The Medial Scaffold of 3D Unorganized Point Clouds

Frederic F. Leymarie and Benjamin B. Kimia

Abstract—We introduce the notion of the medial scaffold, a hierarchical organization of the medial axis of a 3D shape in the form of a graph constructed from special medial curves connecting special medial points. A key advantage of the scaffold is that it captures the qualitative aspects of shape in a hierarchical and tightly condensed representation. We propose an efficient and exact method for computing the medial scaffold based on a notion of propagation along the scaffold itself, starting from initial sources of the flow and constructing the scaffold during the propagation. We examine this method specifically in the context of an unorganized cloud of points in 3D, e.g., as obtained from laser range finders, which typically involve hundreds of thousands of points, but the ideas are generalizable to data arising from geometrically described surface patches. The computational bottleneck in the propagation-based scheme is in finding the initial sources of the flow. We thus present several ideas to avoid the unnecessary consideration of pairs of points which cannot possibly form a medial point source, such as the “visibility” of a point from another given a third point and the interaction of clusters of points. An application of using the medial scaffold for the representation of point samplings of real-life objects is also illustrated.

Index Terms—3D shape understanding, object representation, geometric algorithms, 3D medial axis, shock graphs, flow singularities, 3D bucketing, visibility constraints.

1 INTRODUCTION

Many of the fields of application of computerized technologies, such as machine tooling, biomedical engineering, architectural simulators, and others, require sophisticated interaction with objects using some representation of their shape. Mathematical representations, such as implicit polynomials [9], generalized cylinders [8], superquadrics and splines [33], and others have been studied over the years in trying to address such maturing needs. The Medial Axis (MA; see Table 1 for main symbols) of Blum [10] is a generic representation for describing a variety of shapes ranging from the highly regular and inorganic, such as those crafted by humans or found in crystal aggregates, to the more irregular and organic, such as bodies, cells, leaves, arteries, etc., [32]. The MA has been viewed with great promise as a universal model for shape since

1. it is intuitively appealing in representing elongated and branching objects, such as anthropomorphic forms,
2. it reflects the varying width of forms by incorporating a radius function along the MA trace,
3. important outline features, such as curvature extrema and ridges, are made explicit by the MA branch “tips” [33], [43],
4. a hierarchy of scales is “built-in” via the combination of spatial and width properties, i.e., smaller features can be distinguished from larger ones and ranked accordingly [11],
5. it is complete, i.e., unless one starts pruning the MA’s branch structure, all of the object is represented, ensuring the possibility of an exact reconstruction [56], [20], and
6. the MA provides a powerful framework to model shape dynamics (including perturbations, deformations, and kinematics) and to study object generation [57], [34], [19].

These features of the MA address many of the desirable characteristics: a representation of shape ought to have, e.g., they meet many of Mumford’s proposed criteria [36]. However, compared to the well-studied 2D case, three significant issues arise when dealing with the 3D MA: 1) Its structure is more complicated due to the interactions of medial sheets, curves and vertices, 2) typical applications involve much larger input data sets, e.g., from thousands to millions of points sampling an object’s surface, and 3) it is not a priori clear how to represent the qualitative structure of the shape, such as regional features, in a reduced fashion.

Algorithms for 3D MA computation typically focus on deriving the geometric locus of skeletal surfaces and, thus, usually do not make explicit the local connectivity in the interior of each MA sheet as well as in the joints, where three or more sheets come together (Fig. 1). These methods can be organized into six main approaches (for a more extensive review, see [28, chapter 2]):

1. thinning iteratively peels off discrete layers of objects until one is left with an approximate MA loci, such as in mathematical morphology [41],
2. ridge following on distance maps also uses a discrete grid [35],
3. Partial Differential Equations (PDE) modeling wave propagation are numerically simulated to perform...
grassfire-like transforms in the spirit of Blum’s model [10], where background space is lit up by fires initiated at boundary loci and where quenching wavefronts (or “shocks”) denote MA loci [45].

4. **primitive** shapes for which known medial representations are directly available can be retro-fit to the data [58],

5. the Voronoi Diagram (VD) of either point samplings for arbitrary shapes or of simple polyhedral meshes for solids have been studied as approximations to the MA [42], [2], [16], and

6. full bisector computations are followed by trimming¹ operations to define generalized descriptions of the MA [25], [18].

In addition to difficulties in capturing the topology of 3D MA features, the need for data reduction has not been fully addressed by past methods as it is not a priori clear how to summarize 2D MA sheets into a lower dimensional structure. While the interesting notions of curve skeletons [13], ik-skeletons [55], skeleton graphs [49], and Reeb graphs [54], [24], which provide 3D graph-like descriptions of shapes, have recently been explored, they have important limitations: 1) The resulting graphs are often oversimplified and do not always capture essential geometric features such as surface ridges, 2) a well-segmented 3D object (with an inside and outside) is typically required, and 3) some methods further require user-specified surface features which serve as graph end-points.

The ideal algorithm for the recovery of the 3D MA should combine the advantages of this rich set of approaches while avoiding their limitations. Specifically, we seek a method which, on the one hand, features the exactness of bisector computations [38, chapter 11], but when stripped of some of their computational limitations such as geometric intersections and trimming and, on the other hand, features the flow-based nature of Blum’s grassfire [10], which is useful for qualitative shape descriptions [11], [52] and which underlies thinning, distance transforms, and surface evolutions, but without their connectivity, added dimension, and postprocessing issues. A key insight which unifies these approaches in our work, initiated in 3D in Leymarie’s PhD thesis [28], is motivated by the idea first proposed by Tek and Kimia for 2D problems that the full bisectors need not be considered if a flow-based approach is adopted [50], [52]. Specifically, if the initial sources of flow are completely classified, only portions of the bisectors which initiate from these need to be computed, leading to substantial savings. This also immediately leads to an oriented graph structure on the bisector paths which captures local connectivity and exact results and forms a dynamically computed Lagrangian grid to order computations (Fig. 2). Traditional approaches which use the image grid (Eulerian approach) can waste time and memory in explicitly tracing medial sheets and curves, while the computational bottleneck in the Lagrangian method is in identifying sources of the flow.²

1. When using bisectors to represent MA surfaces, explicit methods are required to 1) carefully trace their intersections forming medial curves and 2) trim away those parts which are not on the MA proper [18].

2. Note that, in generating and using a Lagrangian grid, we do not need to further discretize the results as the grid is the useful output, i.e., a 3D graph structure with all the necessary and sufficient information, which, e.g., can be directly used for visualization.

### Table 1

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>MA</td>
<td>Medial axis</td>
</tr>
<tr>
<td>MS</td>
<td>Medial scaffold</td>
</tr>
<tr>
<td>VD</td>
<td>Voronoi diagram</td>
</tr>
<tr>
<td>SS</td>
<td>Symmetry Set</td>
</tr>
<tr>
<td>G_i</td>
<td>Input generator</td>
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</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1^2</td>
<td>Shock sheet</td>
</tr>
<tr>
<td>A_3^3</td>
<td>Axial shock curve</td>
</tr>
<tr>
<td>A_3</td>
<td>Rib shock curve</td>
</tr>
<tr>
<td>A_1,A_3</td>
<td>Rib end (vertex)</td>
</tr>
<tr>
<td>A_4^4</td>
<td>Axial end (vertex)</td>
</tr>
</tbody>
</table>

The A_i^l notation is detailed in Section 2.
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An iterative process of pairing detected sheets and curves leads to the scaffold nodes and their connectivity. Shock sheets and curves need not be traced explicitly in this approach. (b) In the Eulerian method, input elements are located with respect to a fixed grid. A discrete propagation from each input, $G_i$, is then performed via the grid. The grid nodes closest to each input are then labeled accordingly (indicated by a shade of gray). Elementary cells of the grid where two or more labels arrive at the “same time” are used to compute shocks. Eulerian propagation can lead to connectivity errors while the Lagrangian scheme is exact, but it can be computationally prohibitive if not properly handled.

This paper presents two major contributions: First, we propose a new representation for the 3D medial axis hierarchically organized around a graph: the medial scaffold ($MS$). The term “scaffold” is used in analogy to building constructions, where a set of metallic beams supports relatively weaker materials, so as to indicate the relative significance of curves over sheets, and nodes over curves, in describing the qualitative shape of an object. The graph structure makes the representation useful for analysis and recognition, while preserving $MA$ properties (e.g., uniqueness) and simplifying the computation of the $MA$, its data management, and its visualization since only special points, called medial nodes, need to be explicitly detected. Second, we introduce an efficient method for computing the $MS$ based on a propagation from “shock sources” and implement it for unorganized 3D points [31], e.g., sampling object surfaces whose local connectivity is not available or is deemed unreliable. This special case is significant since it is the native type of free-form data acquired with modern scanning devices, such as laser or computerized tomography (CT) scanners. Local connectivity information, when available, is often unreliable due to gaps or overlaps and inconsistencies in the data [6]. Furthermore, this form of representation allows us to process data “on-the-fly” as they are acquired from scanners and it does not require a “discretization” of the input data in the form of voxels, i.e., sample points can be immediately processed using their real coordinates. We note that the approach developed for computing the $MS$ from unorganized point clouds is fully general and can be applied to other geometric inputs such as unorganized polygonal patches [28, chapter 7].

![Fig. 2. From [52]: Lagrangian versus Eulerian methods of computing the $MS$.](image)

(a) In the Lagrangian method, input elements or generators, $G_i$, are paired to create initial shocks for sheets (shown in a cross-section as thick dotted lines), represented by $A_1 - 2$ (small squares), and the latter are paired to create initial shocks for curves, represented by $A_1 - 2$ (small triangles). An iterative process of pairing detected sheets and curves leads to the scaffold nodes and their connectivity. Shock sheets and curves need not be traced explicitly in this approach. (b) In the Eulerian method, input elements are located with respect to a fixed grid. A discrete propagation from each input, $G_i$, is then performed via the grid. The grid nodes closest to each input are then labeled accordingly (indicated by a shade of gray). Elementary cells of the grid where two or more labels arrive at the “same time” are used to compute shocks. Eulerian propagation can lead to connectivity errors while the Lagrangian scheme is exact, but it can be computationally prohibitive if not properly handled.

The paper is organized as follows: We first define shock points on the basis of order of contact and identify special shocks at the singularities of the associated radius flow (Section 2). This specifies the set of possible medial nodes, which, once connected by following the flow in the direction of increasing radius, leads to a graph, the medial scaffold (Section 3). Then, we consider the case where input generators are sets of unorganized points (Section 4) and develop a Lagrangian scheme for the computation of the $MS$ (Section 5). Finally, we illustrate the method with synthetic and real examples (Section 6).

2 A CLASSIFICATION OF SHOCK POINTS: CONTACT WITH SPHERES

The Symmetry Set ($SS$) is the closure of the loci of centers of spheres tangent to the surface at two or more points; such bitangent spheres are called “contact spheres.” The $MA$ is the subset of the $SS$ for which all such spheres are maximal, i.e., they do not contain any points of the surface other than the contact points. The shock structure arises from a “dynamic” interpretation of the $MA$, as in Blum’s grassfire [10], where the locus of singularities, or shocks, formed in the course of wave propagation from boundaries have an associated direction and speed of flow. In 2D, shock segments are those segments of the $MA$ which have monotonic flow; this grouping of shocks into segments is a more refined partition of the $MA$ than by grouping between $MA$ junctions and endpoints [26]. Shocks are obtained in 2D either by detecting the singularities of the evolving boundary in a curve evolution (PDE) approach [26], [45] or in a mixed Eulerian-Lagrangian propagation which combines wave propagation and computational geometry concepts [50], [52].

The notion of a $MS$ relies on the classification of shock points described by Giblin and Kimia in 2D [21] and in 3D [22], based on the order of contact with tangent balls. Let $A^n_k$ denote the order of contact of a circle (in 2D) or a sphere (in 3D) as tangency to the surface boundary at $n$ distinct points, each with $k + 1$ degree of contact, Fig. 3: $k = 1$ denotes regular tangency at a contact point $p$; $k = 2$ denotes a sphere of curvature (i.e., with principal radius, but $p$ is not part of a ridge); $k = 3$ denotes a sphere of curvature at a ridge point (i.e., such that $p$ is a curvature extremum along a principal curve); $k = 4$ denotes a sphere of curvature at a turning point of a ridge (i.e., where the ridge becomes tangent to the line of curvature at $p$), etc. [23, chapter 6]. Only odd orders of contact (i.e., $k = 1, 3$) can contribute to the $MA$, that is, being the center of a maximal sphere, having a locally isolated intersection with the surface. Then, a classification based on the number and order of contact leads to five principal types of shock points (Fig. 5a) [22].

3. The main symbols used in this paper are listed in Table 1. The medial axis ($MA$) in 3D is made of sets of intersecting medial surfaces, while, in 2D, it is made of intersecting curves. The symbol for the medial scaffold, $MS$, is used both to relate it and to distinguish it from the $MA$.

4. If $n = 1$, we do not write it and use the reduced notation $A_k$. 

Fig. 3. From [22]: Illustration of the notation $A^n_k$ based on contact of a curve (surface) with a circle (sphere); $k + 1$ counts order of contact; $n$ is the number of contact points, e.g., $A^2_1$ means two $A_1$ contacts.

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1. $A^2_1$ contact (sheet): This is a sphere with two ordinary $A_1$ contacts; by adjusting the radius and rolling this sphere between two surface patches, we encounter other $A^2_1$ points such that the local form of $A^2_1$ is a medial sheet which is locally smooth.

2. $A_3$ contact (rib): This is the limiting case when two $A^2_1$ points come together. $A_3$ points form a space curve spanning centers of one of the principal curvatures. This space curve is a rib boundary for an $A^2_1$ sheet, and the associated contact points correspond to surface ridges.

3. $A^2_1$ contact (axial curve): The contact sphere has three ordinary $A_1$ contacts; the local form is a curve where three sheets come together.

4. $A_1 A_3$ contact (rib end): The sphere has $A_1$ contact (i.e., ordinary tangency) at one point and $A_3$ contact at another. The local form is that of an isolated point where a pair of $A^2_1$ and $A_3$ curves meet and terminate.

5. $A^2_1$ contact (axial end): The contact sphere has four ordinary contacts. The local form is also that of an isolated point which is at the intersection of six smooth $A^2_1$ sheets of the $MA$ or, alternatively, at the intersection of four $A^2_1$ curves (i.e., either six distinct pairs or four distinct triplets from four contact points).

Two observations from Giblin and Kimia are significant here [22]. First, the $MA$ points naturally organize into sheets, curves, and points. $A^2_1$ points are interior points of a medial sheet. Each sheet is bounded by a collection of $A^2_1$ and $A_3$ curves. Each $A^2_1$ curve ends at either an $A_1 A_3$ or $A^2_1$ point. Each $A_3$ curve ends at an $A_1 A_3$ point. Second, in addition to a typology based on order of contact and local form, a notion of flow for each $MA$ point in the direction of increasing radius, $r$, of its associated maximal contact spheres leads to a finer classification of the $MA$ in the form of a shock structure both in 2D [26], [21] and in 3D [22]. Flow is a vector field, defined on the $MA$, whose typology we denote via $l$: Shocks can be regular (first order: $l = 1$), act as sources and initiate flow (second order: $l = 2$), act as relays (third order: $l = 3$), or act as sinks and terminate flow (fourth order: $l = 4$), Figs. 4a and 4b. For $A_1 A_3$ vertices where a pair of $A_3$ and $A^2_1$ curves meet, three of four flow configurations are possible [22]: 1) Both curves can flow outward (i.e., a source), 2) the $A^2_1$ can flow outward and the $A_3$ flow inward (i.e., a relay), and 3) both curves flow inward (i.e., a sink), i.e., the configuration where $A^2_1$ is flowing in and $A_3$ is flowing out is not possible. For $A^2_1$ vertices, the classification is based on the number of inward flows as dictated by the four intersecting $A^2_1$ curves, where either two, three, or four curves can flow inward, Fig. 4c, ruling out other configurations. Table 2 summarizes the notation, $A^2_1 - l$, which combines sphere contact $A^2_1$, with flow, $l (=1, 2, 3, 4)$.
Fig. 6. From the “classical” MA static representation to the medial scaffolds. (a) Typical situation in 3D, where three medial sheets intersect into a medial curve. (b) Representation by the augmented MS, where medial nodes along curves are connected by links; the hyperlinks’ cyclic order is indicated by a counterclockwise arrow. (c) Representation by the MS, where the interior of sheets is implicit. (d) The reduced MS is obtained from the MS by discarding the geometry of shock curves. (e) The topological MS is obtained from the MS when only the topology of the graph structure is preserved. Red dots correspond to MA vertices, i.e., \( A_1 \) or \( A_1^3 \). Green triangles correspond to shock sources of curves, e.g., \( A_1^1 - 2 \) points, which are needed for capturing the MS. \( A_1^1 \) and \( A_3 \) curves are shown in red and blue, respectively. (a) \( MA \), (b) \( MS^+ \), (c) \( MS \), (d) \( MS^- \), and (e) \( TMS \).

3 THE MEDIAL SCAFFOLD HIERARCHY

We now define the medial scaffold (MS), a 3D graph representation of a shape. The key insight is that slight deformations of the shape do not generally affect the topological relations among the “special medial points” \( A_1^1 \), \( A_1^3 \), \( A_1^3 - 2 \), \( A_3 - 2 \), as indicated by the medial curves \( A_1^1 \) and \( A_3 \) connecting them, while the geometry and flow dynamics of the medial sheets and curves can change. The MS is a graph representation of all medial points making explicit their hierarchical organization.

The typology described in the previous section is required for a construction of a graph in analogy to the notion of a shock graph in 2D [46] which was successfully used in recognition [40]. Similarly, in 3D we identify flow singularities and organize these in three classes: 1) shock sources: a) \( A_1^1 - 2 \) along a sheet, b) \( A_1^3 - 2 \) and \( A_3 - 2 \) along a curve, and c) \( A_1A_3 - 2 \) at a vertex; 2) shock relays: a) \( A_1^3 - 3 \) along a sheet, b) \( A_1^3 - 3 \) and \( A_3 - 3 \) along a curve, and c) \( A_1^1 - 2 \), \( A_1^3 - 3 \) and \( A_1A_3 - 3 \) at a vertex; and 3) shock sinks: a) \( A_1^1 - 4 \) along a sheet, b) \( A_1^3 - 4 \) and \( A_3 - 4 \) along a curve, and c) \( A_1^3 - 4 \) and \( A_1A_3 - 4 \) at a vertex. The remaining shocks act as linking structures consisting of 1) \( A_1^1 - 1 \) along a sheet and 2) \( A_1^3 - 1 \) and \( A_3 - 1 \) along a curve. The structure which embeds this hierarchical notion is a graph with special medial points serving as its nodes, medial curves acting as links between nodes, and medial sheets acting as hyperlinks [7] bringing together several nodes.

Definition 3.1 (Shock and medial nodes, links, hyperlinks). Shocks of type \{\( A_1A_3 \), \( A_1^1 \), \( A_1^3 - 2 \), \( A_1^3 - 3 \), \( A_1^3 - 4 \), \( A_3 - 2 \), \( A_3 - 3 \), \( A_3 - 4 \), \( A_1^1 - 2 \), \( A_1^3 - 3 \), \( A_1^3 - 4 \)\} will be referred to as shock nodes. A medial node refers to shock nodes of type \( A_1A_3 \), \( A_1^1 \), \( A_1^3 - 2 \), and \( A_3 - 2 \). A medial link is the medial curve segment between medial nodes. A medial hyperlink is an ordered, cyclic set of medial nodes representing a medial sheet via its boundary.

Thus, while a shock (point) is equivalent to an MA point, a shock node is a shock point which is also a singularity of the flow, and medial nodes constitute a subset of all shock nodes. Note that both links and hyperlinks not only represent connectivity among nodes, but also have associated attributes of geometry and dynamics: how shocks flow the geometrically defined region. We can now define the first level in our MS hierarchy, which augments the MA with an oriented graph structure.

Definition 3.2 (Augmented medial scaffold, \( MS^+ \)). The augmented MS is the hypergraph whose nodes are the set of medial nodes connected by medial links and medial hyperlinks.

The advantage of the augmented graph structure over the (“classical”) trace of the MA is that it organizes the MA information into groups and specifies their connectivity. It is precisely the connectivity among these groups which contains the qualitative information, while the remaining information allows for an exact reconstruction or an approximation of the shape from the shock structure [20], [22]. If we make implicit in the \( MS^+ \) the geometry and dynamics of the hyperlinks, which contain the explicit representation of the sheets and their interior, we are left with an “ordinary” oriented graph structure which defines the connectivity among the retained medial nodes via explicit links. This graph summarizes the MA at the next level (Figs. 5 and 6).

Definition 3.3 (Medial scaffold, MS). The MS is a graph made of medial nodes and medial links.

We note that the MS is a geometric graph, i.e., without self-intersections, and is not a tree in general, i.e., it contains circuits (chains of links forming closed loops) bounding MA sheets. Despite the lack of an explicit representation of sheets, the MS alone gives a fairly good idea of the shape of the object due to the remaining connectivity. The MA can be approximated by interpolating the missing MA sheet points by stretching smooth elastic surfaces over the links, much as is done when a “tent” is constructed over a scaffold. If we also make implicit the representation of medial curves or both medial curves and medial nodes, we obtain yet simpler graphs.

Definition 3.4 (Reduced medial scaffold, \( MS^- \)). It is the MS where medial link attributes (i.e., geometry and dynamics) have been discarded retaining only the topological connectivity.

Definition 3.5 (Topological medial scaffold, \( TMS \)). It is the \( MS^- \), where node geometry and dynamics have been discarded.

This hierarchical representation for the MA (Table 3) is illustrated in Fig. 6.
TABLE 3
The Medial Scaffold Hierarchy Is Comprised of Four Levels

<table>
<thead>
<tr>
<th>Level</th>
<th>Symbol</th>
<th>Distinctive features</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\mathcal{MS}^+$</td>
<td>Medial nodes, links &amp; hyperlinks.</td>
</tr>
<tr>
<td>II</td>
<td>$\mathcal{MS}$</td>
<td>Graph: medial nodes &amp; links.</td>
</tr>
<tr>
<td>III</td>
<td>$\mathcal{MS}^-$</td>
<td>Links stripped of geometry.</td>
</tr>
<tr>
<td>IV</td>
<td>$\mathcal{TMS}$</td>
<td>Nodes &amp; links stripped of geometry.</td>
</tr>
</tbody>
</table>

TABLE 4
Point Generators Create Only Eight Shock Types, Six of which Are Shock Nodes, i.e., Singularities of the Flow

<table>
<thead>
<tr>
<th>Shocks</th>
<th>Regular</th>
<th>Source</th>
<th>Relay</th>
<th>Sink</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sheet</td>
<td>$A_1^2-1$</td>
<td>$A_2^2-1$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Axial</td>
<td>$A_1^2-1$</td>
<td>$A_2^2-1$</td>
<td>$A_3^2-3^*$</td>
<td>-</td>
</tr>
<tr>
<td>Axial end</td>
<td>-</td>
<td>-</td>
<td>$A_1^2-2$, $A_1^2-3$</td>
<td>$A_1^2-4$</td>
</tr>
</tbody>
</table>

* The $A_1^2-3$ relay for axial curves can only occur in degenerate configurations.

4 SHOCK NODES FOR UNORGANIZED POINT CLOUDS

In the remainder of this paper, we will consider surface data sets represented by clouds of unorganized points where no a priori proximity relationship is known between the points.\(^5\)

Since points are meant to represent a surface patch whose orientation is not a priori clear, we represent all possible such patches by using an infinitesimal small sphere around each point and call it a point generator, denoted by $G_i((x_i, y_i, z_i)$, where $x_i, y_i,$ and $z_i$ are the coordinates of the point also representing the center of the tiny sphere. We use the name “generator” in analogy to sources when simulating wavefront propagation on a fixed (“Eulerian”) grid from points or spheres to compute the $\mathcal{M}_A$ [50], [28]. For this restricted class of surfaces, i.e., spheres, certain types of shocks do not occur:

There are no $A_3$ curves (ribs) since there are no ridges on spheres nor any $A_1, A_3$ vertices (rib ends) by extension; also, the geometry implies that there are no shock sinks and relays in the interior of sheets nor shock sinks in the interior of curves, so no $A_3^2-4$, $A_3^2-3$, and $A_3^2-4$ occur. Hence, when considering point generators, out of the 18 shock points in Table 2, we are left with only eight possible ones, Table 4, six of which are singularities for the flow, i.e., shock nodes. In the following proposition, we describe the five nondegenerate shock nodes using the barycentric coordinates of Möbius [12], first to easily differentiate their flow topology, e.g., sink versus relay, and second to enjoy the numerical robustness arising from formulae expressed in terms of intergenerator distances, thus avoiding the pitfalls of relying on extrinsic coordinates.\(^7\)

5. Our results are not restricted to surface samplings however, and can also be applied to more general 3D point data sets, such as dense volumetric samplings of cloud data in meteorology.

6. For point generators, the $A_1^2-3$ shock singularity is a degenerate case of a relay, i.e., an $A_1$ curve segment bounded by a pair $A_1$ vertices with equal radius value. It is detected when linking the medial nodes to create the $\mathcal{MS}$.

7. Furthermore, 1) intergenerator distances are often precomputed and reused, 2) overlaps are easily deleted, 3) normalized barycentric coordinates in affine space are used (their sum is always one). Also, $|\cdot|^n$ denotes the use of the $p$-norm or Euclidean metric and $\land$ the vector product.

Proposition 4.1 (Barycentric description and classification of shock nodes).

1. Two generators $G_i$ and $G_j$ give rise to an $A_3^2-2$ shock source candidate located at $O_2$ (Fig. 7a):

$$O_2 = \alpha G_i + \beta G_j, \quad (\alpha, \beta) = (0.5, 0.5).$$

2. Three generators $G_i$, $G_j$, and $G_k$ give rise to an $A_3^2-2$ shock sink candidate at $O_3$ (Fig. 7b):

$$O_3 = \alpha G_i + \beta G_j + \gamma G_k,$$

$$\left(\alpha, \beta, \gamma\right) = \frac{1}{9\Delta^2} \begin{bmatrix} G_i G_j^2 & G_i G_k^2 & G_k G_i^2 \\ G_j G_i^2 & G_j G_k^2 & G_k G_j^2 \\ G_k G_i^2 & G_k G_j^2 & G_i G_k^2 \end{bmatrix},$$

$$16\Delta^2 = \begin{bmatrix} 2\alpha^2 + 2\beta^2 + 2\gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma - 2\left(\alpha + \beta + \gamma\right) \\ 2\alpha^2 + 2\beta^2 + 2\gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma - 2\left(\alpha + \beta + \gamma\right) \\ 2\alpha^2 + 2\beta^2 + 2\gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma - 2\left(\alpha + \beta + \gamma\right) \end{bmatrix},$$

where the last equation is derived from Heron’s formula for the triangle area, $\Delta$. Moreover, a) if all coordinates $(\alpha, \beta, \gamma)$ are positive the triangle is acute and no shock sheet arises at this point, while b) if only one of $(\alpha, \beta, \gamma)$ is negative, the triangle is obtuse and one new shock sheet emerges, not identified by an $A_3^2-2$ source, Fig. 8. Other combinations are not possible.

3. Four generators $G_i$, $G_j$, $G_k$, and $G_l$ give rise to an $A_3^2$ shock vertex candidate at $O_4$ (Fig. 7c):

$$O_4 = \alpha G_i + \beta G_j + \gamma G_k + \delta G_l,$$

$$\alpha = \frac{1}{288V^2} \begin{bmatrix} 2\alpha G_i G_j^2 & 2\alpha G_i G_k^2 & 2\alpha G_i G_l^2 \\ 2\beta G_j G_k^2 & 2\beta G_j G_l^2 & 2\beta G_k G_l^2 \\ 2\gamma G_k G_l^2 & 2\gamma G_k G_i^2 & 2\gamma G_l G_i^2 \end{bmatrix},$$

$$\begin{bmatrix} 2\alpha G_i G_j^2 & 2\alpha G_i G_k^2 & 2\alpha G_i G_l^2 \\ 2\beta G_j G_k^2 & 2\beta G_j G_l^2 & 2\beta G_k G_l^2 \\ 2\gamma G_k G_l^2 & 2\gamma G_k G_i^2 & 2\gamma G_l G_i^2 \end{bmatrix} = \begin{bmatrix} 2\alpha G_i G_j^2 & 2\alpha G_i G_k^2 & 2\alpha G_i G_l^2 \\ 2\beta G_j G_k^2 & 2\beta G_j G_l^2 & 2\beta G_k G_l^2 \\ 2\gamma G_k G_l^2 & 2\gamma G_k G_i^2 & 2\gamma G_l G_i^2 \end{bmatrix} \begin{bmatrix} 2\alpha G_i G_j^2 & 2\alpha G_i G_k^2 & 2\alpha G_i G_l^2 \\ 2\beta G_j G_k^2 & 2\beta G_j G_l^2 & 2\beta G_k G_l^2 \\ 2\gamma G_k G_l^2 & 2\gamma G_k G_i^2 & 2\gamma G_l G_i^2 \end{bmatrix},$$

where $V$ denotes the volume of the tetrahedron defined by the generators. Moreover, a) if all coordinates $(\alpha, \beta, \gamma, \delta)$
Fig. 8. The barycentric coordinate description allows for a classification of the $A^i_k$ curve arising from possible configurations of three generators (gray dots), which can be viewed as the intersection of three shock sheets arising from the three pairings of the generators. (a) When $\alpha > 0$, $\beta > 0$, $\gamma > 0$, the triangle is acute and the flow from the $A^i_k - 2$ shock sources (tiny squares) gives an $A^i_k - 2$ (small triangle); the arrows indicate flow along sheets which are shown by their intersection line with the plane of the generators. (b) When one of $\alpha$, $\beta$, or $\gamma$ is negative, we have an obtuse triangle, e.g., here, $\alpha < 0$ and the midpoint $O_2$ of $G_i G_k$ is not a valid $A^i_k - 2$ (empty square), i.e., the associated contact surface is not maximal due to the proximity of $G_i$. In this case, the $A^i_k - 2$ shock gives rise not only to a new curve, due to the intersection of the two sheets that flow into it, but also gives rise to a new sheet outflowing from it for the pair $\{G_i, G_k\}$. (c) If $\alpha, \beta, \gamma$ are zero, we have a right triangle configuration, the “interface” between the previous two cases. Other combinations are not possible.

Proof. We first derive the geometric locus of the three types of shock singularities and then derive the possible flow configurations. Shock singularities (sources, relays, and sinks) are those shock points (i.e., at equal distance from the generators) that are critical points along the $M_A$ of the associated distance to generator function, $d_i(x)$, between a generator $G_i$ and a point $x$ in space. 1) For a pair of generators $\{G_i, G_j\}$, the $A^i_k - 2$ shock sheet is the bisector surface $d_i(x) - d_j(x) = 0$, which, for point generators, is a plane passing through $O_2$ and orthogonal to $G_i G_j$ [39]. By Pythagoras’ theorem, $O_2$ minimizes the distance to generators, and is the $A^i_k - 2$ source (Fig. 7a). 2) For a triplet of generators $\{G_i, G_j, G_k\}$, the $A^i_k$ shock curve is given as the intersection of two bisector surfaces $d_i(x) - d_j(x) = 0$ and $d_i(x) - d_k(x) = 0$, which for point generators is the straight line passing through the circumcenter $O_3$ of the associated triangle and normal to the plane of the three generators. Again, by Pythagoras’ theorem, $O_3$ minimizes the distance to generators and is the $A^i_k - 2$ source (Fig. 7b). 3) For a quadruplet of generators $\{G_i, G_j, G_k, G_l\}$, its $A^i_k$ shock is a single point given by $d_i(x) - d_j(x) = 0$, $d_i(x) - d_k(x) = 0$, and $d_i(x) - d_l(x) = 0$, which, for point generators and generic configurations, gives the circumcenter $O_4$ of the associated tetrahedron. The formulae in (1), (2), and (3) for $O_2$, $O_3$, and $O_4$ are well-known in classical geometry [4]. For the second part of the proof, we need to study flow configurations at $O_3$ and $O_4$, i.e., whether shocks flow inward or outward.8 Flow configurations have been recently reported by Siersma in 2D who looked at the topological splits of propagating waves [47] and by Giblin and Kimia in 2D and 3D based on a careful analysis of the geometry [21], [22]. Here, we summarize their results in terms of the barycentric formalism. Note that only when a circumcenter is outside its associated triangle 1) or tetrahedron 2) will some of the shock structure flow outward, as one or two generators will prevent an inward flow by being relatively closer to the circumcenter. Consider the case of a triplet of generators. When its triangle is acute, the three associated $A^i_k - 2$ shock sources of sheets have their distance to generator smaller than the circumradius $R_3$ associated to $O_3$; hence, the three sheets are created “before” they flow into $O_3$. When the triangle is obtuse, the edge opposite the generator where the angle is obtuse is longer than twice $R_3$ by simple geometry. Hence, the associated $A^i_k - 2$ and sheet cannot be created before the occurrence of $A^i_k - 2$ at $O_3$. Therefore, the other two sheets must intersect past this edge and $O_3$ is located outside the triangle. Since the barycentric coordinates of $O_3$ are directly proportional to the cosine of the angle (interior to the triangle) at their associated generator, their sign distinguishes the acute from the obtuse case. Consider the case of a quadruplet of generators $\{G_i, G_j, G_k, G_l\}$. According to [22, Section 4.7], an outward flow from $O_4$ along a given $A^i_k$ shock curve, e.g., due to the triplet $\{G_i, G_k, G_l\}$, corresponds to the fourth generator, $G_l$, being on the side of the plane defined by the triplet opposite to $O_i$, i.e., such that $O_i$ is outside the tetrahedron with respect to that face. But, by definition, the barycentric coordinates are proportional to the signed distance of the line subtended from a generator, say $G_i$, through the point

---

8. There is only one possible flow configuration for $O_2$, namely, outflowing along the sheet.
of interest, here $O_3$, to its intercept with the opposite face of the tetrahedron $\{(G_j, G_k, G_l)\}$ [4]. Also, according to [22, Section 4.7], there are either zero, one, or two outward flows at $O_3$ and, hence, the associated barycentric coordinates are either all positive, or one or two are negative and indicate which $A_i^3$ curves flow outward of the $A_i^3$ shock.

Note that, in evaluating (2) and (3), in practice, the reduced forms, $O_3 = G_i + \gamma(G_j - G_i) + \gamma(G_k - G_i)$ and $O_3 = G_i + \beta(G_j - G_i) + \gamma(G_k - G_i) + \delta(G_l - G_i)$, are preferred for better numerical precision as they depend on relative coordinates of generators [44]. Note also that there are special (degenerate) cases of configurations at the interface of those identified above which need to be considered for completeness, e.g., when $O_3$ coincides with the midpoint of two generators (Fig. 8c) or when $O_3$ coincides with $O_2$ or with $O_2$. In practice, these are considered as special cases of one of the existing cases. A direct consequence of the above proposition is the following corollary on the formation of shocks from point generators, which is the key for the efficient computation via propagation from sources, as described in the following section.

**Corollary 4.1 (Shock formation).**

1. An $A_i^3$ shock sheet is originated either by: a) an $A_i^3 - 2$ shock source, b) an $A_i^3 - 2$ shock source corresponding to an obtuse triangular configuration of generators, or c) an $A_i^3 - 2$ shock vertex relay. 2. An $A_i^3$ shock curve is originated either by: a) an $A_i^3 - 2$ shock source or b) an $A_i^3 - 2$ or an $A_i^3 - 3$ shock vertex relay. 3. An $A_i^3$ shock vertex originates from the intersection of at least two $A_i^3$ shock curves.

5 A LAGRANGIAN COMPUTATIONAL SCHEME

We now discuss a computational scheme for the recovery of shock nodes and medial links for an unorganized cloud of point generators, $G = \{G_i, \ i = 1, \ldots, m\}$. Recall that, for a cloud of points there are six types of shock nodes, Table 4 (not counting the two types of regular shock points). At first glance, these can simply be obtained by considering combinations of generators and verifying the maximality of the associated contact sphere against the remaining generators, i.e., that it does not contain other generators. The main computational difficulty in this brute force approach is that, without a priori knowledge of local connectivity among generators, a nonpractical number of combinations should be considered, e.g., $10^{19}$ quadruplets for $10^9$ generators. The algorithm we propose avoids considering all combinations by first seeking the initial sources of the shock flow and then propagating along the MÀ by implicitly detecting intersections of sheets and curves in the spirit of the 2D Lagrangian flow [52]. Note that the vast majority of generator combinations do not yield shocks. In fact, the maximal number of shock nodes can be shown to be bounded and of order $O(m^2)$ [37], [28] so that $O(m^3)$ operations are potentially required for shock computations when validation of each shock is also considered. However, this upper bound is for contrived, highly symmetric generator arrangements and, in practice, we observe that the resulting MS tend to be of order $O(m)$ in size and also in operations required to retrieve the scaffold. The use of propagation in the algorithm is justified in the following proposition.

**Proposition 5.1 (Reachability of medial nodes).** Assuming all $A_i^3 - 2$ sources are known, all medial nodes ($A_i^3 - 2s$ and $A_i^3$s) at a time of formation (i.e., distance to generator) $t = t_k$ are reachable by at least a pair of paths along the MÀ from shock nodes with time of formation $t < t_k$.

**Proof.** We represent “propagation” as traversing a sequence of shock node formations ordered by “time,” which is the distance to the generator (i.e., the radius, $r$, of the associated contact sphere is taken as time). First, we note that any $A_i^3$ shock forming at time $t_k$ has at least a pair of incoming shock paths in the form of $A_i^3$ shock curves [22], i.e., from two previously detected $A_i^3$ curves at some time $t < t_k$. This implies that we can compute $A_i^3$ shocks at time $t_1$ by intersecting pairs of $A_i^3$ curves having outward flows (i.e., not already terminated by $A_i^3$ shocks). Second, an $A_i^3$ shock curve formed at time $t_k$ either has an $A_i^3 - 2$ source as its starting point or, if not, it is an outgoing flow from an $A_i^3$ formed at time $t < t_k$. The special case of an $A_i^3 - 3$ (degenerate) segment is detected as simultaneously flowing out a pair of $A_i^3$ nodes with equal radius value. An $A_i^3 - 2$ shock curve source has at least a pair of incoming shock paths in the form of intersecting $A_i^3$ shock sheets (Corollary 4.1). If we know all existing sheets at time $t < t_k$, then we can compute $A_i^3 - 2$ shocks at time $t_3$ by pairing previously formed $A_i^3$ sheets. Third, an $A_i^3$ shock sheet formed at time $t = t_k$ either has an $A_i^3 - 2$ source as its starting point or, if not, it is an outgoing flow from an $A_i^3 - 2$ or an $A_i^3$ at a time $t < t_k$, i.e., cases already considered. Thus, in all cases, a medial node at time $t = t_3$ is reachable from previously formed shock nodes. 

The above proposition directly leads to a simple flow-based iterative algorithm where we pair shock sheets arising from the $A_i^3 - 2s$ found in an initialization stage to identify new shock curve sources and pair shock curves to identify new shock vertices. Thus, the algorithmic method detailed below consists of two successive stages: 1) initialization (Section 5.1) and 2) propagation (Section 5.2).

5.1 Initialization: Finding the Initial Sources of Flow

The identification of $A_i^3 - 2$ shocks requires finding pairs of generators defining a contact sphere empty of other generators, Fig. 11a. This step is made efficient by the key observation that once a pair of generators creating a shock source candidate is obtained, an entire half-space can be ruled-out from further pairings for each member of the pair, a characteristic we name the “visibility constraint.” A second important observation then is that nearby pairs rule out more space via this visibility constraint than more distant pairs, a phenomenon which calls for an a priori approximate clustering of generators. Furthermore, such clusters of generators can be seen as “metagenerators” to which the same “visibility” rule can be coarsely applied, leading to a search of space for potential shock candidates where a few generators can block the view of numerous metagenerators hence expediting the process of eliminating unnecessary pairings. We describe these ideas in detail below.

Initialization begins with a preprocessing step where we group those generators which are spatially close and consider their interaction to create shock points. The clustering method used for this purpose does not rely on explicit distance computations between generators, as is
common in popular approaches such as \(kd\)-trees, since this would defeat the purpose of working with unorganized data. Rather, we have adapted a 3D bucketing algorithm which is applied to the underlying 3D grid in three steps. First, the volume containing the generator set \(G\) is divided into slabs along one grid direction, say the \(x\)-axis, into \(x\)-buckets each containing roughly the same number of generators. Second, each \(x\)-bucket is divided into strips along another grid direction, say, the \(y\)-axis, once again keeping the number of generators per resulting \(xy\)-bucket roughly the same. Third, each \(xy\)-bucket is divided into smaller strips along the third grid direction, here the \(z\)-axis, resulting in \(xyz\)-buckets, containing roughly the same small number of generators. Each cut through space is taken at integer coordinates and adapted to satisfy, under a tolerance level \(\delta\), a local target density of generators per bucket \(B_k\) (Fig. 10). This method is efficient as it does not order generators within a cluster. Formally speaking, the resulting partition of space is a \textit{paving} in that the buckets tile and entirely cover the volume without overlap. This tight coupling between buckets defines a \textit{coarse-level topology}, which proves useful in finding and validating shocks.

The second part of the initialization stage finds initial shock sources in an iterative manner, starting with \(A^2_i - 2\) shocks which arise from those pairs of generators located \textit{within} a bucket, and then searching for additional pairs with generators located in \textit{different} buckets. This is justified on the basis of “visibility constraints” restricting pairs considered.

**Constraint 1 (Visibility constraint for generator pairs).** Observe that a generator \(G_j\) “may get in between” a pair \(\{G_i, G_k\}\), preventing the formation of an initial shock source. The \(A^2_i - 2\) shock between \(G_i\) and \(G_k\) forms \textit{iff} there does not exist a generator \(G_j\) within the ball of contact with \(G_i G_k\) as diameter, Fig. 11a. The same reasoning, but starting from a given pair \(\{G_i, G_j\}\) and unknown \(G_k\), implies that the presence of \(G_i\) creates a half-space in which no generator \(G_k\) can form a shock with \(G_i\). This \textit{visibility constraint} allows an entire half-space to be ruled-out from further consideration for pairings with \(G_i\) (Figs. 11b and 11c) by applying a simple “projection test.”

**Definition 5.1 (Deadzone, visible zone).** The deadzone of \(G_i\) with respect to \(G_j\), \(D_{G_i, G_j}\), is the region in which a generator \(G_i\) cannot form a shock with \(G_j\) due to the presence of \(G_j\). The complement of this region is the \textit{visible zone} of \(G_i\) with respect to \(G_j\).

**Proposition 5.2 (Deadzone Test).** A generator \(G_k\) is not visible from \(G_i\) in the presence of \(G_j\) \textit{iff} \(G_k G_i \cdot G_j G_k < 0\) or, equivalently, \(\|G_i G_j \cdot G_k G_k\| > \|G_i G_j\|^2\).

**Proof.** The generator \(G_k\) is in the deadzone if the angle \(\theta = \angle G_i G_j G_k\) is obtuse or \(\cos \theta < 0\). Alternatively, this is true if \(G_i G_j \cdot G_k G_k < 0\) or \(G_i G_j \cdot (G_k G_j - G_i G_k) < 0\).

**Corollary 5.1 (Additivity of visibility constraints).** The visibility constraint is cumulative in that the deadzone of \(G_i\) in the presence of generators \(\{G_{i_1}, G_{i_2}, \ldots, G_{i_L}\}\) is the union of the deadzones with respect to the presence of each generator \(G_{i_l}\) (Fig. 12):

\[
D_{G_i, \{G_{i_1}, G_{i_2}, \ldots, G_{i_L}\}} = \bigcup_{l=1}^{L} D_{G_i, G_{i_l}}.
\]

The visibility constraint allows us to explicitly bypass computing and validating \(A^2_i - 2\) shock sources and instead use a simple test to rule out a large number of potential
pairings. Observe also that if the data are spread on the surface of an object, the visibility constraints build up per generator and eventually leave only a relatively small column of visible space for a given generator $G_i$ above and below the local surface’s tangent plane (Fig. 12c). A related formal result has been recently proposed by Amenta and Bern [1], where it is shown that Voronoi cells are elongated along the normal space to a surface at the generators with a sufficiently dense sampling [1].

Constraint 2 (Clusters acting as metagenerators). A second key idea in reducing computational cost is to reason about the visibility of a generator in one cluster from another generator in another cluster at the cluster level without regard to the particular arrangement of generators within each cluster. We define as metagenerator a region of space specified by a grouping of generators. A metagenerator can be entirely in the deadzone of a single generator $G_i$ (Fig. 13a) and one metagenerator can rule out an entire metagenerator’s visibility from another (Fig. 13b). Since the notion of adjacency for arbitrary groups may require an involved computation which is equivalent to the shock computation itself, we rely on the paving property of the bucketing method to determine such a metagenerator and its neighbors. Furthermore, for theoretical and practical reasons, it proves useful to consider convex hulls for meta-generators as described in the following two propositions.

**Proposition 5.3 (Convex shell sufficiency – 1).** Consider the convex hull $A$ of the generators $\{G_{i_1}, \ldots, G_{i_L}\}$ and a generator $G_j$ (Fig. 14), then, $\bigcap_{i=1}^{L} D_{G_{i_i}, G_j} = \bigcap_{i \in A} D_{G_{i_i}, G_j}$.
Proof. Since the vertices \( \{G_{i1}, \ldots, G_{in}\} \) are themselves a subset of points \( G_i \) in the convex hull \( A \), the right-hand side is a subset of the left-hand side. To show the opposite, consider a point \( G_i \in A \) which, due to convexity, can be represented by positive barycentric coordinates: 
\[
G_i = \sum_{l=1}^{L} \alpha_l G_{il}, \quad \sum_{l=1}^{L} \alpha_l = 1, \quad \alpha_l > 0 \text{ for } l = 1, 2, \ldots, L.
\]
Let \( G_k \in \bigcap_{l=1}^{L} D_{G_{il}} \). Then, 
\[
G_k \subset D_{G_{il}}, \quad l = 1, 2, \ldots, L.
\]
using Proposition 5.2. To show that \( G_i \) is also in this set, we must show that \( G_i G_k < 0 \):
\[
G_i G_k = \left( \sum_{l=1}^{L} \alpha_l G_{il} \right) \cdot \left( \sum_{l=1}^{L} \alpha_l G_{jl} \right) < 0, \text{ since } 0 \leq \alpha_i, \alpha_j \leq 1.
\]

Corollary 5.2 (Convex hull sufficiency—1). Consider the convex hull \( \tilde{A} \) of vertices \( \{G_{i1}, \ldots, G_{in}\} \) (not necessarily generators) containing any generator \( G_i \in \tilde{A} \) and a generator \( G_j \), then,
\[
\bigcap_{l=1}^{L} D_{G_{il}}, G_j \subset D_{G_{il}}, G_j.
\]

The above proposition shows that the effect of a generator \( G_j \) blocking the formation of shocks with any other generator inside the convex hull \( \tilde{A} \) can be completely summarized by considering only these vertices. The corollary extends this results to a more conservative approximation for a convex hull \( \tilde{A} \) containing at least one generator and with vertices made of “virtual generators,” i.e., artificially introduced to define a zone whose convex hull bounds actual generators. The advantage of using such virtual generators is in defining a very simple zone, such as a bounding box, with very few vertices, hence leading to few constraints to verify. Below, we show that the blocking effect of a region \( B \), described as the convex hull \( \overline{B} \) of a few vertices \( G_{j1}, \ldots, G_{jn} \), containing an arbitrary but unknown number of generators \( \{G_i\} \), can be bounded as the deadzone generated by its vertices \( G_{j1}, \ldots, G_{jn} \) acting as virtual generators.

Proposition 5.4 (Convex shell sufficiency—2). Consider a generator \( G_j \) and the convex hull \( \overline{B} \) of vertices \( \{G_{j1}, \ldots, G_{jn}\} \) (typically, not generators themselves), then,
\[
\bigcap_{l=1}^{L} D_{G_{il}}, G_j \subset D_{G_{il}}, G_j, \forall G_j \in \overline{B}.
\]

Proof. We can write \( G_j \) in barycentric coordinates: 
\[
G_j = \sum_{l=1}^{L} \alpha_j G_{jl}, \quad \sum_{l=1}^{L} \alpha_j = 1, \quad \alpha_j > 0 \text{ for } l = 1, 2, \ldots, L.
\]
Consider \( G_k \in \bigcap_{l=1}^{L} D_{G_{il}} \) which implies 
\[
G_k \subset D_{G_{jl}}, \quad l = 1, 2, \ldots, L.
\]
We must show 
\[
G_k G_j < 0 \text{ to prove the proposition:}
\]

\[
G_k G_j = \left( \sum_{l=1}^{L} \alpha_j G_{jl} \right) \cdot \left( \sum_{l=1}^{L} \alpha_j G_{jl} \right) < 0,
\]

\[
G_k G_j = \left( \sum_{l=1}^{L} \alpha_j G_{jl} \right) \cdot \left( \sum_{l=1}^{L} \alpha_j G_{jl} \right) < 0.
\]

Corollary 5.3 (Convex shell sufficiency—2). Consider the convex hull \( \tilde{A} \) of vertices \( \{G_{i1}, \ldots, G_{in}\} \) and convex hull \( \overline{B} \) of vertices \( \{G_{j1}, \ldots, G_{jn}\} \) (not necessarily generators themselves), then,
\[
D_{\tilde{A} \overline{B}} = \bigcap_{l=1}^{L} D_{G_{il}}, G_j \subset D_{G_{il}}, G_j, \forall G_j \in \tilde{A}, G_j \in \overline{B}.
\]

The proposition provides a lower bound on the smallest possible deadzone, given that there is at least one generator in each metagenerator, namely, the deadzone arising from the least favorable arrangement of the least favorable set of generators (i.e., one generator). Nevertheless, entire regions can be ruled out based on this conservative estimate. A metagenerator \( \overline{C} \) can be ruled out as a potential candidate to form shocks with generators in metagenerator \( \tilde{A} \) because metagenerator \( \overline{B} \) blocks it (Fig. 14d). In our implementation, buckets serve as metagenerators where the bucket’s eight vertices are used to check for visibility. Since buckets are expected to contain tens of hundreds of generators, this metalevel visibility constraint greatly reduces the number of computations.

Constraint 3 (Multiresolution metagenerator visibility). Observe that building zones of visibility constraint at the metagenerator level is independent of the number of generators within the nonempty metagenerator. In particular, if a metagenerator \( \overline{C} \) had been ruled out as nonvisible from metagenerator \( \tilde{A} \) because of an intermediary metagenerator
Covered (Fig. 16).

...also proves useful for this step since we can iteratively check candidates, we still need to validate the remaining ones by ensuring that their associated contact sphere does not remain empty of other generators. Here, the \( A_i^1 - 2 \) shock of a pair of generators in \( T \) will be invalidated due to the presence, in the domain of the associated contact sphere, of a third generator member of (a) \( T \) and (b) of \( T \).

5.2 Propagation: Shock Scaffold Formation from Initial Shock Sources

We now describe the process of pairing initial shock sources \( \{A_i^1 - 2\} \) to search for other medial nodes according to Proposition 5.1 and Corollary 4.1 which state that shock curves begin either at an \( A_i^1 - 2 \) or at an \( A_i^1 \). First, the \( A_i^1 - 2 \) points correspond to a pair of sheets sharing one generator and together identifying a unique triplet of generators \( \{G_i, G_j, G_k\} \) whose circumcenter is the \( A_i^1 - 2 \) candidate (Figs. 17a and 17b). This \( A_i^1 - 2 \) candidate needs to be validated, i.e., by ensuring no other generators exist in the circumsphere of the generator triplet. The valid \( A_i^1 - 2 \) shock sources always produce two opposite new shock curve flows (normal to the triangular plane of the generator triplet, Fig. 17a), and those corresponding to obtuse triangular configurations also produce a new shock sheet (Fig. 17b).

Constraint 4 (Spiral search to rule out generator-metagenerator interactions). In general, the closest pairs of generators rule out the most space. Thus, we first construct the shock sources within each metagenerator. Second, we consider interactions with neighboring metagenerators and then proceed with interactions with neighbors of neighbors, which very often are already entirely ruled out as nonvisible. Thus, an iterative process of visiting neighboring metagenerators in a spiral search rapidly converges and terminates, in practice, in a few steps (Fig. 15).

While these constraints rule out a majority of shock source candidates, we still need to validate the remaining ones by ensuring that their associated contact sphere does not include any other generators. The brute force approach would increase the computational complexity by \( O(m) \) for \( m \) generators. However, the formation of metagenerators also proves useful for this step since we can iteratively check in a layered fashion whether any generator falls in the associated contact sphere by first considering the metagenerator containing the shock candidate, the metagenerator neighbors and, so on, until the contact sphere is entirely covered (Fig. 16).
algorithm will always converge in a finite number of steps. The system parameters are summarized in Table 5 and the pseudocode is given in the thesis of Leymarie [28].

6 Experimental Results

We now examine the theoretical notion of an MS in application to realistic data sets using the algorithm for unorganized clouds of points. We examine three issues here: 1) Is the method practical for large data sets? 2) Can the MS of a surface sampled by unorganized points be recovered? And, 3) is the resulting scaffold potentially useful in object recognition and other applications?

First, while the propagation along the MS is relatively fast, the bottleneck which dominates the overall computational cost is the computation of $A^3_i - 2$ shocks. Fig. 18 summarizes timing results of the overall MS computations (initialization and propagation) as a function of the number of generators. Note that, when the unorganized points are not samples from a geometric structure, the behavior is nearly linear, Fig. 18a. In contrast, for samples from real structures where points arise from complex surfaces, the dependence on the spatial relations gives rise to a roughly linear “beam,” Fig. 18b. Note that our timings are similar to those obtained via QHULL [5], a popular set of free software used to compute Voronoi/Delaunay graphs in 3D and higher dimensions for point generators. This is significant since our code computes a substantially richer structure, it can dynamically be updated with additionally acquired data, and it can be generalized to handle richer classes of input generators such as unorganized polygonal shapes [28, chapter 7].

Second, we address the question of computing the MS of a surface when only unorganized samples are available from it. The MS of a densely sampled surface retains the original surface symmetries (albeit in a slightly deformed version), but it also includes symmetries pertaining to the interaction of nearby sampled points. Thus, we need to identify which portion of the MS is a result of the surface sampling process, the “sampling artifact scaffold,” and segregate it from the symmetries of the original surface, the “surface scaffold.”

A proper theoretical investigation of how to segregate the MS requires a model of the sampling process together with a study of transitions of the 3D MS [19]. However, when the surface is densely sampled with respect to its local geometry and intersurface distances, there is sufficient evidence to segregate the MS into its constituent components based on the following observation: For a dense sampling of the surface, the triangles which correspond to $A^3_i - 2$ shocks and which interpolate the corresponding triplet of generators $\{G_i, G_j, G_k\}$ form two classes: 1) triangles with small sides arising from the surface and 2) triangles with two long edges, corresponding to MA symmetries of the original surface. This observation permits a segregation of these two groups by rank-ordering the associated $A^3_i$ shock curves (Fig. 7b) by a geometric measure of their associated triangle (e.g., longest side), and using a threshold $t_s$ to identify the two classes [28, chapter 6].

12. Leymarie’s thesis is available online from: www.lems.brown.edu/~leymarie/phd/.

13. The related issue of the compression rate achievable by this segregation process, i.e., in how the resulting surface scaffold captures the relevant shape attributes, also requires a careful study of transitions of the 3D MS coupled with a geometric sampling theorem for which theoretical results are needed. For recent advances on the latter topic, refer, e.g., to Chazal et al. [15].
An identification of the sampling artifact scaffold not only gives the surface scaffold, but also meshes the unorganized points into triangulated surfaces: when an A³[1] curve is identified as a member of the sampling artifact scaffold, the corresponding triangle is locally interpolating the surface. Fig. 19f shows the result of this process, where blue curves indicate A³[1] ribs of ridges, red curves indicate A³[1] axial shock curves, and the surface meshes are a result of the algorithm presented here.

Third, we examine the recovered $\mathcal{MS}$ for six data sets and comment on its object recognition potential. We already previewed the $\mathcal{MS}$ of a branching shape part of a human aorta, Fig. 5. The branches which are roughly elliptical cylinders are represented by elongated shock sheets bounded by A³[1] curve pairs. Our second example is for a set of “heads” depicting concavities, convexities, and ridges (Fig. 20);
observe how the \( A_3 \) “rib curves” identified by the \( MS \) indicate ridges on the surface, e.g., for the nose, lips, chin, and eyebrows. That such significant surface features can be robustly recovered in all face data sets indicates that the \( MS \) can potentially serve as basis for recognition.

The third example shows results from unorganized point samplings of two full human body scans from Cyberware, Fig. 21. The overall graph structure is evident: Observe that the arms and the legs, roughly elliptical cylinders, are represented by elongated shock sheets bounded by \( A_3 \) curve pairs. The torso is also a single broad sheet with structure, e.g., corresponding to the protruding breasts, hanging off it. The \( MS \) as a network of \( A_1 \) and \( A_3 \) shock curves can potentially be used in deforming the limbs and their spatial relationships for object animation in computer graphics [55].

The apparent complexity of the \( MS \) for the above shapes is reminiscent of a similar picture for 2D \( MA \) and 2D shock graphs [21], [40] and illustrates the need for regularizing the \( MS \). The \( MA \) is inherently unstable and, as the surface is deformed, the graph structure of the \( MS \) can abruptly change. A successful solution for the practical use of 2D shock graphs in applications employs the instabilities as key points.
of the representation itself. The set of such instabilities, or transitions, was identified and paths of deformations were characterized as a set of transitions. The optimal path relating two shapes in 2D was then found in the space of sequences of transitions [51], [21], [40]. Fortunately, the transitions arising from small deformations in 3D shape has already been classified [19]. Our fourth example simulates all the MS transitions in a stable manner, as shown in Fig. 22 for two of the seven possible transitions: 1) the $A_1A_3$ transition corresponds to a surface being pulled out slightly, which results in a new shock sheet being generated in the direction of the pull (Fig. 19f), and 2) the $A_1^3$ transition shows how compressing a shape can abruptly change its MS.

The fifth example illustrates how these transitions can be used to regularize realistic data by comparing a rectangular box perturbed by a few bumps and a pot sherd laser scanned for applications to digital archaeology [29], Fig. 23. We can think of this sherd as a highly deformed version of the box, which consists of a network of significant $A_1^3$ and $A_3$ curves, much like the box, but with the addition of numerous small $A_1^3$-$A_3$ combination “tabs” of the sort shown in the first transition of Fig. 22 growing out of sheets due to small perturbations. Once these “tabs” are identified, they can be considered as small scale additions to a coarse scale structure and removed, thus smoothing the sherd’s surface in the process while preserving its main geometric features [30].

Fig. 22. Simulation of two of the seven 3D transitions (after [19]) using our approach, where the shape deforms from left to right passing through (a) $A_1A_3$-1 and (b) $A_1^3$ transitions.

Fig. 23. An example of how transitions can be used to regularize the MS. (a) A rectangular box deformed by five protrusions (four on top, one on a side) is uniformly sampled by 11,200 point generators. (b) The initial internal MS; note the small loop structures due to the protrusions. (c) MS after transition removal [30]. (d) The pot sherd of Fig. 1 is shown as the surface mesh recovered from the associated sampling artifact scaffold for this data set. (e) The internal MS contains a perturbed form of the “box” scaffold together with numerous “looped spikes” associated to small medial sheets transversal to a large medial sheet, i.e., toward a local surface bump. The junction of each small sheet with the larger medial sheet is “sitting” on an $A_1^3$ shock curve. (f) After transition removal (of the “spikes”), the resulting MS is simplified, but the most significant medial structures remain, e.g., the network of long ribs associated to breaks of the pot surface.
much in the manner of the 2D symmetry transforms used to structurally regularize 2D outlines [51].

Finally, Fig. 24 shows a rather challenging example of a tube-like structure arranged in a knot where the surface closely passes nearby other portions of itself. The iterative application of transition removal reveals the “generalized cylinder nature” of this object. We expect, in future work, to be able to relate such geometric graph structures to perform shape comparison [14], morphing, and analysis, in analogy to the similar use of 2D shock graphs [40].

7 Conclusion

We proposed a 3D graph model, the medial scaffold (MS), to represent 3D shapes. The MS is a hierarchical representation of Blum’s medial axis (MA) [10], where at one level a network of medial curves and their intersections define a scaffold and the rest of the MA information is built on top of this scaffold. Since the MS captures qualitative aspects of shape, such as regional features like long surface ridges, we expect it to be useful in object recognition, modeling, animation, and other pattern analysis tasks. We also presented a scheme for computing the MS of an unorganized set of points which is practical even for large data sets. We expect the availability of qualitative aspects of the MA as a graph together with an understanding of its instabilities [19], [30], [14] is a basis for its practical use in fields such as CAD, computer graphics, computer vision, and medical imaging.

We also note that the MS is part of a superstructure, the shock scaffold (SC), which features additional nodes (Table 2) and curves along medial sheets. We expect in the future to describe these other types of curves, i.e., shock links connecting all shock nodes and, thereby, define the SC. Toward this goal, the classification of curves connecting to A2 type nodes needs to be completed. Both the SC and MS are based on augmenting the traditional MA with a notion of dynamics via singularities of the flow associated with the radius function.

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References

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