of random numbers from this family is easy and fast.

For the power simulation the values of $\rho$ and $\theta$ are selected in such a way that Spearman’s coefficient of correlation is equal to $\rho_{Sp} = 0.2, (0.2), 0.8$. The probability of rejection under $H_0$ for cases A and B can be seen in Table 2. The results show again that there is an effect of replacing $F_X$ and $F_Y$ by their empirical counterparts. Indeed, the RTT is not in a position to discriminate between a Gaussian and a Clayton copula if $\rho_{Sp}$ is of small or medium size. The power, however, increases with increasing $\rho_{Sp}$.
Table 2
Rejection probabilities in setting I when the true copula is of the Clayton type

<table>
<thead>
<tr>
<th></th>
<th>Case A</th>
<th>Case B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 0.1$</td>
<td>$\alpha = 0.05$</td>
</tr>
<tr>
<td>$\rho_S = 0.2$</td>
<td>0.16</td>
<td>0.09</td>
</tr>
<tr>
<td>$\rho_S = 0.4$</td>
<td>0.25</td>
<td>0.16</td>
</tr>
<tr>
<td>$\rho_S = 0.6$</td>
<td>0.56</td>
<td>0.44</td>
</tr>
<tr>
<td>$\rho_S = 0.8$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Fig. 3. Probability of rejection for $H_0$ is setting II as a function of the degrees of freedom $\nu$. $\rho = 0.5$ is used for the simulations. The sample size is $n = 2500$ and number of MC replications is $M = 2000$. —– refers to case A and —— refers to case B. The prescribed rejection probabilities are $\alpha = 0.01$, 0.05 and 0.1.

**Setting II: $t_3$-copula:**

The null hypothesis to be tested in this setting is

$$H_0 : (X, Y) \text{ has a } t_3\text{-copula } C_{\rho, 3}.$$  

For the definition of $C_{\rho, v}(u, v)$ see above.

The rejection probabilities of $H_0$ have been simulated using $\rho = 0.5$. The sample size is $n = 2500$ and the number of MC replications was $M = 2000$. $\rho$ has to be estimated under $H_0$ for case B. This is done in the following way. Kendall’s $\tau$ is estimated by

$$\hat{\tau} = \frac{2}{n^2 - n} \sum_{1 \leq i < j \leq n} \text{sign}[R(X_i) - R(X_j)] \cdot \text{sign}[R(Y_i) - R(Y_j)],$$

where $R(X_i) = n \cdot \hat{F}_X(X_i)$ and $R(Y_i) = n \cdot \hat{F}_Y(Y_i)$ for $i = 1, \ldots, n$. Then $\rho$ is estimated (see Embrechts et al., 2002) by

$$\hat{\rho} = \sin\left(\frac{\hat{\tau} \cdot \pi}{2}\right).$$

Rejection probabilities are displayed in Fig. 3 as a function of $\nu$. Looking at $\nu = 3$ (which corresponds to the null hypothesis) one can see that the test keeps the prescribed error probability of the first kind fairly well in case A. It is 0.00 in case B.
4. A bootstrap version of the test

We have shown in the previous section that the goodness of fit test based on the Rosenblatt transformation works in the standard case A, when parameters and marginal distribution functions (which can be viewed as nuisance parameters) are known. In case B, however, the null distribution of the AD test statistic is very different from that in the standard case A. In the latter case the critical values are given in Section 3. The true critical values in case B have been determined by simulation. The AD test statistics for testing \( H_0 \) with \( n = 2500 \) are simulated \( M = 2000 \) times for setting I, case B and \( \rho = 0, 0.4 \) and 0.8. Quantiles are obtained by the empirical \((1 - \alpha)\)-quantile of the values in increasing order. They are displayed in Table 3.

It can be seen that they are much smaller than in case A. Further, there is a strong dependence of the critical values (i.e. \((1 - \alpha)\)-quantiles) on \( \rho \). As \( \rho \) is unknown in case B these critical values cannot be used for testing. Therefore, the critical values have to be determined by bootstrapping (see Efron and Tibshirani, 1993).

A parametric bootstrap procedure for setting I to determine the critical value (i.e. the \((1 - \alpha)\)-quantile) can be described as follows:

1. Estimate \( \rho \) from the original observations \((x_1, y_1), \ldots, (x_n, y_n)\) by
   \[
   \hat{\rho} = 2 \sin\left(\frac{\hat{S}_{Sp} \cdot \pi}{6}\right).
   \]

2. Generate i.i.d. observations \((x^*_1, y^*_1), \ldots, (x^*_n, y^*_n)\) from a Gaussian copula with parameter \( \hat{\rho} \).
3. Estimate \( \rho \) by \( \hat{\rho}^* \) from \((x^*_i, y^*_i), \ i = 1, \ldots, n\), as above and compute \( \hat{S}^*(x^*_1, y^*_1), \ldots, \hat{S}^*(x^*_n, y^*_n) \). The latter are used to compute the value \( AD^* \) of the AD test statistic.
4. Repeat steps (2) and (3) \( N_B \) times, with \( N_B \) the number of bootstrap repetitions. The desired critical value is determined as the \((1 - \alpha)\)-quantile of the values \( AD^{*(1)}, \ldots, AD^{*(N_B)} \).

The null hypothesis of a Gaussian copula is rejected if \( AD \) computed from the original observations \((x_i, y_i), i = 1, \ldots, n\), is larger than the critical value determined in step (4).

We have not yet proved the asymptotic validity of this bootstrap procedure. This can probably be done along the lines suggested in Politis et al. (1999, Chapter 1.8). We have found a very similar parametric bootstrap in Genest et al. (2006) for which the asymptotic validity has already been established (see Genest and Rémillard, 2006).

We have investigated this bootstrap version of the RTT in an MC simulation study. The sample size is \( n = 2500 \) again. Due to the quite substantial computation time the number of MC replications is reduced to \( M = 1000 \). In order to see the effect of the number of bootstrap replications we used \( N_B = 500, 1000 \) and 2000. The results are displayed in Table 4. It can be seen that our bootstrap version of the RTT keeps the prescribed values for \( \alpha \) sufficiently well even for \( N_B = 500 \).

Further, it has power to detect a wrong null hypothesis. There is, however, some effect of \( N_B \) on the power of the test. Indeed, for \( \alpha = 0.05 \) the probability of rejecting \( H_0 \) if the true copula is \( t_{10} \) is 0.88 for \( \rho = 0.4 \) and 0.94 for \( \rho = 0.8 \) if \( N_B = 500 \). It is slightly higher for \( N_B = 1000 \) and 2000. If the true copula is of Clayton type the probability of rejecting \( H_0 \) is 0.63 for \( \rho_{Sp} = 0.4 \) and 1 for \( \rho_{Sp} = 0.8 \) if \( N_B = 500 \). Again power is slightly higher for \( N_B = 1000 \) and 2000. Therefore, looked at from the point of view of power, a large number of bootstrap replications such as \( N_B = 2000 \) are preferable.
Table 4
Performance of the bootstrap version of the RTT for setting I in case B

<table>
<thead>
<tr>
<th>NB</th>
<th>x</th>
<th>Rejection probabilities</th>
<th>For ( t_{10} ) alternatives</th>
<th>Clayton alternatives</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \rho = 0.4 )</td>
<td>( \rho = 0.8 )</td>
<td>( \rho = 0.4 )</td>
</tr>
<tr>
<td>500</td>
<td>0.09</td>
<td>0.10</td>
<td>0.93</td>
<td>0.97</td>
</tr>
<tr>
<td>1000</td>
<td>0.10</td>
<td>0.10</td>
<td>0.95</td>
<td>0.98</td>
</tr>
<tr>
<td>2000</td>
<td>0.10</td>
<td>0.10</td>
<td>0.95</td>
<td>0.98</td>
</tr>
<tr>
<td>500</td>
<td>0.04</td>
<td>0.05</td>
<td>0.88</td>
<td>0.94</td>
</tr>
<tr>
<td>1000</td>
<td>0.05</td>
<td>0.05</td>
<td>0.91</td>
<td>0.95</td>
</tr>
<tr>
<td>2000</td>
<td>0.05</td>
<td>0.05</td>
<td>0.91</td>
<td>0.95</td>
</tr>
<tr>
<td>500</td>
<td>0.01</td>
<td>0.01</td>
<td>0.73</td>
<td>0.82</td>
</tr>
<tr>
<td>1000</td>
<td>0.01</td>
<td>0.01</td>
<td>0.74</td>
<td>0.85</td>
</tr>
<tr>
<td>2000</td>
<td>0.01</td>
<td>0.01</td>
<td>0.74</td>
<td>0.85</td>
</tr>
</tbody>
</table>

Table 5
Number of rejections of the null hypotheses \( H_0 : (X, Y) \) has a \( t_v \)-copula (for \( v = 1, \ldots, 10 \)), using the bootstrap version of the RTT

<table>
<thead>
<tr>
<th>Degrees of freedom ( v )</th>
<th>Number of rejections</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>28</td>
</tr>
<tr>
<td>2</td>
<td>28</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>21</td>
</tr>
<tr>
<td>5</td>
<td>21</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
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<tr>
<td>7</td>
<td>25</td>
</tr>
<tr>
<td>8</td>
<td>27</td>
</tr>
<tr>
<td>9</td>
<td>28</td>
</tr>
<tr>
<td>10</td>
<td>28</td>
</tr>
</tbody>
</table>

Data are daily returns of eight German assets.

5. Application of the RTT to financial data

We consider the daily returns of eight German assets which are included in the German DAX 30. These assets are Deutsche Bank, Hypovereins Bank, BASF, Bayer, BMW, VW, SAP and Siemens. The daily returns are from 28.02.1992 to 01.03.2002, giving \( n = 2610 \). It is a stylized fact of daily asset returns that their marginal distributions are not Gaussian. Indeed, we observe strong leptokurtosis which entails more peakedness and fatter tails than that of a Gaussian distribution. Consequently, the joint distribution of asset returns cannot be Gaussian either, because this would entail Gaussian margins.

Another problem is whether the copula of the joint distribution is of the Gaussian type. Note that the Gaussian copula of the joint distribution is well compatible with non-Gaussian margins. We will investigate this problem by applying the bootstrap version of RTT. In order to keep things as simple as possible we investigate only the bivariate distributions of the eight assets under study. For every pair of asset returns \((X, Y)\), we test the null hypothesis \( H_0 : (X, Y) \) has a Gaussian copula. There are \( \binom{8}{2} = 28 \) tests. The prescribed level is \( \alpha = 0.05 \).

Our empirical investigation yields a clear result: the null hypothesis of a Gaussian copula is rejected in every case. The Gaussian copula is therefore unsuitable for the description of the dependence structure of daily returns. These findings confirm the results of Dobrić and Schmid (2005). In the latter paper a modified \( \chi^2 \)-test of fit was applied to test the above null hypothesis with the same set of data. The null hypothesis was also rejected in every case. Our empirical results are in line with those of Mashal and Zeevi (2002) who also claim that a Gaussian dependence structure is constantly rejected.

In order to find a more suitable model we now test the null hypothesis \( H_0 : (X, Y) \) has a \( t_v \)-copula. The degrees of freedom considered are \( v = 1, \ldots, 10 \). The empirical results are summarized in Table 5.

The second line of Table 5 contains the number of rejections of \( H_0 \) using the bootstrap version of RTT (which was modified for the \( t_v \)-copula in a straightforward way). It can be seen that there are now some pairs of assets where a \( t_v \)-copula with low degrees of freedom such as \( v = 4 \) or \( 5 \) is a possible model for their dependence structure. Note that similar findings have been made in Dobrić and Schmid (2005) and Mashal and Zeevi (2002).

However, the question “which copula fits the data” remains. Possible candidates are general elliptical copulas (see Fang et al., 2002; Frahm et al., 2003) or mixtures of two families of copulas which describe the dependence structure
more flexibly than a $t_r$-copula. Due to the large sample size $n = 2610$ and the good power properties of the bootstrap version of the RTT it is expected, however, that even more flexible models will be rejected for some combinations of two assets.

6. Conclusion and outlook

The Rosenblatt transformation can be applied to copulas in order to obtain a test of fit for copulas. This approach to goodness of fit testing can in principle be used for every parametric family of copulas. The computation of the test statistic is in general not difficult even in the case of copulas of high dimension. Procedures based on Rosenblatt’s transformation involve conditioning on successive components of the random vector and depend on the order in which this conditioning is done.

A serious problem, however, arises with the determination of the distribution of the test statistic. If marginal distributions are unknown (case B)—which is always the case in empirical applications—one has to use the empirical distribution functions, which means that the test is based on ranks and the distribution of the test statistic greatly differs from the standard case where the marginal distribution is known (case A).

Using the bivariate Gaussian copula as an example in this paper we demonstrated by simulation that using critical values of the standard case (case A) makes the test useless because the true rejection probability under $H_0$ is zero and there is reduced power for rejecting a wrong null hypothesis.

A remedy is the parametric bootstrap which we suggest for the determination of the critical values. Again using the bivariate Gaussian copula as an example we show by simulation that the bootstrap version of the RTT works well. A generalization of this parametric bootstrap to further families of copulas is straightforward.

The present paper is about goodness of fit testing for bivariate copulas. But the bootstrap version of the RTT can be extended to higher dimensions $d > 2$ though this will require a much higher computational effort. An important prerequisite is that the conditional distribution function of $U_i$ given $U_{i-1} = u_{i-1}, \ldots, U_1 = u_1$, i.e.

$$C(u_i|u_{i-1}, \ldots, u_1) = P(U_i \leq u_i|U_{i-1} = u_{i-1}, \ldots, U_1 = u_1)$$

can be computed efficiently for $i = 2, \ldots, d$. Therefore explicit formulas are useful. These formulas are available for the Gaussian copula and the Clayton copula. A more serious problem seems to be, however, the estimation of parameters. This has to be done once in step (1) of the bootstrap procedure for the empirical data and $B$ times in step (3) for the bootstrapped samples. As the number of parameters usually increases with $d$, computational problems will occur. For the Gaussian copula in $d$ dimensions, e.g. there are $d(d-1)/2$ parameters to be estimated. Further, the matrix of estimated parameters has to be checked for positive definiteness. Implementation of such a procedure is difficult and easily becomes numerically instable. The number of parameters should therefore be kept as small as possible in order to obtain a procedure that works within reasonable computation time. Nevertheless, implementation of the bootstrap version of the RTT in higher dimensions should be addressed in further work.

References


