Characterization of symmetric monotone metrics on the state space of quantum systems

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Abstract

The quantum Fisher information is a Riemannian metric, defined on the state space of a quantum system, which is symmetric and decreasing under stochastic mappings. Contrary to the classical case such a metric is not unique. We complete the characterization, initiated by Morozova, Chentsov and Petz, of these metrics by providing a closed and tractable formula for the set of Morozova-Chentsov functions. In addition, we provide a continuously increasing bridge between the smallest and largest symmetric monotone metrics.

1 Introduction

In the geometric approach to classical statistics the canonical Riemannian metric is given by the Fisher information, and it measures the statistical distinguishability of probability distributions. The Fisher metric is the unique Riemannian metric contracting under Markov morphisms [1].

In quantum mechanics the probability simplex is replaced by the state space of density matrices (positive semi-definite trace one matrices), and Markov morphisms are replaced by stochastic mappings. A linear map $T: M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is said to be stochastic if it is completely positive and trace preserving. Since stochastic mappings (like their Markovian counterparts) represent coarse graining or randomization, one would expect statistical distinguishability of states to decrease under stochastic mappings.

These considerations lead Chentsov and Morozova to define a monotone metric as a map (or rather a family of maps) $\rho \to K_\rho$ from the set $M_n$
of positive definite $n \times n$ density matrices to sesquilinear$^1$ forms $K_\rho(A, B)$ defined on $M_n(C)$ satisfying:

(i) $K_\rho(A, A) \geq 0$, and equality holds if and only if $A = 0$.

(ii) $K_\rho(A, B) = K_\rho(B^*, A^*)$ for all $\rho \in \mathcal{M}_n$ and all $A, B \in M_n(C)$.

(iii) $\rho \rightarrow K_\rho(A, A)$ is continuous on $\mathcal{M}_n$ for every $A \in M_n(C)$.

(iv) $K_{T(\rho)}(T(A), T(A)) \leq K_\rho(A, A)$ for every $\rho \in \mathcal{M}_n$, every $A \in M_n(C)$ and every stochastic mapping $T : M_n(C) \rightarrow M_m(C)$.

It is understood that these requirements should hold for all $n$ and $m$. The condition (ii) is sometimes omitted, but we shall only consider symmetric metrics. Since condition (iv) implies unitary covariance we may in all calculations assume that $\rho$ is a diagonal matrix. Chentsov and Morozova proved that there to each monotone metric $K$ is a positive function $c(\lambda, \mu)$ defined in the first quadrant and a positive constant $C$ such that

\begin{equation}
K_\rho(A, A) = C \sum_{i=1}^{n} \lambda_i^{-1} |A_{ii}|^2 + \sum_{i \neq j} |A_{ij}|^2 c(\lambda_i, \lambda_j)
\end{equation}

for each diagonal matrix $\rho \in \mathcal{M}_n$ with diagonal $(\lambda_1, \ldots, \lambda_n)$ and every $A$ in $M_n(C)$. The metric is therefore fully described by the so called Morozova-Chentsov function $c$ which is symmetric in its two variables and satisfy

\[ c(\lambda, \lambda) = C\lambda^{-1} \quad \text{and} \quad c(t\lambda, t\mu) = t^{-1}c(\lambda, \mu) \]

for all $t, \lambda, \mu > 0$. At the time it was completely unsettled which type of functions $c$ would, through the formula (1), give rise to a monotone metric. It was not even clear that there existed a single Morozova-Chentsov function (and thus a monotone metric) although Morozova and Chentsov had a few candidates.

Petz connected the theory of monotone metrics with the theory of connections and means by Kubo and Ando$^8$ and was able to prove that the set of Morozova-Chentsov functions are given on the form

\begin{equation}
c(\lambda, \mu) = \frac{1}{\mu f(\lambda \mu^{-1})} \quad \lambda, \mu > 0,
\end{equation}

where $f$ is a positive operator monotone function defined on the positive half-axis satisfying the functional equation

\begin{equation}
f(t) = tf(t^{-1}) \quad t > 0.
\end{equation}

$^1$We use the complexification proposed by Petz$^{12}$. 

2
Petz was by this result able to give some examples of Morozova-Chentsov functions, and they included the candidates put forward by Morozova and Chentsov. The existence of monotone metrics was then established.

It is however a problem that the class of operator monotone functions satisfying (3) is largely unknown. The aim of the present paper is to provide a closed and tractable formula for the set of Morozova-Chentsov functions and in this way complete the characterization of (symmetric) monotone metrics given by Morozova, Chentsov and Petz. The main result is the formula given in Theorem 2.2. In addition, we provide a continuously increasing bridge (9) between the smallest and largest (symmetric) monotone metrics.

2 Statement of the main results

Let $f$ be a positive operator monotone function defined on the positive half-axis. It has a canonical representation of the form

$$f(t) = \int_{0}^{\infty} \frac{t(1+s)}{t+s} d\mu(s), \quad t > 0,$$

where $\mu$ is a positive (non-vanishing) finite measure on the extended half-line $[0, \infty]$. The function $f^\#$ defined by setting

$$f^\#(t) = t f(t^{-1}) \quad t \in \mathbb{R}_+$$

is operator monotone. This follows easily from the above integral representation, but may also be inferred by much simpler algebraic arguments (6) 2.1. Theorem (v) and 2.5 Theorem without the use of Löwner’s deep theory. Since $f^{##} = f$ the operation $f \to f^\#$ is an involution on the set of positive operator monotone functions defined on the positive half-axis. The harmonic mean is separately (operator) increasing, hence also the function

$$\tilde{f}(t) = H(f(t), f^\#(t)) = \frac{2f(t)f^\#(t)}{f(t) + f^\#(t)}$$

is operator monotone. It is an easy calculation to show that $(\tilde{f})^\# = \tilde{f}$. The formula (5) therefore associates a positive operator monotone function $\tilde{f}$ satisfying the functional equation (3) to any positive operator monotone function $f$, and $\tilde{f} = f$ if already $f^\# = f$. This procedure is implicitly applied in (12) Formula (12)] where Petz calculates a Morozova-Chentsov function from an operator monotone function not necessarily satisfying the functional equation (3).
There are several problems with this method, although it may be useful to calculate explicit examples. Firstly, the mapping $f \to \bar{f}$ is not injective. There are in general infinitely many operator monotone functions which are mapped to the same function. Secondly, it seems difficult to specify the induced equivalence relation on the set of positive finite measures on the extended half-line $[0, \infty]$.

**Theorem 2.1.** Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying

(i) $f$ is operator monotone,

(ii) $f(t) = tf(t^{-1})$ for all $t > 0$.

Then $f$ admits a canonical representation

$$f(t) = e^{\beta} \frac{1 + t}{\sqrt{2}} \exp \int_0^1 \frac{\lambda^2 - 1}{\lambda^2 + 1} \frac{1 + t^2}{(\lambda + t)(1 + \lambda t)} h(\lambda) \, d\lambda$$

where $h : [0, 1] \to [0, 1]$ is a measurable function and $\beta$ is a real constant. Both $\beta$ and the equivalence class containing $h$ are uniquely determined by $f$. Any function on the given form maps the positive half-axis into itself and satisfy (i) and (ii).

**Proof.** The result follows by applying Lemma 4.4 and Theorem 4.4. QED

Note that $\exp \beta = f(i) \exp(-i\pi/4)$. We may adjust the constant $\beta$ such that $f(1) = 1$. This corresponds to setting the constant $C = 1$ in formula (4.1) and gives rise to a so called Fisher adjusted metric. We are now able to calculate the set of Morozova-Chentsov functions.

**Theorem 2.2.** A Morozova-Chentsov function $c$ admits a canonical representation

$$c(x, y) = \frac{C_0}{x + y} \exp \int_0^1 \frac{1 - \lambda^2}{\lambda^2 + 1} \frac{x^2 + y^2}{(x + \lambda y)(\lambda x + y)} h(\lambda) \, d\lambda$$

where $h : [0, 1] \to [0, 1]$ is a measurable function and $C_0$ is a positive constant. Both $C_0$ and the equivalence class containing $h$ are uniquely determined by $c$. Any function $c$ on the given form is a Morozova-Chentsov function.

Note that $c$ is increasing in $h$ and that the constant $C_0$ may be adjusted such that $c(x, x) = x^{-1}$. For Morozova-Chentsov functions $c_h$ with a fixed constant $C_0$ and $h$ as in formula (7) we have $c_{sh+(1-s)g} = c_h^{1-s}c_g^{1-s}$, $0 \leq s \leq 1$. 


Proposition 2.3. Let the exponent $\gamma \in [0,1]$. The functions

$$f_\gamma(t) = \frac{1}{2}(1 + t) \left( \frac{4t}{(t + 1)^2} \right)^\gamma = t^\gamma \left( \frac{1 + t}{2} \right)^{1-2\gamma}, \quad t > 0$$

are operator monotone, normalized in the sense that $f(1) = 1$ and satisfy the functional equation $f(t) = tf(t^{-1})$ for $t > 0$.

Proof. We first calculate the integral

$$\int_0^1 \frac{\lambda^2 - 1}{\lambda^2 + 1} \cdot \frac{1 + t^2}{(\lambda + t)(1 + \lambda t)} d\lambda = \log \frac{2t}{(1 + t)^2}$$

and then by setting $h(\lambda) = \gamma$ in (6) obtain the operator monotone function

$$f(t) = e^{\beta \frac{1 + t}{\sqrt{2}} \left( \frac{2t}{(1 + t)^2} \right)^\gamma}$$

satisfying the functional equation (3). The result now follows by setting $\beta = (\gamma - 1/2) \log 2$. QED

Remark 2.4. Once we found the functions $f_\gamma$ above, we may directly verify that they are operator monotone and satisfy the functional equation (3) without making use of Theorem 2.1. Indeed, it is trivial that they satisfy equation (3). To see that they are operator monotone we take a complex number $z = r \exp(i\theta)$ in the upper half plane, that is $0 < \theta < \pi$. Since $1 + z$ is translated one unit to the right as compared with $z$ it is still located in the upper half plane, but the angle with the real axis has decreased. It can therefore be written on the form $1 + z = r_1 \exp(i\theta_1)$ where $0 < \theta_1 < \theta$ and we notice that $0 < \theta - \theta_1 < \theta < \pi$. The analytic continuation of $f_\gamma$ to $z$ may thus be written on the form

$$f_\gamma(z) = \frac{4^\gamma r r_1^{1-2\gamma}}{2} \exp(i(\gamma \theta + (1-2\gamma)\theta_1)).$$

But since $\gamma \theta + (1-2\gamma)\theta_1 = \gamma(\theta - \theta_1) + (1-\gamma)\theta_1 \in (0,\pi)$, we derive that the imaginary part of $f(z)$ is positive. But this proves that $f_\gamma$ is operator monotone by Löwner’s theorem [9,3].

A third proof is obtained by considering the set $E$ of exponents $\gamma \in [0,1]$ such that $f_\gamma$ is operator monotone. Since $f_{(\gamma+\delta)/2} = (f_\gamma f_\delta)^{1/2}$ and the geometric mean is operator increasing, we derive that $E$ is mid-point convex, and since $E$ is closed and contains 0 and 1, we obtain $E = [0,1]$. 

5
The set of operator monotone functions \( f \) defined on the positive half-axis such that \( f(1) = 1 \) and \( f(t) = tf(t^{-1}) \) for all \( t > 0 \) has a minimal and a maximal element \([13][3]\). These extremal functions are given by

\[
f_1(t) = \frac{2t}{1 + t} \quad \text{(min)} \quad \text{and} \quad f_0(t) = \frac{1 + t}{2} \quad \text{(max)}.
\]

Since \( 4t(t + 1)^{-2} \leq 1 \) for all \( t > 0 \) we deduce that

\[
0 \leq \gamma \leq \delta \leq 1 \quad \Rightarrow \quad f_\gamma(t) \geq f_\delta(t) \quad \forall t > 0.
\]

The family \( (f_\gamma(t))_{\gamma \in [0,1]} \) therefore provides a continuously decreasing bridge between the above extremal functions. Note also that \( f_{1/2}(t) = t^{1/2} \). The Morozova-Chentsov functions corresponding to the family \( (f_\gamma(t))_{\gamma \in [0,1]} \) are given by

\[
c_\gamma(x, y) = x^{-\gamma}y^{-\gamma}\left(\frac{x + y}{2}\right)^{2\gamma - 1} \quad \gamma \in [0,1],
\]

and they provide a continuously increasing bridge between the smallest and largest symmetric monotone metrics. Finally, since \( c_{\gamma + (1-s)\delta} = c_\gamma^s c_\delta^{1-s} \) for \( \gamma, \delta, s \in [0,1] \) we realize that the mapping \( \gamma \to c_\gamma \) is log-affine.

3 The exponential order relation

We introduced in an earlier paper \([3]\) the exponential ordering \( \preceq \) between linear self-adjoint operators on a Hilbert space by setting \( A \preceq B \) if \( \exp A \leq \exp B \). It is easily verified that \( \preceq \) is an order relation, and since the logarithm is operator monotone it follows that \( A \preceq B \) implies \( A \leq B \). This is expressed by saying that the order relation \( \preceq \) is stronger than \( \leq \).

We also introduced and studied the set \( E \) of real functions \( F : \mathbb{R} \to \mathbb{R} \) which are monotone with respect to the exponential ordering.

**Proposition 3.1.** The mapping \( \Phi \) defined by setting

\[
\Phi(F)(t) = \exp F(\log t) \quad x \in \mathbb{R}
\]

is a bijection of \( E \) onto the set \( \mathcal{P} \) of positive operator monotone functions defined on the positive half-axis.

**Proof.** Let \( A \) and \( B \) be positive invertible operators. Then

\[
\Phi(F)(A) \leq \Phi(F)(B) \quad \Leftrightarrow \quad \exp F(\log A) \leq \exp F(\log B)
\]
\[
\Leftrightarrow \quad F(\log A) \preceq F(\log B),
\]

and the assertion follows since the logarithm maps the positive half-line onto the real line. \( \text{QED} \)
The next result was proved in [3, Theorem 2.3].

**Theorem 3.2.** A non-constant function \( F : \mathbb{R} \to \mathbb{R} \) belongs to the class \( \mathcal{E} \) if and only if it admits an analytic continuation into the strip \( \{ z \in \mathbb{C} \mid 0 < \Im z < \pi \} \) which leaves the strip invariant.

Based on this result and by applying the theory of analytic functions we obtained [3, Theorem 2.4] the following representation theorem.

**Theorem 3.3.** A function \( F : \mathbb{R} \to \mathbb{R} \) is in the class \( \mathcal{E} \) if and only if it admits a canonical representation

\[
F(x) = \beta + \int_{-\infty}^{0} \left( \frac{1}{\lambda - \exp x} - \frac{\lambda}{\lambda^2 + 1} \right) h(\lambda) \, d\lambda \quad x \in \mathbb{R},
\]

where \( h : (-\infty, 0] \to [0, 1] \) is a measurable function and \( \beta \) is a real constant. The constant \( \beta \) and the equivalence class containing \( h \) are uniquely determined by \( F \).

4 The functional equation

The next result is the key observation in the present article.

**Lemma 4.1.** Let \( F \) be a function in \( \mathcal{E} \). The function \( f = \Phi(F) \in \mathcal{P} \) satisfies the functional equation [3] if and only if \( F(x) = x + F(-x) \) for every \( x \in \mathbb{R} \).

**Proof.** By taking the logarithm in the equation

\[
\exp F(\log t) = \Phi(F)(t) = f(t) = tf(t^{-1}) = t \exp F(\log t^{-1}),
\]

we realize that the functional equation [3] for \( f \) is equivalent to

\[
F(\log t) = \log t + F(-\log t),
\]

or to \( F(x) = x + F(-x) \) by setting \( x = \log t \). \hspace{1cm} \text{QED}

We need the following lemma as a preparation to the main theorem in this section.

**Lemma 4.2.**

\[
\int_{-1}^{0} \frac{2 \sin \theta}{\lambda^2 - 2 \lambda \cos \theta + 1} \, d\lambda = \theta \quad 0 < \theta < \pi.
\]
Proof. Since the integrand can be written as \(2\Im(\lambda - e^{i\theta})^{-1}\) the integral is calculated to be
\[
2\Im \left[ \log(\lambda - e^{i\theta}) \right]_{-1}^0 = 2\Im \log \frac{e^{i\theta}}{1 + e^{i\theta}} = 2\Im \log \frac{e^{i\theta/2}}{2\cos(\theta/2)} = 2\Im \left( -\log 2\cos(\theta/2) + \log e^{i\theta/2} \right) = \theta,
\]
where we used the complex logarithm. \(\Box\)

**Theorem 4.3.** A function \(F \in \mathcal{E}\) with canonical representation as given by (11) satisfies the functional equation
\[
F(x) = x + F(-x) \quad \forall x \in \mathbb{R},
\]
if and only if \(h(\lambda^{-1}) = 1 - h(\lambda)\) for almost all \(\lambda \in [-1, 0)\).

Proof. Suppose that a function \(F \in \mathcal{E}\) satisfies the given functional equation. Applying analytic continuation into the strip \(\{z \in \mathbb{C} \mid 0 < \Im z < \pi\}\) and setting \(x = i\theta\), we thus obtain
\[
F(i\theta) = i\theta + F(-i\theta) \quad 0 < \theta < \pi.
\]
Inserting the integral expression (11) we then get
\[
F(i\theta) - F(-i\theta) = \int_{-\infty}^0 \left(\frac{1}{\lambda - \exp(i\theta)} - \frac{1}{\lambda - \exp(-i\theta)}\right) h(\lambda) d\lambda = \int_{-\infty}^0 \frac{2i\sin \theta}{\lambda^2 - 2\lambda \cos \theta + 1} h(\lambda) d\lambda = i\theta,
\]
or equivalently
\[
\int_{-\infty}^0 \frac{2 \sin \theta}{\lambda^2 - 2\lambda \cos \theta + 1} h(\lambda) d\lambda = \theta \quad 0 < \theta < \pi.
\]
We split the range of integration at the point \(\lambda = -1\) and make the variable change \(\lambda \rightarrow \lambda^{-1}\) in the first term and calculate
\[
\theta = \int_{-\infty}^0 \frac{2 \sin \theta}{\lambda^2 - 2\lambda \cos \theta + 1} h(\lambda) d\lambda = \int_{-1}^0 \frac{2 \sin \theta}{\lambda^{-2} - 2\lambda^{-1} \cos \theta + 1} h(\lambda^{-1}) \frac{-1}{\lambda^2} d\lambda + \int_{-1}^0 \frac{2 \sin \theta}{\lambda^2 - 2\lambda \cos \theta + 1} h(\lambda) d\lambda.
\]
By applying Lemma 4.2 we therefore obtain
\[ \int_{-1}^{0} \frac{2 \sin \theta}{\lambda^2 - 2 \lambda \cos \theta + 1} (h(\lambda) + h(\lambda^{-1}) - 1) \, d\lambda = 0 \quad 0 < \theta < \pi, \]
which is simplified to
\[ \int_{-1}^{0} \frac{1}{\lambda^2 - 2 \lambda \cos \theta + 1} (h(\lambda) + h(\lambda^{-1}) - 1) \, d\lambda = 0 \quad 0 < \theta < \pi. \]

We introduce the change of variable \( u = 2\lambda(\lambda^2 + 1)^{-1} \) and note that \( u(-1) = -1 \) and \( u(0) = 0 \). Since the derivative
\[ u'(\lambda) = \frac{2(1 - \lambda^2)}{(\lambda^2 + 1)^2} > 0 \quad -1 < \lambda \leq 0, \]
we may write \( \lambda = \lambda(u) \) as an increasing function of \( u \). By introducing the function
\[ g(\lambda) = \frac{\lambda^2 + 1}{2(1 - \lambda^2)} (h(\lambda) + h(\lambda^{-1}) - 1) \quad -1 \leq \lambda < 0, \]
we may write the above integral on the form
\[ \int_{-1}^{0} \frac{1}{1 - 2\lambda(\lambda^2 + 1)^{-1} \cos \theta} g(\lambda) u'(\lambda) \, d\lambda, \]
hence
\[ \int_{-1}^{0} \frac{1}{1 - u \cos \theta} g(\lambda(u)) \, du = 0 \quad 0 < \theta < \pi. \]

Setting \( t = \cos \theta \) we obtain that the function
\[ \varphi(t) = \sum_{n=0}^{\infty} t^n \int_{-1}^{0} u^n g(\lambda(u)) \, du = 0 \quad -1 < t < 1, \]
hence the derivatives
\[ \varphi^{(n)}(0) = n! \int_{-1}^{0} u^n g(\lambda(u)) \, du = 0 \quad n = 0, 1, 2, \ldots. \]

We conclude that the function \( u \to g(\lambda(u)) \) vanish for almost all \( u \), and since \( \lambda \to u(\lambda) \) maps sets with positive Lebesgue measure to sets with positive Lebesgue measure, we derive that \( g(\lambda) = 0 \) for almost all \( \lambda \in [-1, 0] \). But this shows that \( h(\lambda^{-1}) = 1 - h(\lambda) \) for almost all \( \lambda \in [-1, 0] \).

If on the other hand this relationship is assumed, we may calculate backwards and obtain that \( F \) satisfies the functional equation (12). The assertion then follows by applying analytic continuation and setting \( \theta = -ix \). QED
**Theorem 4.4.** A function $F : \mathbb{R} \to \mathbb{R}$ is in the class $\mathcal{E}$ and satisfy the functional equation $F(x) = x + F(-x)$ for all $x \in \mathbb{R}$ if and only if it admits a canonical representation

$$F(x) = \beta + \log \frac{1 + \exp x}{\sqrt{2}} + \int_0^1 \frac{\lambda^2 - 1}{\lambda^2 + 1} \cdot \frac{1 + \exp 2x}{(\lambda + \exp x)(1 + \lambda \exp x)} h(\lambda) \, d\lambda$$

where $h : [0, 1] \to [0, 1]$ is a measurable function and $\beta \in \mathbb{R}$. The equivalence class containing $h$ is uniquely determined by $F$, and $\beta = \Re F(i\pi/2)$.

**Proof.** Take a function $F \in \mathcal{E}$ with canonical representation as given by (3) and satisfying the functional equation. Then $h(\lambda^{-1}) = 1 - h(\lambda)$ for almost all $\lambda \in [-1, 0)$ by Theorem 4.3. By splitting the integral at the point $\lambda = -1$ in the integral representation (3) and making the substitution $\lambda \to \lambda^{-1}$ in the first term, we obtain

$$F(x) = \beta - \int_0^{-1} \left( \frac{1}{\lambda - \exp x} - \frac{\lambda^{-1}}{\lambda^2 + 1} \right) h(\lambda^{-1}) \frac{d\lambda}{\lambda^2}$$

$$+ \int_{-1}^0 \left( \frac{1}{\lambda - \exp x} - \frac{\lambda}{\lambda^2 + 1} \right) h(\lambda) \, d\lambda$$

$$= \beta + \int_{-1}^0 \left( \frac{\lambda^{-1}}{1 - \lambda \exp x} - \frac{\lambda^{-1}}{1 + \lambda^2} \right) (1 - h(\lambda)) \, d\lambda$$

$$+ \int_{-1}^0 \left( \frac{1}{\lambda - \exp x} - \frac{\lambda}{\lambda^2 + 1} \right) h(\lambda) \, d\lambda,$$

where we used Theorem 4.3. Consequently

$$F(x) = \beta + \int_{-1}^0 \left( \frac{\lambda^{-1}}{1 - \lambda \exp x} - \frac{\lambda^{-1}}{1 + \lambda^2} \right) \, d\lambda$$

$$+ \int_{-1}^0 \left( \frac{1}{\lambda - \exp x} - \frac{\lambda^{-1}}{1 - \lambda \exp x} - \frac{\lambda}{\lambda^2 + 1} + \frac{\lambda^{-1}}{1 + \lambda^2} \right) h(\lambda) \, d\lambda,$$

and since

$$\int_{-1}^0 \lambda^{-1} \left( \frac{1}{1 - \lambda \exp x} - \frac{1}{1 + \lambda^2} \right) \, d\lambda = \int_0^1 \frac{\exp x - \lambda}{(1 + \lambda \exp x)(1 + \lambda^2)} \, d\lambda$$

$$= \left[ \log(1 + \lambda \exp x) - \frac{1}{2} \log(1 + \lambda^2) \right]_{\lambda=0}^{\lambda=1}$$

$$= \log(1 + \exp x) - \frac{1}{2} \log 2 = \log \frac{1 + \exp x}{\sqrt{2}}$$

10
we obtain

\[ F(x) = \beta + \log \frac{1 + \exp x}{\sqrt{2}} + \int_{-1}^{0} \frac{1 - \lambda^2}{1 + \lambda^2} \cdot \frac{1 + \exp 2x}{(\lambda - \exp x)(1 - \lambda \exp x)} h(\lambda) \, d\lambda. \]

Defining \( h : [0, 1] \to [0, 1] \) by setting \( h(\lambda) = h(-\lambda) \) we may write

\[ F(x) = \beta + \log \frac{1 + \exp x}{\sqrt{2}} + \int_{0}^{1} \frac{\lambda^2 - 1}{\lambda^2 + 1} \cdot \frac{1 + \exp 2x}{(\lambda + \exp x)(1 + \lambda \exp x)} h(\lambda) \, d\lambda \]

which is the desired expression. Calculating backwards we first extend \( h \) to the interval \([-1, 0]\) by setting \( h(-\lambda) = h(\lambda) \), and then to the interval \([-\infty, 0]\) by setting \( h(\lambda^{-1}) = 1 - h(\lambda) \). We arrive in this way at the integral expression \([11]\), and the sufficiency thus follows by Theorem \([1.3]\) \footnote{QED}

**References**


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