Quantitative accelerated degradation testing: Practical approaches

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\begin{abstract}
The concept of accelerated testing by tracking degradation of samples over test time needs to be developed for reliability estimation. This paper aims at proposing practical approaches to conduct accelerated degradation testing on new and available used samples. For this purpose, product failure is related to a suitable physical property. Then, its failure time is defined as the expected time in which its property reaches the critical level. Degradation model of field samples returned from service due to a degrading failure mode has been estimated based on the least square method, and available gap between manufacturer criterion and user’s claim (to report a failure) has also been discussed. For a product under some stresses, a general formula has been proposed by the superposition principle in order to estimate its degradation for independent and dependent failure modes. If used samples are available, and acceleration factor of the related test is unknown, partial aging method has been presented to considerably shorten the test time.
\end{abstract}

\section{1. Introduction}

For a product with a degrading failure mode exposed to an accelerated degradation testing (ADT), there is an uncertainty in estimating its failure time because of any obvious failures in operation. This problem could become more complicated for a multi-failure mode product under some stresses, especially in the case when there are interactions between its failure modes. On the other hand, carrying out an ADT might take long time to degrade samples. Furthermore, there might be no known relationship between the test and service times. Accordingly, most of ADTs are mainly conducted for studying the degradations of physical properties over time rather than estimating reliability and lifetime. The sample which is returned from service due to a degrading failure mode (claimed by its user) might be unacceptable as failed by the manufacturer (the contrast between user and manufacturer’s criteria).

ADT has widely been used by manufacturers and testers for qualitative explanation of a degradation process, and comparative analysis. The degradation process obtained for different design aspects could be used to predict the most robust design to tolerate service stresses. Knox and Cowling \cite{1} compared the durability of two surface pre-treatments of adhesives. The effects of different stresses (e.g. \cite{2}) and different stress levels (e.g. \cite{3}) on degradations of physical properties of test samples can contribute to a modification to design in order to remove (or moderate) their effects on the product.

Although natural and outdoor tests are time consuming and limited to only special levels of stresses (depending on the geographical situation), their results could be useful to make a comparison with corresponding results obtained from the ADT, especially if no field data are available. Such comparison might be used to recognize whether a physical property could be a reasonable representative of product degradation in service (e.g. \cite{4}). One of the most probable reasons of deviation of an ADT (from natural test) could be due to its excessive high stress levels than normal (e.g. \cite{5,3}). For more realistic results, degradation of samples could be achieved by a combination of accelerated and natural testing (e.g. \cite{6}).

Conducting an ADT is a challenge of physical properties, stress levels and aging of samples (e.g. \cite{7}). In order to characterize the reliability in an ADT, every failure mode has to be related to a suitable physical property (performance factor). Then, a critical level of the property has to be specified as the failure criterion. The failure mode is considered in the failing state if its property is below the critical level (e.g. \cite{8,9,10}). Such definition has been presented by Meeker et al. \cite{11} as soft failure for degrading failure modes. This level could be specified based on customer

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expectation, the physics of the problem, the interaction of failure modes, or safety considerations. It might also be suggested by related international and local standards (e.g. [12]).

A performance factor could be measured by a destructive or a non-destructive test. The estimation of yield stress [12,13], cycles to failure in low cycle fatigue test [14] and tensile strength [15] are destructive, whereas the measurement of volume and pressure [4], flow rate of an oil pump [16], remaining weight of steel due to the corrosion and wear [12,16], non-corroded area of coatings [6], and the capacity of batteries [10] are considered as non-destructive.

Used samples (returned from service due to a degrading failure mode) could be used to estimate product reliability, either by a destructive test (e.g. [17]) or a non-destructive test (e.g. [18]). The estimation of failure time is achieved by relating residual lifetime of each sample with its performance factor. If field samples are still in operation in service, some specimens could be taken for testing their materials (e.g. [19]). Kivtsov et al. [20] tested used and new car tyres of different brands in order to measure their physical properties. Then, the Cox survival regression model was employed to estimate the most sensitive properties to the test time. Ranganathan et al. [16] measured different properties of used journal bearings in oil pumps. They also estimated corresponding results for new journal bearings, and concluded that the service and test results of journal volume loss are consistent, so a correlation was made between test time (in terms of hours) and service time (in terms of kilometres).

The main aim of this paper is to present practical approaches to conduct ADT on a product to characterize its failure time and reliability. ADT is categorized into single and multi-aging process for a failure mode under a single stress. For each category, sample-based and interval-based techniques are proposed to estimate failure time. Degradation diagrams of field samples returned from service (due to a probable degrading failure mode claimed by users) are estimated by the least square method. The contrast between manufacturer’s criterion to accept a failure, and reported failures by users are characterized by relating their lifetimes and physical properties. A general formula for a failure mode of a multi-failure mode product under some stresses has been derived by the superposition principle. The formula is also extended for dependent failure modes. Partial aging method, as a new approach to estimate product reliability by an unknown aging process, is dependent failure modes. So, the results of ADT might need to be extrapolated to the critical performance for estimating failure time. Every ACT is limited to catastrophic failure modes, but ADT could be conducted for both degrading and catastrophic failure modes, however in this paper, we only considered degrading failure modes for ADT. The application of ADT for catastrophic failure modes is the challenge of finding a property of the product whose degradation leads to the catastrophic failure mode. For example, tyre tread and its depth could be considered as one of the parameters to result in tyre split.

For highly reliable products with catastrophic failure modes in an ACT, there might be many right censored times (i.e. the samples that survive the ACT) that might result in inaccurate statistical analysis. Censored data are for binary-state products with failing and operating states, and they are meaningless for degrading failure modes. So, the results of ADT might need to be extrapolated to the critical performance for estimating failure times of test samples.

Performance diagram is usually different from sample to sample, but every diagram might belong to a common performance model (if there is any). For example, the unknown parameters a and b in the quadratic performance model \( P(t) = a - b^t \) specifies different performance diagrams for different samples. For the sample i, its performance values at time \( t = 0 \) and \( t = t_0 \) are denoted by \( P^i \) and \( P' \), respectively, so

\[
\begin{align*}
    a &= P^i \\
    b &= \frac{P' - P^i}{t_0}
\end{align*}
\]

In this section, the product is considered to be single-failure mode under a single stress with a known performance factor and its related non-destructive PT. The values of nominal and critical performance must also be identified prior to the test. Related aging process is assumed to be known, i.e. its time (test time) can be related to service time somehow as follows. In a q-based aging process, samples are under normal service levels of stresses and degradation mechanism is stimulated by increasing the usage frequency of the stress which is defined as the ratio of the time in which service samples are under this stress level to the total service time. In an AF-based aging process, acceleration factor of the aging process (the ratio of the service time to its equivalent test time to acquire the same level of degradation) is assumed to be known. In a stress-based aging process, the results of the test at high levels of stress could be used to estimate acceleration factor by known life-stress model (e.g. [21,22]).
2.2. Single-aging process

In the single-aging process, each new sample must first pass the PT in order to obtain its initial performance (before the aging process). According to the above-mentioned assumption, the amount of initial performance must be the same and equal to the nominal performance for every new sample. Accordingly, the implementation of the PT in this stage aims at estimating the nominal performance which could be the mean value of the measured initial performance values. If many samples are available for ADT, only a few of them are needed to pass the PT (at this stage) to save time.

Then, the sample has to pass the only aging process for the specified period of time \( t_i \). Finally, it must again pass the PT to obtain its performance after the aging process. For the new sample \( i \), \( P^N \) and \( P \) denote its performance factors before and after the aging process, respectively. Performance test is conducted twice on each new sample in a single-aging process, so the performance model could simply be assumed to be linear (Miner’s rule). Failure time of the sample is proposed to be estimated by the applied performance model. The estimation of failure time in the single-aging process based on the linear performance model by sample-based technique for used samples.

2.3. Multi-aging process

To improve the accuracy of estimated failure time, the aging process must be divided into some time intervals, so that a more realistic performance model (including more unknown parameters) could be obtained (compared to the single-aging process). Therefore, the whole aging process is divided into \( I \) time intervals \((I > 1)\), and performance must be measured at the beginning of the test, and after each time interval (totally \( I + 1 \) times for each sample). Like the single-aging process, the sample-based or the interval-based technique could be used to estimate failure time.
In the sample-based technique, for each new sample, according to its performance values, the best fitted diagram has to be estimated by regression analysis. Then, its failure time is defined as the time in which the diagram intersects the horizontal critical performance line as shown in Fig. 4. All such estimated failure times must be used for statistical analysis of reliability. Because regression analysis is individually done for each sample, there might be no performance model, i.e. each sample has its own performance diagram which might inherently be different from others.

In the interval-based technique, at the end of each time interval, the measured performance must statistically be analyzed to estimate performance probability diagram at the time as shown in Fig. 5. Depending on required accuracy, an equal number of representative values ($S^i$) must be taken from every performance probability diagram, and such representative values for each diagram must be sorted in descending order. Accordingly, $L$ sets of representative values in descending order are obtained. The virtual sample $i$ is defined as the sample including every value of the sets listed in the ranking $i$ at its own time. Then, its performance diagram and failure time could be estimated by regression analysis. The virtual samples 1 and $S^i$, respectively, have the longest and the shortest failure times among the virtual samples. Then, the non-parametric reliability of the product must be estimated by Eq. (2).

3. Degradation analysis of field samples

Used samples of a product returned from service due to a degrading failure mode claimed by users and/or technicians, potentially possess valuable information about their degradation processes in their own fields. Qualitative explanation of these samples (excellent, good, fair or poor) is not sufficient to predict their lifetimes for an exact reliability analysis. In addition, such explanation is not able to describe the available usual gap between manufacturer and user’s criteria to report a failure.

Primary requirements for reliability analysis of the samples returned from their own fields are the identification of performance factor and its corresponding PT for the failure mode under study. In order to identify the frontier between operating and failing states, critical performance has to be specified. The implementation of the PT on a service sample (with its known age) reveals its location in the performance-time coordinate system. Furthermore, in order to identify nominal performance, the PT must be conducted on some new samples, so their average performance values could be considered as the nominal performance.

In this section, in order to clarify different aspects in field failure analysis of degrading failure modes, the results of an empirical study on 12-V valve regulated lead acid (VRLA) batteries used in 24-V powered wheelchairs (two batteries for each wheelchair) are presented. The failure nature of such batteries is discharging, and battery capacity, which is the time required to discharge a battery from fully charged to fully discharged, is selected as the performance factor (e.g. [23,10]). Battery capacity is capable of fulfilling the main requirements of a performance factor as expressed by Mohammadian et al. [24]. It is measurable and sensitive to service time, and it continuously decreases over time. For our application in powered wheelchairs, the above definition has been re-defined as the time required to discharge the battery from 12.7 V (not fully charged) to 12.0 V (not fully discharged), because most powered wheelchair batteries are used in this range of voltage. Such modification represents a real connection between the performance factor of batteries and their required application. Critical performance is defined as 80% of the nominal capacity for such batteries by many manufacturers (e.g. [23,25]). Hereafter, this issue is referred to as manufacturer’s criterion.

Performance test including both charging and discharging tests are carried out in an approximate ambient temperature of 23°C, whereas discharging current rate is kept constant at 6 A.
Performance test has been conducted on 17 used batteries (claimed as failed by users) and two new batteries in order to obtain their capacities as shown in Fig. 6 by Mohammadian et al. [18]. Nominal performance ($P_N$) is calculated as the average capacity values of new batteries ($P_N = 366$ min), so critical performance is calculated as $P_c = (0.8)(366) = 292.8$ min according to the above-mentioned manufacturer’s criterion. The age and performance of the used battery i are denoted by $L_i$ and $P_i$, respectively. The batteries below the horizontal critical performance line are considered as failed, so only five out of the 17 batteries are failed based on the manufacturer’s criterion.

3.1. The best fitted performance model

The least square method is used to obtain the best fitted function (curve) among m mathematical functions as $T_i(t)$, $j=1, 2, \ldots, m$. The square estimator of the $j$th function ($SE_j$) is defined as

$$SE_j = \sum_{i=1}^{S} \left( \frac{T_i(L_i) - P_i}{L_i} \right)^2$$

where $T_i(L_i)$ is the estimated performance of the used sample i by the function $j$ at the service time $L_i$, and $S$ is the number of used samples. According to the least square method, the function $j$ is the best fitted function (among m functions) if

$$SE_j < SE_k, \quad k = 1, 2, \ldots, m, \quad k \neq j$$

The least square method could also be used to estimate unknown parameters of a specified performance model. Then, all parameters must be kept constant at their own estimated values except one of them that has to be obtained for each used sample individually (based on its performance value at its age). As an illustrative example, we present the model $P(t) = P_N - at^b$ with two unknown parameters $a$ and $b$ for the wheelchair batteries. According to the least square method, the parameters $a$ and $b$ should be obtained so that they could minimize the square estimator $SE$. By a simple MATLAB program, the parameters $a$ and $b$ are estimated as 91.5(10^-5) and 1.38, respectively, as illustrated in Fig. 6. The square estimator of the function $T_i(t) = 366 - 0.00915t^{1.38}$ is estimated as $SE = 0.0151$ by Eq. (3). The performance model $P(t) = P_N - at^{1.38}$ (by keeping $b$ at its estimated value) is one of the best fitted models for the wheelchair batteries with the unknown parameter $a$. For the sample i, its unknown parameter $a_i$ is obtained so that its performance sets to $P_i$ at the age $L_i$. Consequently, related unknown parameter and lifetime of the sample i are obtained as

$$a_i = \frac{P_N - P_i}{(L_i)^{1/1.38}} \quad \text{and} \quad t_{ij} = \left( \frac{P_N - P_{ij}}{a_i} \right)^{1/1.38}$$

The reliability of wheelchair batteries for the best fitted diagram at the first year in service is estimated by the Weibull pdf diagram on the 17 used wheelchair batteries, and compared with other known performance models as illustrated in Fig. 7. It can be concluded that the selected performance model can considerably affect the estimated reliability of the product. Note that, the best fitted model has the least square estimator among the presented functions.

3.2. User-based criterion

For most catastrophic failure modes, the failure reported by a user is often acceptable by the manufacturer because the failure is obvious. For degrading failure modes, there might be a contrast between user and manufacturer’s decisions to report a failure. Usually, critical performance is the base criterion to define failure from the manufacturer point of view. Such criterion is often in contrast to user expectation. In addition, the user satisfaction level might be different from one to another. For example, a sample of a product reported as failed by its user might still be in operating state according to the manufacturer’s criterion. On the other hand, another sample might have already failed according to the manufacturer’s criterion, but it is still satisfactorily used by its own user. Such contrasts motivate manufacturers and retail users (not real users) to specify the gap between manufacturer and users’ criteria.

Returning a sample from service due to a probable degrading failure mode is the result of the customer and/or technician’s decision to report it as a failed sample. The user-based criterion states that the age of a field-returned sample must be considered as its lifetime regardless of the manufacturer’s decision to accept or refuse the reported failure. So, for the used sample i, $t_{ij} = L_i$. Then, all of these failure times are used for statistical analysis.

There are 17 used batteries belong to 12 different wheelchairs. To avoid entering extra failure time data, for each pair of batteries used in the same wheelchair, one failure time is considered. Fig. 8 illustrates the Weibull failure probability diagrams of batteries based on the best fitted performance model and the user-based criterion. The reliability is estimated as 97.4% and 93.2% at the first year based on the users and the manufacturer’s criteria, respectively, so fewer failed batteries are reported during the
first year by users compared to the expected value estimated by the manufacturer's criterion. Surprisingly, more than 96% of batteries are returned between the first and the second years, whereas only 42.2% of them are expected to be returned during this period of time. The main reason of this major contrast might be due to users' perception that battery capacity considerably decreases after the first year.

The user-based criterion could be stated in terms of performance values of samples. If each field-retumed sample is accepted as a failed sample, then its performance must be considered as a critical performance. Hence, the critical performance is defined as a random variable introduced by field samples. The capacities of 17 field batteries are used to estimate the normal distribution diagram for critical performance as shown in Fig. 9. The average value of the random critical performance is accepted as a failed sample, then its performance must be considered as a critical performance. Hence, the critical performance is defined as a random variable introduced by field samples. The capacities of 17 field batteries are used to estimate the normal distribution diagram for critical performance as shown in Fig. 9. The average value of the random critical performance is estimated as 310.8 min, so there is a huge gap between critical performance values in damage-time coordinate system. The best fitted damage model of the failure mode $q$-based, the time $q_{l}$ must be replaced with $q_{t}$, where $q_{t}$ is the usage frequency of the stress $l$, and $t$ is the service time. Service

4. Generalized ADT concept

For a single-failure mode product under a single stress, ADT can estimate its performance model, failure times of its samples, and finally product reliability. Such problem could become more complicated if the product is multi-failure mode, and its samples are under more than one stress in service. For each stress, an exclusive known aging process must be defined while all other stresses are inactive. Then, the effect of the stress must individually be estimated on every failure mode in order to estimate its performance model due to the stress. The existence of random variables inside performance models and usage profile make reliability estimation more complicated (or impossible) to be solved by analytical methods.

Unit-less damage factor (or simply damage) at the time $t$ is defined as $[26-28]$:

$$D(t) = \frac{P_{0} - P(t)}{P_{0}}$$

(5)

where damage value is zero at the beginning of its operation and one at the failure time. After the failure, the amount of damage stands constant at one for catastrophic failure modes, whereas it tends to increase over one for degrading failure modes. Damage diagram (damage versus time) is assumed to be a continuous ascending diagram. Damage factor $D(t)$ and performance factor $P(t)$ of a failure mode are related together by Eq. (5), so failure time could be estimated by using either performance or damage diagram.

For a physical property in a multi-failure mode product, performance-time coordinate system is scaled based on its physical unit. The performance diagram of another failure mode cannot be drawn in this coordinate system, because the coordinate system has already been scaled based on the first physical unit which might be different from the second one. The damage diagrams of all physical properties could be drawn together in a single coordinate system, because damage is a unit-less factor. Therefore, such illustration facilitates the observation and comparison of physical properties.

For a failure mode under a specified stress, damage diagram could be different from sample to sample, but all diagrams might belong to a common damage model. According to the applications and stresses, variety of damage models is available [29,30]. A damage model depends on material, design, manufacturing process, applications and users. The most suitable damage model could be recognized by an apparent observation of available damage values in damage-time coordinate system. The best fitted model could also be detected by the least square method.

4.1. Independent failure modes

A failure mode is called independent if other failure modes cannot have any effects on its damage factor. ADT on a product with $N$ independent failure modes under $M$ stresses must individually be performed for every stress. It is assumed that every combination of stresses in service levels is not able to activate another failure mechanism than the ones that have already been detected under the individual stresses, otherwise such combination must be considered as a new stress, and the new failure mode has to be included in the product.

Destructive nature of PT increases sample size and total time of accelerated testing. If the PT of every failure mode is non-destructive, for each stress, one set of test samples is needed for the ADT. An exclusive set of test samples is needed for any performance factor whose PT is destructive.

The mathematical function $G_{j}(\tau_{l})$ denotes the test damage model of the failure mode $j$ under the stress $l$ (as the only stress while others are not active) in terms of the time of the aging process $\tau_{l}$. This model obtained from ADT must be extended to its corresponding service model as follows. If the aging process is q-based, the time $\tau_{l}$ must be replaced with $q_{l}t$, where $q_{l}$ is the usage frequency of the stress $l$, and $t$ is the service time. Service
where $AF_{ij}$ is the acceleration factor of the aging process. It is concluded that for the failure mode $j$ under the stress $l$, the constant acceleration factor in AF-based aging process and the random variable usage frequency in corresponding $q$-based aging process have an inverse relationship in the above formulas.

The expression $dD_j/l(t)$ is the time derivative of the damage model that is called damage rate (e.g. [31]), and it is assumed to be a function of $D_j$ rather than the service time $t$. The damage rates of the $q$-based and AF-based aging processes are, respectively, obtained from Eqs. (6) and (7):

$$
\frac{dD_j(t)}{dt} = q_jG_j(D_j)
$$

(8)

$$
\frac{dD_j(t)}{dt} = \frac{1}{AF_{ij}}G_j(D_j)
$$

(9)

where $G_j(D_j)$ is the time derivative of $G_j(t)$. For instance, if the test damage model of the failure mode $j$ under the stress $l$ is obtained by an ADT as quadratic form $G_j(t) = ct^2$ where $c$ is the model parameter, the service damage rates of the failure mode for $q$-based and AF-based aging processes are obtained as

$$
\frac{dD_j(t)}{dt} = q_j(2\sqrt{cD_j})
$$

(10)

$$
\frac{dD_j(t)}{dt} = \frac{1}{AF_{ij}}(2\sqrt{cD_j})
$$

(11)

where damage rates are not dependent on time.

The damage rate $dD_j/l(t)$ of the failure mode $j$ is only one caused by the stress $l$. According to the superposition principle, the total damage rate of this failure mode is equal to the summation of every damage rate, each caused by its own related stress. Therefore, the general differential equation for the failure mode $j$ is expressed as

$$
\frac{dD_j(t)}{dt} = \sum_{i=1}^{N} D_j(t)
$$

(12)

where $D_j(t)$ is the (total) damage rate of the $j$th physical property. As you can see from the above formula, the damage rate of $j$ is a function of damage factor $j$ under different stresses, and it is independent of other damage factors than $j$ (the definition of independent failure mode). The above equation for all failure modes, i.e. $j = 1, 2, \ldots, N$ makes a system of $N$ non-linear independent first order differential equations for the product.

In general, there exists no analytical solution for the above system of equation regarding the existence of many random variables, so it should be solved numerically. Here, the virtual sample method is used by representative values of probability diagrams. For every stress, if its related aging process is $q$-based, a number of representative values must be taken from its usage frequency probability diagram. Representative values must also be taken from each failure probability diagram obtained from its related ADT. Then, such failure times should be used to estimate representative damage diagrams based on the known damage models as typically illustrated in Fig. 10 for the failure mode $j$ under the stress $l$ by the linear damage model.

For an AF-based aging process, each failure probability diagram must be modified based on its known service-related diagram in order to achieve a unique acceleration factor. For example, for Weibull function, the test diagram must set parallel to the service diagram by a common shape factor.

A virtual sample is identified by its damage diagrams (a damage diagram from each damage model) and usage frequencies (a value from each set of usage frequency representative values of a stress if its related aging process is $q$-based).

Regarding the number of representative values of usage frequencies and representative damage diagrams, there might be huge number of virtual samples as

$$
S^\prime = \prod_{j=1}^{N} \prod_{l=1}^{M} S_j^l = \prod_{l=1}^{M} S_j^l
$$

(13)

where $S_j^l$ is the number of representative damage diagrams for the damage model of the $j$th physical property under the stress $l$, and $S_j$ is the number of representative values of the usage frequency of the stress $l$ if the related aging process is $q$-based. Hence, for each virtual sample, a system of equations (like Eq. (12)) is obtained. The system can be solved by the finite difference approximation as

$$
\frac{D_j'(t+\Delta t) - D_j'(t)}{\Delta t} = \frac{N}{\sum_{i=1}^{N} \frac{dD_j'(t)}{dt}}
$$

for $j = 1, 2, \ldots, N$

(14)

where the accent $'$ represents a virtual sample. The time interval $\Delta t$ specifies the precision and the speed of calculation. The step by step calculation leads to obtain the damage factor of each failure mode at any time. According to the overall assumption in this paper, the initial value of damage for each physical property should be considered zero, i.e. $D_j'(0) = 0$. $j = 1, 2, \ldots, N$, otherwise initial damage must be considered as a random variable, and its representative values must also be applied to the system of equations.

The failure mode $j$ of a sample occurs whenever its damage $D_j$ reaches 1, but it does not mean the sample fails. For this purpose, failure of the product has to be defined based on its failure modes and their connections which might be series, parallel, $k$-out-of-$n$, stand-by, etc. According to the definition of the product, the damage factor of the product could be defined as its proximity to the failing state. For example, for series and parallel products, their damage factors could be defined as the highest and the lowest damage factors of their failure modes, respectively. At any given time $t$, according to the number of virtual samples ($S^\prime$) and the number of survived virtual samples ($S_j^l$), non-parametric reliability of the product is estimated by Eq. (2).

As an illustrative example, consider a product with two independent series failure modes under two stresses. The first...
ADT is an AF-based aging process that has no effect on the second failure mode, whereas the second ADT is a q-based aging process. The characteristics of both aging processes, scale and shape factors of each failure probability diagram, and service damage models are presented in Table 1.

For each virtual sample, the random variable q must be taken from related probability diagram of the usage frequency, and unknown parameters a, b and c have to be obtained from the test failure probability diagrams. In other words, each set of (q, a, b, c) specifies an exclusive virtual sample. For every virtual sample, its system of equations is written as (see Eqs. (10) and (11))

\[
\frac{dD_1(t)}{dt} = \frac{a}{AF} + 2q\sqrt{bD_1}
\]
\[
\frac{dD_2(t)}{dt} = cq
\]

Because the product is assumed to be series, the time of the first failure mode occurrence must be considered as the failure time of the virtual sample. Finite difference approximation has been used to solve the above system for each virtual sample in time step \(\Delta t = 1\) day.

By a MATLAB program, reliability of the product at the first year is estimated and drawn in Fig. 11 in terms of the number of virtual samples. The ability of ADT to estimate reliability for a product with dependent failure modes is one of ADT’s most distinctive advantages over ACT. If the failure mode \(k\) can stimulate the degradation process of the physical property \(j\), then the failure modes \(j\) and \(k\) are called dependent and cause failure modes, respectively. The general formula presented for independent failure modes (Eq. (12)) is not valid for dependent failure modes, and it must be modified based on the physics of the problem and the type of the cause failure mode (catastrophic or degrading).

### 4.2. Dependent failure modes

The catastrophic cause failure mode \(k\) during its operating state usually has no effect on the dependent failure mode \(j\). The effect of the failure of the cause failure mode could be like a physical impact on the dependent failure mode so that it could suddenly increase the stress level afterwards, and consequently, change its damage diagram.

The system of beam and rope shown in Fig. 13 under the mechanical force \(F\), has two failure modes which are the dependent failure mode \(j\) due to the breaking the beam at point I, and the cause failure mode \(k\) due to the catastrophic failure of the rope. Once the rope fails at the time \(t_k\), the stress at the point I suddenly increases so that it will result stimulating degradation of the beam at the point I. It is assumed that the physical impact on the failure mode \(j\) at the time \(t_k\) cannot activate degradation mechanism of other failure mode (than the one under study), otherwise the new failure mode has to be considered in the analysis.

In order to obtain the damage rate formula for the dependent failure mode \(j\), ADT must individually be done on the system for both stress levels (stress level \(l\) before and stress level \(l\) after the catastrophic failure \(k\)) in order to obtain their damage models. Then, the general formula for the independent failure mode (Eq. (12)) must be modified to be applicable for the dependent

### Table 1

The characteristics of the aging processes, failure probability diagrams, and damage models, where \(\alpha\) and \(\beta\) are scale and shape factors of the Weibull distribution function, respectively, and \(\beta_1\) is the shape factor for field failure data.

<table>
<thead>
<tr>
<th>Failure mode</th>
<th>First ADT (under the stress I)</th>
<th>Second ADT (under the stress II)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AF-based aging process</td>
<td>q-base aging process</td>
<td></td>
</tr>
<tr>
<td>(\alpha) (day)</td>
<td>(\beta)</td>
<td>AF</td>
</tr>
<tr>
<td>I</td>
<td>190</td>
<td>7.5</td>
</tr>
<tr>
<td>II</td>
<td>65</td>
<td>7.5</td>
</tr>
</tbody>
</table>
failure mode \( j \) as
\[
\frac{dD_j(t)}{dt} = (1 - \phi^0(D_h)) G_{j,l}(D_l) + \phi^0(D_h) H_{j,l}(D_l)
\]  
(15)
where \( \phi^0(D_h) \) and \( H_{j,l}(D_l) \) are damage rate diagrams (Fig. 13) due to the stresses \( l \) and \( t \), respectively, and \( \phi^0(D_h) \) is the zero order singularity function defined as
\[
\phi^0(D_h) = \begin{cases} 
0 & \text{if } D_h < 1 \\
1 & \text{if } D_h \geq 1
\end{cases}
\]
As you can see from Eq. (15), the damage rate of \( j \) is a function of damage factor \( k \) (the definition of dependent failure mode).

4.2.2. Degradation cause failure mode

Regarding the physics of the product and the connection between its failure modes, the damage of the degrading failure mode \( k \) (not its failure) might continuously stimulate degradation mechanism of \( j \). In other words, the total damage rate of the failure mode \( j \) could be affected by the damage level of \( k \).

The original system of beam and spring shown in Fig. 14 includes the cause failure mode \( k \) due to continuous degradation of the spring constant \( K \) and the dependent failure mode \( j \) at point \( I \). The supported mechanical force by the spring decreases over time because of degradation in the spring constant. Consequently, the level of mechanical stress at the point \( I \) continuously increases, and it will cause speeding up the damage rate of \( j \). The general formula for the failure mode \( j \) could be modified as
\[
\frac{dD_j(t)}{dt} = A_{j,k}(D_h) G_{j,l}(D_l)
\]  
(16)
where \( A_{j,k}(D_h) \) is called dependency function of the failure mode \( j \) on the failure mode \( k \), and it shows the tendency of the failure mode \( j \) to increase its damage level in comparison to the situation it is independent. Note that the function \( G_{j,l}(D_l) \) is the damage rate of \( j \) without any effects from \( k \). For each level of \( D_h \), \( A_{j,k}(D_h) \) must be greater than one. As you can see from Eq. (16), the damage rate of \( j \) is a function of damage factor \( k \) (the definition of dependent failure mode).

In order to estimate the dependency function, ADT must individually be conducted for the original and modified products (Fig. 14) as follows:

- ADT on the original product: The ADT test on the original product results in the damage rate diagram of the failure mode \( j \) by considering the effect of the failure mode \( k \).
- ADT on the modified product: In order to remove the effect of the failure mode \( k \), the original product must be modified so that the stress level on the failure mode \( j \) is kept constant and equal to its level at the beginning of its operation in which the damage of the failure mode \( k \) is zero.

The diagram in Fig. 14 clearly shows the difference between damage rate diagrams obtained from the above ADTs. The dependency function should be estimated by dividing the first damage rate diagram by the second one. The diagrams also show the individual effects of the stress \( F \) and the failure mode \( k \) on the dependent failure mode \( j \). This issue is useful for design engineers to decide necessary modifications to design in order to decrease the damage of \( j \).

5. Partial aging method

The concept of the partial aging method proposed here for a single-failure mode product is based on aging a few new and some available used samples (within their known ages) in a relatively shorter time than a regular ADT by an unknown aging process. If many used samples are available, some of them should be selected for testing so that they could cover a wide range of ages and applications. Critical performance must be identified prior to the test. In order to make an estimation of the time of a partial aging process, there should be an approximate prediction of the whole aging process \( L_a \), i.e. required time to fail a new sample.

To implement the method, each sample (new and used) must firstly pass the PT in order to estimate its damage before the aging process. Then the sample must pass a portion of the whole aging process for the time period of \( \delta t \) (\( \delta t \ll L_a \)). Finally the sample has to again pass the PT to estimate its damage after the partial aging process. Note that, the unknown aging process used in this method distinguishes it from the single-aging process.

Let \( D_i^1 \) and \( D_i^2 \) be the damages of the sample \( i \) before and after the partial aging process, respectively, so the approximate damage...
rate of the sample at the damage level $D^{i,1}$ could be estimated as

$$D^{i,1} = \frac{D^{i,2} - D^{i,1}}{\delta t}$$  \hspace{1cm} (17)$$

The geometrical definition of damage rate for a sample is the gradient of its linear damage diagram (over the span of $\delta t$) as shown in Fig. 15 for the new sample $j$ and the used sample $k$.

The main assumption in this method is that the damage rate of the product $dD/dt$ is a function of the damage level $D$ (not the age). Accordingly, for each sample, its damage before the aging process and its damage estimated by Eq. (17) make a couple (a point) in the two dimensional coordinate system in which the damage rate is expressed in terms of damage. By the least square method, the best fitted function (curve) could be estimated as

$$\frac{dD}{dt} = g_{a}(D)$$  \hspace{1cm} (18)$$

where the subscript $a$ denotes aging process, and the mathematical function $g_{a}(D)$ is called damage rate model (DRM).

Unlike performance and damage models, there are not any unknown parameters in DRM, because it represents an average function of damage rate. In order to obtain the damage rate diagram for the $i$ th used sample, i.e. $g_{a}(D)$, the difference between its damage rate (estimated by Eq. (17)) and the damage rate value obtained from DRM at the damage level $D^{i,1}$ specifies the deviation of $g_{a}(D)$ from DRM as illustrated in Fig. 16 and estimated by

$$g_{a,i}(D) = \frac{D^{i,1}}{g_{a}(D^{i,1})} g_{a}(D)$$  \hspace{1cm} (19)$$

The above differential equation could analytically (or numerically) be solved for the sample $i$ in order to estimate its service damage rate. For this purpose, the time in test $t$ must be transferred to the corresponding time in service $s$ by defining the acceleration factor of the sample $i$ as $AF_{i} = t_{f,i}/t_{s,i}$ which is unknown. Note that, $t_{f,i}$ is the lifetime of the sample $i$. Then (see Eq. (9))

$$\frac{dD}{dt} = \frac{1}{AF_{i}} g_{a,i}(D)$$  \hspace{1cm} (20)$$

The above differential equation should be solved so that service damage diagram (in terms of time $t_{s}$) must include the point $(L_{i}, D^{i,1})$ where $L_{i}$ is the age of the sample $i$. Then, the unknown acceleration factor of the sample $i$, and consequently, its service damage could be obtained.

The damage factors of 7 used samples and a new sample before and after the partial aging process are presented in Table 2. The approximate whole aging process time is considered as 1000 h, so the partial aging process time has been selected as $\delta t = 100$ h. Although the initial damage of used samples is considered zero, but this value for the only new sample is not zero. The Weibull pdf is used to statistically analyze the test and service failure times. Scale parameters of the aging process and service are obtained as $x = 884.9$ h and $x = 5798.1$ h, respectively, so the acceleration factor of the whole aging process is estimated as 6.55. In fact, the samples are aged for only 100 h, so the acceleration factor of the partial aging is estimated as $\frac{x}{100} = 58.0$ which is much higher than the one for the whole aging process.

### Table 2

<table>
<thead>
<tr>
<th>Sample</th>
<th>Damage</th>
<th>$D^{i,1}(10^{-4})$</th>
<th>$g_{a,i}(D)$</th>
<th>$D^{i,1}/g_{a}(D^{i,1})$</th>
<th>$t_{f,i}$</th>
<th>$AF_{i} = t_{f,i}/t_{s,i}$</th>
<th>$t_{s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01</td>
<td>0.06</td>
<td>5</td>
<td>by least square method based on linear damage rate model:</td>
<td>0.9288</td>
<td>829.0</td>
<td>4.13</td>
</tr>
<tr>
<td>2</td>
<td>0.14</td>
<td>0.22</td>
<td>8</td>
<td>$1.3034$</td>
<td>500.8</td>
<td>13.60</td>
<td>8031.7</td>
</tr>
<tr>
<td>3</td>
<td>0.13</td>
<td>0.24</td>
<td>11</td>
<td>$0.7718$</td>
<td>997.8</td>
<td>5.05</td>
<td>5042.8</td>
</tr>
<tr>
<td>4</td>
<td>0.39</td>
<td>0.49</td>
<td>10</td>
<td>$0.6604$</td>
<td>1166.0</td>
<td>3.67</td>
<td>4280.6</td>
</tr>
<tr>
<td>5</td>
<td>0.69</td>
<td>0.81</td>
<td>12</td>
<td>$1.4295$</td>
<td>538.7</td>
<td>10.32</td>
<td>5558.4</td>
</tr>
<tr>
<td>6</td>
<td>0.73</td>
<td>0.75</td>
<td>22</td>
<td>$x = 17.38\times10^{-4}$</td>
<td>849.1</td>
<td>5.94</td>
<td>5041.8</td>
</tr>
<tr>
<td>7</td>
<td>0.85</td>
<td>1.04</td>
<td>19</td>
<td>$y = 6.18\times10^{-4}$</td>
<td>650.7</td>
<td>8.55</td>
<td>5563.2</td>
</tr>
<tr>
<td>8</td>
<td>0.86</td>
<td>1.11</td>
<td>25</td>
<td>$z = 5798.1$</td>
<td>884.9</td>
<td>5798.1</td>
<td>50</td>
</tr>
</tbody>
</table>
6. Conclusion

ADT has been categorized based on the number of its aging process and performance test as single-aging and multi-aging processes. For both sample and interval-based techniques to estimate failure time, the multi-aging process is suggested in order to achieve more accurate estimation of reliability.

The accuracy of analyzing the results of field samples of a product strongly depends on its estimated performance model. The best fitted diagram (obtained from the least square method) has been proposed to be selected as the performance model. The gap between the estimated reliability by the manufacture’s criterion and reported failures by users has been characterized by their estimated failure and performance probability diagrams.

A general formula is developed for failure modes of a multi-failure mode product under multiple stresses. Regarding the complexity of analytical methods to solve the system of differential equations due to the existence of random variables, virtual sample method has been introduced as a numerical technique to estimate non-parametric reliability. It has been concluded that the accuracy of virtual sample method has a direct relationship with selected number of virtual samples. The general formula has also been extended for a dependent failure mode. If new and used samples of the product are available, the partial aging method is proposed for an unknown aging process to considerably decrease required aging time.

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References