The Ordered Median Euclidean Straight-line Location Problem *

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Abstract

An optimal line is sought in the Euclidean plane in terms of the ordered median criterion, when the set of existing facilities are points with arbitrary positive associated weights. By way of geometric duality a dominating set for such a line is found. This allows the construction of an algorithm to solve the problem in $O(n^4)$.

Keywords  Line-location, ordered median problem, geometric duality.

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1 Introduction

The ordered median objective function has been introduced and analyzed in several papers and contexts in Location Theory, see Nickel and Puerto [11]. This criterion is a generalization of the most popular objective functions: median, center, centdian, $k$-centrum, amongst many others and permits the development of a unified theory for location problems.

When the facility cannot be modeled as an isolated point but needs to be represented by some geometrical figure like a straight line, extensive facility location and location of structures arise. Thus, the median straight-line problem, i.e. finding a line minimizing the sum of weighted distances to the points of a set in the plane, appeared within the context of transportation in the paper [9]. The first exact algorithm for this problem was devised by Wesolowsky [14]. As far as the authors are aware, the current lowest time algorithm is the one provided by Lee and Ching [7], that finds the weighted median straight line in $O(n^2)$ time. For the unweighted version the best known subquadratic algorithm have been proposed by Dey [2], which solves the problem in $O(n^{4/3} \log n)$ time. The center straight-line problem, i.e. finding a line minimizing the maximum weighted distance to the points of a given set, was first addressed by Morris and Norback [10] and an $O(n \log n)$ optimal time algorithm was proposed by Edelsbrunner [5]. More information about median and center line location problems can be found in [13] and [3]. In the end, in [8], Lozano, Mesa and Plastria studied the weighted version of the $k$-centrum straight-line problem, i.e. finding a line minimizing the sum of $k$ maximum weighted Euclidean distances to $n$ given points. For this problem they devise an $O((k + \log n)n^3)$ time algorithm to find all $t$-centrum lines for $1 \leq t \leq k$.

In this paper we investigate the properties of the ordered median objective problem for straight lines and multiplicatively weighted Euclidean distances, by considering it in a dual space, leading to a quite efficient algorithm. The problem and its geometrical dual are stated in Section 2. In Section 3 we study the tesselations induced by the dual of the given points and their bisectors. This allows us to derive a characterization of the solutions in Section 4, leading to a finite set containing at least one solution. An algorithm based on the results given in Sections 2, 3 and 4, is provided in Section 5, together with an analysis of its complexity. In section 6 the nondegenaracy assumptions used earlier are lifted, then under some mild conditions the
characterization of solutions is shown to be strong, and, finally, the same finite dominating set is shown to remain valid when an arbitrary norm is considered instead of the Euclidean distance.

2 The problem and its geometrical dual

A point set \( P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2 \), its associated (positive) weight set \( W = \{w_1, \ldots, w_n\} \) and a positive weight vector \( \Lambda = (\lambda_1, \ldots, \lambda_n) \) are given. The ordered median straight-line problem consists in finding a straight line \( \ell \) that minimizes the function

\[
 f(\ell) = \sum_{i=1}^{n} \lambda_i w_{\sigma(i)} d(p_{\sigma(i)}, \ell)
\]

where \( d(\cdot, \cdot) \) is the point-line Euclidean distance: \( d(p, \ell) = \min_{q \in \ell} d(p, q) \), and \( \sigma \) is a permutation of the elements of the set \( \{1, \ldots, n\} \) such that

\[
 w_{\sigma(1)} d(p_{\sigma(1)}, \ell) \leq \ldots \leq w_{\sigma(n)} d(p_{\sigma(n)}, \ell).
\]

In order to solve it, the problem will be transformed into an equivalent one stated in the geometrical dual of the Euclidean plane. For this purpose, let us consider the duality map that associates each point \( p = (p_x, p_y) \in \mathbb{R}^2 \) with the dual (non-vertical) straight line \( p^* : y = p_x x - p_y \), and the non-vertical straight line \( \ell : y = mx + n \) with the point \( \ell^* = (m, -n) \). This map has the following relevant properties [1]:

1. Incidence between points and straight-lines is preserved: \( p \in \ell \) if and only if \( \ell^* \in p^* \).

2. The relative vertical position between points and straight lines is preserved: \( p \) is above \( \ell \) if and only if \( \ell^* \) is above \( p^* \).

3. The map is idempotent, i.e. the bi-dual of either a point or a straight line coincides with the original.

4. For the Euclidean distance

\[
 d(p, \ell) = \frac{d_v(\ell^*, p^*)}{\sqrt{1 + \ell^*_x^2}}
\]

where \( \ell^*_x \) denotes the abcissa of \( \ell^* \) and \( d_v(\cdot, \cdot) \) is the vertical distance, which for points \( p = (p_x, p_y), q = (q_x, q_y) \in \mathbb{R}^2 \) is defined as:
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\[ d_v(p, q) = \begin{cases} |p_y - q_y|, & \text{when } p_x = q_x \\ \infty, & \text{when } p_x \neq q_x \end{cases} \]

and for point \( p = (p_x, p_y) \) and a non-vertical straight line \( \ell : y = mx - n \) as:

\[ d_v(p, \ell) = \min_{q \in \ell} d_v(p, q) = |p_y - mp_x + n|. \] (2)

The duality map is a bijective mapping between the primal and the dual planes, transforming points in non-vertical straight lines and vice-versa. This fact along with the above properties allow us to state an equivalent problem in the dual. If \( P^* = \{p^*_1, \ldots, p^*_n\} \) is the dual image of the set \( P \), then finding a minimum of the function (1) is equivalent to finding a minimum of the function:

\[ f^*(\ell^*) = \sum_{i=1}^{n} \lambda_i \frac{w_{\sigma(i)}d_v(\ell^*, p^*_i)}{\sqrt{1 + \ell^* x^2}} \]

where \( \sigma \) is a permutation of the set \( \{1, 2, \ldots, n\} \) such that \( w_{\sigma(1)}d_v(\ell^*, p^*_1) \leq \ldots \leq w_{\sigma(n)}d_v(\ell^*, p^*_n) \), except when the minimum of (1) is a vertical straight line, in which case it corresponds to some point at infinity of the dual plane.

3 Tessellating the dual plane

We will now define three tessellations that will be useful for solving the ordered median straight-line problem.

First, consider the set \( P^* \). This set induces a partition of the dual plane in regions forming a Dual Tessellation that will be denoted by \( DT(P) \).

Lemma 1

Given \( R^* \), a region of \( DT(P) \), for all \( p^* \in P^* \) the function \( d_v(\cdot, p^*) : R^* \to \mathbb{R} \) is a linear function.

Proof

Since no line \( \ell^* \in P^* \) crosses \( R^* \), the result follows. \( \square \)

Next, define the bisector of the weighted straight lines \( p^*_i, p^*_j \in P^* \) by

\[ bis_v(p^*_i, p^*_j) = \{ \ell^* \mid w_i d_v(\ell^*, p^*_i) = w_j d_v(\ell^*, p^*_j) \} \].
Using (2) we can see that \( \ell^* = (m, n) \in \text{bis}_v(p^*_i, p^*_j) \) corresponds to the equations

\[
\| (p_i)_y - m(p_i)_x + n \| = \| (p_j)_y - m(p_j)_x + n \|.
\]

It follows that the bisector of two straight lines \( p^*_i \) and \( p^*_j \) with different weights consists of two non-vertical straight lines in the dual plane with the same intersection point as \( p^*_i \) and \( p^*_j \), and when \( p^*_i \) and \( p^*_j \) are parallel the two branches of the bisector are also parallel to them. When the weights are equal, one of these bisector branches is vertical in case \( p^*_i \) and \( p^*_j \) intersect, and when they are parallel the bisector is reduced to a single line (the second being the line at infinity). See Figure 1.

**Lemma 2**

The set \( B_v(P^*) = \{ \text{bis}_v(p^*_i, p^*_j) \mid i \neq j, 1 \leq i \leq n, 1 \leq j \leq n \} \) induces a partition of the dual plane into polygonal regions. The points within each region are dual images of straight lines for which the ordering of the weighted distances to the elements of \( P \) remains constant.

**Proof**

Since the vertical distance function is continuous with respect to \( \ell^* \), the order between \( w_i d_v(\ell^*, p^*_i) \) and \( w_j d_v(\ell^*, p^*_j) \) may only be inverted by crossing some line of \( \text{bis}_v(p^*_i, p^*_j) \). Therefore, for each region \( R^* \) determined by \( B_v(P^*) \), there exists a permutation \( \sigma \) of the set \( \{1, 2, \ldots, n\} \) so that
from which it follows that

\[
\frac{w(\ell^*, P_{\sigma(1)})}{\sqrt{1 + \ell^2}} \leq \frac{w(\ell^*, P_{\sigma(2)})}{\sqrt{1 + \ell^2}} \leq \cdots \leq \frac{w(\ell^*, P_{\sigma(n)})}{\sqrt{1 + \ell^2}}
\]

or equivalently,

\[
w(\sigma(1))d(p(\sigma(1), \ell)) \leq w(\sigma(2))d(p(\sigma(2), \ell)) \leq \cdots \leq w(\sigma(n))d(p(\sigma(n), \ell)).
\]

We now study the tessellation induced by \(B_v(P^*) \cup P^*\), the regions of which are the intersections of regions of both previous tessellations.

For the sake of simplicity, from here on, all the results assume that points in \(P\) are in general position, i.e. that the vertices of the tessellation induced by \(B_v(P^*) \cup P^*\) are uniquely determined. This restriction will be removed in the last Section. More precisely, the general position means that:

1. There are no three aligned points in \(P\), nor four points at equal weighted distance from some straight line.
2. If \(\ell\) crosses \(p_i, p_j \in P\) then, for each pair \(\{p_s, p_t\} \subseteq P \setminus \{p_i, p_j\}\), \(w_s d(p_s, \ell) \neq w_t d(p_t, \ell)\) holds.
3. If a straight line \(\ell\) satisfies \(w_i d(p_i, \ell) = w_j d(p_j, \ell)\) and \(w_s d(p_s, \ell) = w_t d(p_t, \ell)\) for four points \(p_i, p_j, p_s, p_t \in P\) then \(w_i d(p_i, \ell) \neq w_m d(p_m, \ell)\) for any \(p_m \in P \setminus \{p_i, p_j, p_s, p_t\}\) and no point \(p \in P \setminus \{p_i, p_j, p_s, p_t\}\) lies on \(\ell\).
4. If a straight line \(\ell\) satisfies \(w_i d(p_i, \ell) = w_j d(p_j, \ell) = w_k d(p_k, \ell)\) for three points \(p_i, p_j, p_k \in P\) then \(w_s d(p_s, \ell) \neq w_m d(p_m, \ell)\) for any \(p_s, p_m \in P \setminus \{p_i, p_j, p_k\}\) and no point \(p \in P \setminus \{p_i, p_j, p_k\}\) lies on \(\ell\).

From these assumptions we immediately obtain

**Lemma 3**

*Extreme points of regions of the tessellation induced by \(B_v(P^*) \cup P^*\) are one of the four following types:*

- **Type 1 points:** Vertices of a bisector of two elements of \(P^*\), which are exactly duals of lines connecting two points of \(P\).
Time 2 points: Intersections of two straight-lines corresponding to the bisectors of two pairs of elements of $P^*$ with a common element, i.e. of the form $\{p_i^*, p_j^*\}$ and $\{p_i^*, p_t^*\}$.

• Type 3 points: Intersections of two straight-lines corresponding to the bisectors of two pairs of elements of $P^*$ without common elements, i.e. of the form $\{p_i^*, p_j^*\}$ and $\{p_s^*, p_t^*\}$.

• Type 4 points: Intersections of a bisector of two elements of $P^*$ and an element of $P^*$ different from those generating the bisector.

In Figure 2 the tessellation induced by $B_v(P^*) \cup P^*$ for a given set $P = \{p_1, p_2, p_3, p_4\}$ is partially depicted: for the sake of simplicity not all the branches of the bisectors have been included in the picture. One region of this tessellation is shaded for which each vertex $v_i$ is a type i point ($i = 1, 2, 3, 4$).

4 A finite dominating set

Lemma 4
The objective function $f^*$ in the dual plane is quasiconcave in each cell of the tessellation induced by $B_v(P^*) \cup P^*$.
Proof

Let $R^*_C$ be a connected component of the tessellation induced by $B_v(P^*) \cup P^*$; the ordering of the weighted vertical distances from lines $p^*_i$ to points in it remains constant, i.e. there is a fixed permutation $\sigma$ of $\{1, 2, \ldots, n\}$ so that $\forall \ell^* \in R^*_C$:

$$w_{\sigma(1)}d_v(\ell^*, p^*_\sigma(1)) \leq \ldots \leq w_{\sigma(n)}d_v(\ell^*, p^*_\sigma(n))$$

In $R^*_C$ all the vertical distances to lines of $P^*$ are linear (Lemma 1). Therefore, the vertical point-line distance given in (2) becomes a linear function and the function

$$f^*(\ell^*) = \sum_{i=1}^{n} \lambda_i w_{\sigma(i)} d_v(\ell^*, p^*_\sigma(i)) \sqrt{1 + \ell^*_x^2}$$

on $R^*_C$ is a sum of nonnegative linear functions divided by a positive convex function, yielding a quasiconcave function.

Theorem 5

A finite dominating set for the ordered median straight-line problem is composed of the primals of all type 1, 2, 3 and 4 points. Therefore at least one solution of the problem exists satisfying one of the following conditions

1. Passes through two points of $P$.
2. Are at equal weighted distance of three points of $P$.
3. Are at equal weighted distance of two points $p_i, p_j \in P$ and at equal weighted distance of two other points $p_s, p_t \in P \setminus \{p_i, p_j\}$.
4. Are at equal weighted distance of two points of $P$ and passes through another point of $P$.

Proof

Candidate straight lines correspond to candidate points for the dual objective function, under the geometrical duality map. By Lemma 4, and since a quasiconcave function attains its minimum in an extreme point of any polygonal region, the set of extreme points (possibly at infinity) of each region of the tesselation induced by $B_v(P^*) \cup P^*$, i.e. type 1, 2, 3, 4 points and extreme points at infinity, form a dominating set for $f^*$.

Let us assume that the minimum of $f^*$ is not attained in a type 1, 2, 3 or 4 vertex of the tesselation induced by $B_v(P^*) \cup P^*$. Then there exists a minimum of $f^*$ reached at the infinity point of a half straight line of the tesselation. In this case the solutions to the ordered median straight-line
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problem must all be vertical straight lines in the primal plane. Let us denote $V$ as the set of such vertical lines and $G_{\alpha}$ as a rotation around the origin; its angle $\alpha$ is chosen arbitrarily but nonzero, so that the transformed set $G_{\alpha}(V)$ does not contain any vertical line. Then, since the straight lines of $G_{\alpha}(V)$ are optima of the problem associated to $G_{\alpha}(P)$ and are not vertical, there is a dual point to a line which is a type 1, 2, 3 or 4 point and, therefore, at least one line of $G_{\alpha}(V)$ satisfies one of the announced conditions in the theorem.

\[\Box\]

5 An algorithm

For each $p_i = (x_i, y_i)$ and the corresponding $p^*_i : y^* = x_ix^* - y_i$, the function $\alpha^*_i$ is defined as

$$\alpha^*_i(\ell^*) = \begin{cases} 1 & \text{when } \ell^*_y > x_i\ell^*_x - y_i \\ 0 & \text{when } \ell^*_y = x_i\ell^*_x - y_i \\ -1 & \text{when } \ell^*_y < x_i\ell^*_x - y_i \end{cases} \quad (3)$$

where $\ell^* = (\ell^*_x, \ell^*_y)$ is a dual point. Using (3) the function $f^*$ may be rewritten as

$$f^*(\ell^*) = \frac{1}{\sqrt{1 + \ell_x^2}} \left( \sum_{i=1}^{n} \lambda_i w_{\sigma(i)} \alpha^*_{\sigma(i)}(\ell^*) \right) \ell^*_y - \left( \sum_{i=1}^{n} \lambda_i w_{\sigma(i)} \alpha^*_{\sigma(i)}(\ell^*) \alpha_{\sigma(i)}(\ell^*) \right) \ell^*_x$$

$$= \sum_{i=1}^{n} \lambda_i w_{\sigma(i)} \alpha^*_{\sigma(i)}(\ell^*) \ell^*_y - \sum_{i=1}^{n} \lambda_i w_{\sigma(i)} \alpha^*_{\sigma(i)}(\ell^*) \alpha_{\sigma(i)}(\ell^*) \ell^*_x$$

where $\sigma$ is a permutation of the set $\{1, 2, \ldots, n\}$ such that

$$w_{\sigma(1)}d_v(\ell^*, p^*_{\sigma(1)}) \leq \ldots \leq w_{\sigma(n)}d_v(\ell^*, p^*_{\sigma(n)})$$
For each straight-line $\gamma^* \in B_v(P^*) \cup P^*$, the set $(B_v(P^*) \cup P^*)\setminus\{\gamma^*\}$ induces a partition of $\gamma^*$ into $O(n^2)$ segments and two half straight-lines (in what follows we will talk about segment in both cases). For each such segment $sg^*$ no element of $(B_v(P^*) \cup P^*)\setminus\{\gamma^*\}$ crosses it and therefore the numerator of $f^*$ is a sum of linear functions on it. As a consequence, $A, B, C \in \mathbb{R}$ exist such that

$$f^*(\ell^*) = \frac{A\ell_y^* + B\ell_x^* + C}{\sqrt{1 + \ell_y^*}} \quad \forall \ell^* = (\ell_x^*, \ell_y^*) \in sg^*. \quad (4)$$

Note that under the general position assumption the coefficients $A, B, C$ and, therefore, the objective function also can be updated in constant time when traversal of $\gamma^*$ results in moving from a segment to an adjacent segment.

A description of the algorithm follows. In order to compute the order of (increasing) distances to the elements of $P^*$ from each segment, an ordered list $order$ will be used.

**Algorithm**

**Input:** A point set $P$, its associated (positive) weight set $W$ and a positive weight vector $\Lambda$.

**Output:** An ordered median straight-line.

- **Step 0.** Compute the arrangement of the plane induced by $B_v(P^*) \cup P^*$.

- **Step 1.** Compute the best primal vertical line.

- **Step 2.** For each non-vertical straight-line $\gamma^* \in B_v(P^*) \cup P^*$, repeat:

  - **Step 2.1** Compute the intersections of $\gamma^*$ with each line of $B_v(P^*) \cup P^* \setminus \{\gamma^*\}$. Split $\gamma^*$ into segments according to these intersections, yielding the ordered list $sg_o(\gamma^*)$.

  - **Step 2.2** Walk forward along segments in $sg_o(\gamma^*)$, from left to right, performing the following steps

    - **Step 2.2.1** Compute the list $order$ for the first segment of $sg_o(\gamma^*)$.

    - **Step 2.2.2** Compute $A, B, C$ in the first segment of $sg_o(\gamma^*)$ and evaluate the objective function on its right extreme.
Step 2.2.3. Walk to the next segment in \( s_{g_o}(\gamma^*) \). If more segments exist in the list, update \( A, B, C \) and order; otherwise, loop Step 2.

Step 2.2.4. Evaluate the objective function in the right extreme of the current segment, update the best solution found if necessary and return to Step 2.2.3.

Step 3. Compute the best solution found in previous steps.

\[ \text{Theorem 6} \]
An ordered median straight line may be computed in \( O(n^4) \) time.

\[ \begin{proof} \]
Step 0 can be performed in \( O(n^4) \) [4]. The subproblem solved in Step 1 is equivalent to solving the ordered median problem on a path. The latter problem can be solved in \( O(n^2 \log n) \) time [6]. Step 2.1: The output of the algorithm used in Step 0 gives, for each \( \gamma^* \), the ordered list of the \( O(n^2) \) intersection points of all lines in the set \( B_v(P^*) \cup P^* \) with \( \gamma^* \). Step 2.2.1 requires \( O(n \log n) \) time. Steps 2.2.2 to 2.2.4: There are \( O(n^2) \) segments in \( s_{g_o}(\gamma^*) \) and the coefficients \( A, B \) and \( C \) and the list order can be updated in constant time. Therefore Step 2.2 takes \( O(n \log n) + O(n) + O(n^2) = O(n^2) \) time. Finally, since there are \( O(n^2) \) straight-lines in \( B_v(P^*) \cup P^* \), Step 2 is repeated \( O(n^2) \) times, so the complete algorithm takes \( O(n^4) \) time.

\[ \end{proof} \]

6 Particular cases and extensions

6.1 Degeneracies

In this subsection the case where points are not in general position is discussed. In order to include cases of degeneracy, the classification of the extreme points of regions of the tessellation induced by \( B_v(P^*) \cup P^* \) will be reformulated:

Definition 7
If points in \( P \) are not in general position we will say that a extreme point of a region of the tessellation induced by \( B_v(P^*) \cup P^* \) is:

- A type 1 point if it is the vertex of a bisector of two elements of \( P^* \), or, equivalently, the dual of a line connecting two points of \( P \).
• A type 2 point if it is the intersection of two straight-lines corresponding to bisectors of two pairs of elements of $P^*$, with a common element, and it is not a type 1 point.

• A type 3 point if it is the intersection of two straight-lines corresponding to the bisectors of two pairs of elements of $P^*$, without common elements, and it is neither a type 1 point nor a type 2 point.

• A type 4 point if it is the intersection of a bisector of two elements of $P^*$ and an element of $P^*$, different from those generating the bisector, and it is neither a type 1 point nor a type 2 point nor a type 3 point.

Although the points of $P$ are not in general position the function $f^*$ remains quasiconcave in each cell of $B_v(P^*) \cup P^*$ and, therefore, the finite dominating set of Theorem 5 remains also in such a case. The only issue to be checked is the complexity of the algorithm. Since there are $O(n^2)$ lines in $B_v(P^*) \cup P^*$ the global number of updates of $A$, $B$, $C$ and order along $\gamma^*$, in the Step 2.2, is $O(n^2)$ (note that some of these updates should be performed simultaneously when several intersections coincide). Finally, since the number of segments in the partition of $\gamma^*$ remains $O(n^2)$, the Step 2 takes $O(n^2)$ and the algorithm solves the problem in $O(n^4)$ time when the points of $P$ are not in general position.

6.2 Strong characterization of solutions

When $\lambda_2 = \ldots = \lambda_n = 0$ the problem consists in finding a straight-line minimizing the distance to the closest point of $P$. This is a trivial problem for which any straight-line passing through a point of $P$ is a solution. In this subsection we will assume that not all $\lambda_2, \ldots, \lambda_n$ are null simultaneously.

If weights are pairwise different (i.e. $w_i \neq w_j$, for $i \neq j$) there is no vertical bisector in the dual plane. Let $sg^*$ be a segment (or half-line) of a non-vertical straight line of $B_v(P^*) \cup P^*$ in which $f^*$ attains a minimum. Since $sg^*$ is a line segment the coordinates of any point $\ell^* \in sg^*$ are related by $\ell^*_y = a\ell^*_x + b$. Therefore, from (4)

\[ f^*(\ell^*) = \frac{A\ell^*_x + B(a\ell^*_x + b) + C}{\sqrt{1 + \ell^*_x^2}} = \frac{A\ell^*_x + C}{\sqrt{1 + \ell^*_x^2}} \forall \ell^* = (\ell^*_x, \ell^*_y) \in sg^*. \]
Under the assumption that not all \( \lambda_2, \ldots, \lambda_n \) are null simultaneously, since the numerator is the sum of weighted vertical distances to non-vertical straight lines it cannot be constantly 0 on \( sg^* \), so \((A', C') \neq (0, 0)\); on the other hand, since \( sg^* \) is non-vertical, the denominator is not constant and, as a consequence, the function \( f^*(\ell^*) \) is strongly quasiconcave on \( sg^* \). Therefore, given two points \( \ell^*_1, \ell^*_2 \in sg^* \), \( f^*(\lambda \ell^*_1 + (1-\lambda) \ell^*_2) > \min \{ f^*(\ell^*_1), f^*(\ell^*_2) \} \), \( \forall \lambda \in (0, 1) \), and \( f^* \) attains its minimum only in an extreme point (or infinite point) of \( sg^* \). As a consequence, the following strong property of the solutions follows:

**Corollary 8**

If the weights of points in \( P \) are pairwise different and \( \lambda_2, \ldots, \lambda_n \) are not null simultaneously, then any optimal ordered median straight-line \( \ell_{\text{opt}} \) satisfies at least one of the following conditions:

1. \( \ell_{\text{opt}} \) crosses two points \( p_i, p_j \in P \).
2. \( \ell_{\text{opt}} \) is at equal weighted distance from three points \( p_i, p_j, p_t \in P \):
   \[
   w_i d(p_i, \ell_{\text{opt}}) = w_j d(p_j, \ell_{\text{opt}}) = w_t d(p_t, \ell_{\text{opt}}).\]
3. \( \ell_{\text{opt}} \) is at equal weighted distance from two points \( p_i, p_j \in P \) and at equal weighted distance of two points \( p_s, p_t \in P \setminus \{ p_i, p_j \} \):
   \[
   w_i d(p_i, \ell_{\text{opt}}) = w_j d(p_j, \ell_{\text{opt}}), w_s(p_s, \ell_{\text{opt}}) = w_t(p_t, \ell_{\text{opt}}).\]
4. \( \ell_{\text{opt}} \) crosses a point \( p_i \in P \) and is at equal weighted distance of two points \( p_j, p_t \in P \setminus \{ p_i \} : w_j d(p_j, \ell_{\text{opt}}) = w_t d(p_t, \ell_{\text{opt}}).\)

\( \square \)

### 6.3 General norms

For a general norm \( \mu \), the expression (4) becomes (see e.g. [12]):

\[
d_\mu(p, \ell) = \frac{d_v(\ell^*, p^*)}{\mu^0(u)}
\]

where \( \mu^0 \) is the polar norm of \( \mu \), which is convex, and \( u = (\ell^*_x, -1) \) with \( \ell^*_x \) the abcissa of \( \ell^* \). Therefore, the dual objective function is:

\[
f^*_\mu(\ell^*) = \sum_{i=1}^{n} \lambda_i w_{\sigma(i)} d_v(\ell^*, p^*_{\sigma(i)}) / \mu^0(\ell^*_x, -1).
\]
This is a quasiconcave function on each cell of the tesselation induced by $B_v(P^*) \cup P^*$ and, therefore, attains its minimum at one of its vertices or at the point at infinity of one of the edges. As a consequence, the finite dominating set of Theorem 5 remains valid for an arbitrary norm. If $\mu$ is a smooth norm, then $\mu^0$ is strictly convex and the Corollary 8 holds.

References


