Constructing an efficient parametrization of a large, noisy data set of points lying close to a smooth manifold in high dimension remains a fundamental problem. One approach consists in recovering a local parametrization using the local tangent plane. Principal component analysis (PCA) is often the tool of choice, as it returns an optimal basis in the case of noise-free samples from a linear subspace. To process noisy data, PCA must be applied locally, at a scale small enough such that the manifold is approximately linear, but at a scale large enough such that structure may be discerned from noise. Using eigenspace perturbation theory, we study the stability of the subspace estimated by PCA as a function of scale, and bound (with high probability) the angle it forms with the true tangent space. By adaptively selecting the scale that minimizes this bound, our analysis reveals the optimal scale for local tangent plane recovery.

1. Local Tangent Plane Recovery.

1.1. Introduction. Large data sets of points in high-dimension often lie close to a smooth low-dimensional manifold. A fundamental problem in processing such data sets is the construction of an efficient parameterization that allows for the data to be well represented in fewer dimensions. Such a parameterization may be realized by exploiting the inherent manifold structure of the data. However, discovering the geometry of an underlying manifold from only noisy samples remains an open topic of research.

The case of data sampled from a linear subspace is well studied (see [13, 15, 22], for example). The optimal parameterization is given by principal component analysis (PCA), as the singular value decomposition (SVD) produces the best low-rank approximation for such data. However, most interesting manifold-valued data organize on or near a nonlinear manifold. PCA, by projecting data points onto the linear subspace of best fit, is not
optimal in this case, as curvature may only be accommodated by choosing a subspace of dimension higher than that of the manifold. Algorithms designed to process nonlinear data sets typically proceed in one of two directions. One approach is to consider the data globally and produce a nonlinear embedding. Alternatively, the data may be considered in a piecewise-linear fashion and linear methods such as PCA may be applied locally. The latter is the subject of this work.

There have been several versions of localized PCA for tangent plane recovery proposed in the literature. While the need for locality has been acknowledged, a precise treatment of the size of the local neighborhood is often not addressed. The appropriate neighborhood size must be a function of intrinsic (manifold) dimensionality, curvature, and noise level. Despite the fact that these properties may change as different regions of the manifold are explored, locality is often defined via an \textit{a priori} fixed number of neighbors or as the output of an algorithm (e.g., [24, 33, 16, 31]). Other methods [2, 23, 18] adaptively estimate local neighborhood size but are not equipped with optimality guarantees.

The selection of the optimal scale, or neighborhood size, for local tangent plane recovery is the key contribution of this paper. What is novel about our approach is that we use the geometry of the data to guide our definition of locality. On the one hand, a neighborhood must be small enough so that it is approximately linear and avoids curvature. On the other hand, a neighborhood must be large enough to overcome the effects of noise. We use eigenspace perturbation theory to study the stability of the tangent plane as the size of the neighborhood varies. We bound, with high probability, the angle between the recovered linear subspace and the true tangent plane. In doing so, we are able to adaptively select the neighborhood that minimizes this bound, yielding the best approximate tangent plane. Further, the behavior of this bound demonstrates the non-trivial existence of such an optimal scale. We are also able to accurately and efficiently estimate the curvature of the local neighborhood. Finally, we introduce a geometric uncertainty principle quantifying the limits of noise-curvature perturbation for tangent plane recovery.

Our approach is similar to the analysis presented by Nadler in [22], who studies the finite-sample properties of the PCA spectrum. Through matrix perturbation theory, Nadler examines the angle between the leading finite-sample-PCA eigenvector and that of the leading population-PCA eigenvector. As a linear model is assumed, perturbation results from noise only. Despite this key difference, the two analyses utilize similar techniques to bound the effects of perturbation on the PCA subspace and our results re-
cover those of Nadler in the curvature-free setting. Nadler also reports that sample-PCA suffers from a sudden “loss of tracking” of the true dominant eigenvector due to a crossover between signal and noise eigenvalues. We demonstrate a similar phenomenon, owing to geometry rather than noise. The present work therefore generalizes the study of Nadler to noisy samples from a nonlinear manifold model.

Other recent related works include that of Singer and Wu [27], who use local PCA to build a tangent plane basis and give an analysis for the neighborhood size to be used in the absence of noise. Using the hybrid linear model, Zhang, et al. [32] assume data are samples from a collection of “flats” (affine subspaces) and choose an optimal neighborhood size from which to recover each flat by studying the least squares approximation error in the form of Jones’ β-number (see [14] and also [8] in which this idea is used for curve denoising). We also note the work of Maggioni and coauthors [4], in which multiscale PCA is used to discover the intrinsic dimensionality of a data set.

The paper is organized as follows. The remainder of this section provides the intuition and assumptions of our approach and introduces the geometric model that is used throughout this work. We frame the problem as one of subspace perturbation in Section 2 and study the size of the perturbation as a function of scale in Section 3. The selection of the optimal scale is our main result and is presented in Section 4, along with the necessary geometric conditions for tangent plane recovery. Numerical results are given in Section 5. We conclude with algorithmic considerations and a discussion of future directions in Section 6.

1.2. Problem Setup. Our goal is to recover the best approximation to a local tangent space of a nonlinear d-dimensional Riemannian manifold \( \mathcal{M} \) from noisy samples presented in dimension \( D > d \). Working about a reference point \( x_0 \), an approximation to the linear tangent space of \( \mathcal{M} \) at \( x_0 \) is given by the span of the top \( d \) singular vectors of the centered data matrix (where “top” refers to the \( d \) singular vectors associated with the \( d \) largest singular values). The question becomes: how many neighbors of \( x_0 \) should be used (or in how large of a radius about \( x_0 \) should we work) to recover the best approximation?

To answer this question, we examine the “noise-curvature trade-off.” Given noisy samples of a linear subspace, the quality of PCA approximation improves as more points are included. However, the curvature of \( \mathcal{M} \) prevents the inclusion of a large number of points. Similarly, there exists a local scale about \( x_0 \) such that the effects of curvature are small, as \( \mathcal{M} \) locally resembles
Euclidean space. This suggests allowing only a very small radius about \( x_0 \), yet at small scales, the sample points are indistinguishable from noise. We therefore seek a balance and assume there exists a scale large enough to be above the noise level, but still small enough to avoid curvature. This scale reveals a linear structure that is sufficiently decoupled from both the noise and the curvature to be well approximated by a tangent plane. We note that the concept of noise-curvature trade-off has been a subject of interest for decades in dynamical systems theory [9].

1.3. Geometric Data Model. A \( d \)-dimensional manifold of codimension 1 may be described locally by the surface \( y = f(\ell_1, \ldots, \ell_d) \), where \( \ell_i \) is a coordinate in the tangent plane. After translating the origin, a rotation of the coordinate system can align the coordinate axes with the principal directions associated with the principal curvatures at the given reference point \( x_0 \). Aligning the coordinate axes with the plane tangent to \( M \) at \( x_0 \) gives a local quadratic approximation to the manifold. Using this choice of coordinates, the manifold may be described locally [11] by the Taylor series of \( f \) at the origin \( x_0 \):

\[
y = f(\ell_1, \ldots, \ell_d) = \frac{1}{2}(\kappa_1 \ell_1^2 + \cdots + \kappa_d \ell_d^2) + o(\ell_1^2 + \cdots + \ell_d^2),
\]

where \( \kappa_1, \ldots, \kappa_d \) are the principal curvatures of \( M \) at \( x_0 \). In this coordinate system, \( x_0 \) has the form

\[
x_0 = [\ell_1 \ell_2 \cdots \ell_d f(\ell_1, \ldots, \ell_d)]^T
\]

and points in a local neighborhood of \( x_0 \) have similar coordinates. Generalizing to a \( d \)-dimensional manifold of arbitrary codimension in \( \mathbb{R}^D \), there exist \((D-d)\) functions

\[
f_i(\ell) = \frac{1}{2}(\kappa^{(i)}_1 \ell_1^2 + \cdots + \kappa^{(i)}_d \ell_d^2) + o(\ell_1^2 + \cdots + \ell_d^2)
\]

for \( i = (d + 1), \ldots, D \), with \( \kappa^{(i)}_1, \ldots, \kappa^{(i)}_d \) representing the principal curvatures in codimension \( i \) at \( x_0 \). Then, given the coordinate system aligned with the principal directions, a point in a neighborhood of \( x_0 \) has coordinates \([\ell_1, \ldots, \ell_d, f_{d+1}, \ldots, f_D]\). We truncate this Taylor expansion and use the quadratic approximation

\[
f_i(\ell) = \frac{1}{2}(\kappa^{(i)}_1 \ell_1^2 + \cdots + \kappa^{(i)}_d \ell_d^2),
\]

\( i = (d + 1), \ldots, D \), as the local model for our analysis.
Consider now discrete samples from $\mathcal{M}$ that are contaminated with an additive Gaussian noise vector $e$ drawn from the $\mathcal{N}(0,\sigma^2 I_D)$ distribution. Each sample $x$ is a $D$-dimensional vector and $N$ such samples may be stored as columns of a matrix $X \in \mathbb{R}^{D \times N}$. The coordinate system above allows the decomposition of $x$ into its linear (tangent plane) component $\ell$, its quadratic (curvature) component $c$, and noise $e$, three $D$-dimensional vectors

\begin{align}
\ell &= [\ell_1 \ell_2 \cdots \ell_d 0 \cdots 0]^T \\
c &= [0 \cdots 0 c_{d+1} \cdots c_D]^T \\
e &= [e_1 e_2 \cdots e_D]^T
\end{align}

such that the last $(D - d)$ entries of $c$ are of the form

\begin{equation}
c_i = \frac{1}{2}(\kappa_1^{(i)} \ell_1^2 + \cdots + \kappa_d^{(i)} \ell_d^2).
\end{equation}

We may store the $N$ samples of $\ell$, $c$, and $e$ as columns of matrices $L$, $C$, $E$, respectively, such that our data matrix is decomposed as

\begin{equation}
X = L + C + E.
\end{equation}

**Remark.** Of course it is unrealistic for the data to be observed in the described coordinate system. As noted, we may use a rotation to align the coordinate axes with the principal directions associated with the principal curvatures. Doing so allows us to write (2) as well as (7). Because we will ultimately quantify the norm of each matrix using the unitarily-invariant Frobenius norm, this rotation will not affect our analysis. We therefore proceed by assuming that the coordinate axes align with the principal directions.

The true tangent plane we wish to recover is given by the PCA of $L$. Because we do not have direct access to $L$, we work with $X$ as a proxy, and instead recover a subspace spanned by the corresponding eigenvectors of $XX^T$. We will study how close this recovered invariant subspace of $XX^T$ is to the corresponding invariant subspace of $LL^T$ as a function of scale. Throughout this work, scale refers to the number of points $N$ in the local neighborhood within which we perform PCA. Given a fixed density of points, scale may be equivalently quantified as the radius $r$ about the reference point $x_0$ defining the local neighborhood.

**2. Perturbation of Invariant Subspaces.** Given the decomposition of the data (7), we have

\begin{equation}
XX^T = LL^T + CC^T + EE^T + LC^T + CL^T + LE^T + EL^T + CE^T + EC^T.
\end{equation}
To account for the centering required by PCA, define the sample mean of \( N \) realizations of random variable \( Y \) as

\[
\hat{E}[Y] = \frac{1}{N} \sum_{i=1}^{N} Y^{(i)},
\]

where \( Y^{(i)} \) denotes the \( i \)th realization. Let the mean of a matrix \( M \) be the matrix \( \hat{E}[M] \) such that each entry of row \( i \) is the sample mean of the \( i \)th row of \( M \). Let \( \tilde{M} \) denote the centered version of \( M \):

\[
\tilde{M} = M - \hat{E}[M].
\]

Thus we have

\[
\tilde{X}\tilde{X}^T = \tilde{L}\tilde{L}^T + \tilde{C}\tilde{C}^T + \tilde{E}\tilde{E}^T + \tilde{L}\tilde{C}^T + \tilde{C}\tilde{L}^T + \tilde{L}\tilde{E}^T + \tilde{E}\tilde{L}^T + \tilde{C}\tilde{E}^T + \tilde{E}\tilde{C}^T.
\]

The problem may be posed as a perturbation analysis of invariant subspaces. Rewrite (8) as

\[
\frac{1}{N} \tilde{X}\tilde{X}^T = \frac{1}{N} \tilde{L}\tilde{L}^T + \Delta,
\]

where

\[
\Delta = \frac{1}{N}(\tilde{C}\tilde{C}^T + \tilde{E}\tilde{E}^T + \tilde{L}\tilde{C}^T + \tilde{C}\tilde{L}^T + \tilde{L}\tilde{E}^T + \tilde{E}\tilde{L}^T + \tilde{C}\tilde{E}^T + \tilde{E}\tilde{C}^T)
\]

is the perturbation that prevents us from working directly with \( \tilde{L}\tilde{L}^T \). The dominant eigenspace of \( \tilde{X}\tilde{X}^T \) is therefore a perturbed version of the dominant eigenspace of \( \tilde{L}\tilde{L}^T \). Seeking to minimize the effect of this perturbation, we look for the scale \( N^* \) at which the dominant eigenspace of \( \tilde{X}\tilde{X}^T \) is closest to that of \( \tilde{L}\tilde{L}^T \). Before proceeding, we review material on the perturbation of eigenspaces relevant to our analysis. The reader familiar with this topic is invited to skip directly to Theorem 1.

The distance between two subspaces of \( \mathbb{R}^D \) can be defined as the spectral norm of the difference between their respective orthogonal projectors [12]. As we will always be considering two equidimensional subspaces, this distance is equal to the sine of the largest principal angle between the subspaces. We state our results in terms of the Frobenius norm as it will provide a simplification of Theorem 1. Then, by the equivalence of norms, we may define the optimal scale \( N^* \) as

\[
N^* = \arg\min_N \|P - \tilde{P}\|_F,
\]
where $P$ and $\hat{P}$ are the orthogonal projectors onto the subspaces computed from $L$ and $X$, respectively. The solution to (14) is the main goal of this work.

The distance $\|P - \hat{P}\|_F$ may be bounded by the classic $\sin \Theta$ theorem of Davis and Kahan [6]. We will use a version of this theorem presented by Stewart (Theorem V.2.7 of [28]), modified for our specific purpose. First, we establish some notation, following closely that found in [28]. Consider the eigendecompositions

$$\frac{1}{N} \tilde{L} \tilde{L}^T = U \Lambda U^T = [U_1 \ U_2] \Lambda [U_1 \ U_2]^T,$$

$$\frac{1}{N} \tilde{X} \tilde{X}^T = \hat{U} \hat{\Lambda} \hat{U}^T = [\hat{U}_1 \ \hat{U}_2] \hat{\Lambda} [\hat{U}_1 \ \hat{U}_2]^T,$$

such that the columns of $U$ are the eigenvectors of $\frac{1}{N} \tilde{L} \tilde{L}^T$ and the columns of $\hat{U}$ are the eigenvectors of $\frac{1}{N} \tilde{X} \tilde{X}^T$. The columns of $U_1$ are those eigenvectors associated with the $d$ largest eigenvalues in $\Lambda$ arranged in descending order. The columns of $U_2$ are then those eigenvectors associated with the smallest $(D - d)$ eigenvalues, and $\hat{U}$ is similarly partitioned. The subspace we recover is spanned by the columns of $\hat{U}_1$ and we wish to have this subspace as close as possible to the tangent space spanned by the columns of $U_1$. The orthogonal projectors onto the tangent and computed subspaces, $P$ and $\hat{P}$ respectively, are given by

$$P = U_1 U_1^T \quad \text{and} \quad \hat{P} = \hat{U}_1 \hat{U}_1^T.$$

Define $\lambda_d$ to be the $d$th largest eigenvalue of $\frac{1}{N} \tilde{L} \tilde{L}^T$, or the last entry on the diagonal of $\Lambda_1$. Note that $\lambda_d$ will not be small since it corresponds to variance in a tangent plane direction.

We are now in position to state the theorem. Note that we have made use of the fact that the columns of $U$ are the eigenvectors of $\tilde{L} \tilde{L}^T$, that $\Lambda_1, \Lambda_2$ are Hermitian (diagonal) matrices, and that the Frobenius norm is used to measure distances. The reader is referred to [28] for the theorem in its original form.

**Theorem 1.** (Davis & Kahan [6], Stewart [28])

Let $\delta = \lambda_d - \|U_1^T \Delta U_1\|_F - \|U_2^T \Delta U_2\|_F$ and consider

- (Condition 1) $\delta > 0$
- (Condition 2) $\|U_1^T \Delta U_1\|_F \|U_2^T \Delta U_2\|_F < \frac{1}{4} \delta^2$. 


Then, provided that conditions 1 and 2 hold,

(17) \[ \| P - \hat{P} \|_F \leq 2\sqrt{2} \frac{\| U^T \Delta U \|_F}{\delta}. \]

The two conditions of the theorem have important geometric interpretations. Informally, condition 1 requires that the linear structure we seek to recover be sufficiently decoupled from both the noise and curvature (this is consistent with our assumption of the existence of a scale yielding sufficient decoupling). We may consider \( \delta^{-1} \) to be the condition number for subspace recovery. When \( \delta \) approaches zero, the condition number becomes large, and bound (17) loses meaning as we cannot recover an approximating subspace. In Section 4 we will see that condition 1 naturally gives rise to an uncertainty principle that quantifies the limits of noise-curvature perturbation for tangent plane recovery. We will also see that the second condition naturally implies that the manifold be sufficiently sampled.

The solution to (14) is impractical to compute. However, (17) is a tight bound, as will be demonstrated by the experiments (Section 5.) Thus, a solution may be approximated by minimizing the right-hand side of (17). To do so, and to give each quantity in the theorem a geometric interpretation, we must first understand the behavior of the perturbation \( \Delta \) as a function of the scale parameter \( N \).

3. Bounding the Effects of Noise and Curvature. In this section we study the behavior of each term in (13) as a function of the scale parameter \( N \). First, we provide insight as to their leading order behavior. As explained by Fukunaga [10] (Chapter 5), estimator bias and estimator variance depend on the Hessian and gradient, respectively, of the function being estimated. Consider the local manifold model (2). This second order approximation is presented in a coordinate system such that its gradient is zero and its Hessian is a diagonal matrix with the principal curvatures as its entries. We therefore expect perturbation terms associated with variance to tend to zero as the scale parameter \( N \) increases. Likewise, we expect pure curvature terms to grow with \( N \). Formal calculations will show that \( \frac{1}{N} CC^T \), the term associated purely with curvature, has nonzero expectation that increases with \( N \). Note that while the diagonal entries of \( \frac{1}{N} EE^T \) also have nonzero expectation, these terms do not grow with \( N \) and are therefore associated with a noise-floor rather than with estimator bias. All other terms in (13) are zero in expectation, and thus only carry variance. Accordingly, these terms decay as \( 1/\sqrt{N} \).
3.1. **Preliminaries.**

3.1.1. **Sampling a Linear Subspace.** Consider sampling a linear subspace by uniformly sampling points inside \(B^d(x_0, r)\), the \(d\)-dimensional ball of radius \(r\) centered at \(x_0\). We drop the dependence on \(x_0\) from our notation for the remainder of this analysis. Because we are sampling from a noise-free linear subspace, the number of points \(N\) captured inside \(B^d(r)\) is a function of the sampling density \(\rho\):

\[
N = \rho v_d r^d,
\]

where \(v_d\) is the volume of the \(d\)-dimensional unit ball. As we wish to maintain a local analysis, we must enforce that \(r\) be small. To make this explicit, denote by \(r_{\text{max}}\) the largest radius within which the local model (2) holds and compute the number of points captured in \(B^d(r_{\text{max}})\):

\[
N_{\text{max}} = \rho v_d r_{\text{max}}^d.
\]

Then rescale (18) by dividing by (19) and solve for \(r\):

\[
r = r_{\text{max}} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{d}}.
\]

**Remark.** Equation (20) suppresses the dependence on sampling that is captured by the \(\rho v_d\) term. Note that because \(r\) is small, the sampling density \(\rho\) may have to be large to allow for large \(N\), as is explicitly seen in equation (18). The volume of the unit ball, \(v_d\), is very small for even reasonable values of \(d\), further necessitating a large sampling density. The analysis in this section may be performed entirely in the context of equation (18) provided that \(r\) is taken to be small. In doing so, the dependence on sampling density \(\rho\) is clear in all steps. We prefer to instead perform the analysis in the context of the rescaled equation (20), explicitly forcing the analysis to the local scale by considering \(r\) to be a fraction of the largest radius allowed by the local model.

3.1.2. **Notation.** In this section and throughout the remainder of this work, we will make use of the following definitions involving the principal curvatures:

\[
K_i = \sum_{n=1}^{d} \kappa_n^{(i)},
\]

\[
K = \left( \sum_{i=d+1}^{D} K_i^2 \right)^{\frac{1}{2}},
\]
\[ K_{ij}^{nn} = \sum_{n=1}^{d} \kappa_n^{(i)} \kappa_n^{(j)}, \quad K_{mn}^{ij} = \sum_{m,n=1}^{d} \kappa_n^{(i)} \kappa_n^{(j)}. \]

The constant \( K_i \) quantifies the curvature in codimension \( i \), for \( i = (d + 1), \ldots, D \). Note that given our choice of coordinate system in Section 1.2, \( K_i \) is the trace of the Hessian in the \( i \)th codimension. The overall curvature of our local model is quantified by \( K \) and is a natural result of our use of the Frobenius norm. We note that \( K_i K_j = K_{ij}^{nn} + K_{ij}^{mn} \).

Due to the choice of coordinate system, \( U_1 \) is the \( D \times d \) matrix whose columns are the first \( d \) columns of \( I_D \), the identity matrix of order \( D \). Similarly, \( U_2 \) is the \( D \times (D - d) \) matrix whose columns are the last \( (D - d) \) columns of \( I_D \). Therefore \( U_1^T C, C U_1^T, U_2^T L, \) and \( L U_2^T \) are all zero matrices of the appropriate size. Because \( \Delta \) is a symmetric matrix, we have that \( \| U_2^T \Delta U_1 \|_F = \| U_1^T \Delta U_2 \|_F \).

Finally, we will work with projections of vector \( a \) onto \( U_1 \) and \( U_2 \), where \( a \) takes the form of \( \ell, c, \) or \( e \) (equations (3)–(5)), and denote such projections by

\[ U_p^T a = a_{up}, \quad \text{for } p = \{1, 2\}. \]

### 3.2. Analysis of Perturbation Terms.

We begin by presenting our general strategy for bounding terms of the form \( \| U_p^T \frac{1}{N} \tilde{A} \tilde{B}^T U_q \|_F \) for \( p, q = \{1, 2\} \) where \( A \) and \( B \) are general matrices of size \( D \times N \). The key observation is that \( \frac{1}{N} \tilde{A} \tilde{B}^T \) is a sample mean of \( N \) outer products of vectors \( a \) and \( b \), each sampled from a given distribution:

\[
\frac{1}{N} \tilde{A} \tilde{B}^T = \hat{E}[(a - \hat{E}[a])(b - \hat{E}[b])^T],
\]

where \( \hat{E}[Y] \) is the sample mean defined in (9). We therefore expect that \( \frac{1}{N} \tilde{A} \tilde{B}^T \) will converge toward the centered outer product of \( a \) and \( b \).

We will use the following result of Shawe-Taylor and Cristianini [26] to bound, with high probability, the norm of the difference between this sample mean and its expectation,

\[
\| \hat{E}[U_p^T (a - \hat{E}[a])(b - \hat{E}[b])^T U_q] - \hat{E}[U_p^T (a - \hat{E}[a])(b - \hat{E}[b])^T U_q] \|_F
\]

where \( \hat{E}[Y] \) is the expectation of the random variable \( Y \in \mathcal{Y} \).

**Theorem 2.** (Shawe-Taylor & Cristianini, [26]). Given \( N \) samples of a random variable \( Y \) generated independently at random from \( \mathcal{Y} \) according
to the distribution $P_Y$, with probability at least $1 - e^{-\eta^2}$ over the choice of the samples, we have

$$\| \mathbb{E}[Y] - \hat{\mathbb{E}}[Y] \|_F \leq \frac{R}{\sqrt{N}} \left( 2 + \eta \sqrt{2} \right)$$

where $R = \sup_{\text{supp}(P_Y)} \| Y \|_F$ and $\text{supp}(P_Y)$ is the support of distribution $P_Y$.

**Remark.** With a slight abuse of notation, we note that the “Frobenius norm of a vector” is equivalent to the vector’s Euclidean norm, and thus we use $\| \cdot \|_F$ for both matrices and vectors.

**Remark.** The choice of $R$ in (26) need not be unique. Our analysis will proceed by using upper bounds for $\| Y \|_F$ which may not be suprema.

Continuing from (25),

$$\| \mathbb{E}[U_p^T (a - \hat{\mathbb{E}}[a])(b - \hat{\mathbb{E}}[b])^T U_q] - \hat{\mathbb{E}}[U_p^T (a - \hat{\mathbb{E}}[a])(b - \hat{\mathbb{E}}[b])^T U_q] \|_F$$

$$= \| \mathbb{E}[a_{up} b_{uq}^T] - \hat{\mathbb{E}}[a_{up} b_{uq}^T] + \hat{\mathbb{E}}[a_{up}] \hat{\mathbb{E}}[b_{uq}^T] - \mathbb{E}[a_{up}] \mathbb{E}[b_{uq}^T] \|_F$$

$$\leq \| \mathbb{E}[a_{up} b_{uq}^T] - \hat{\mathbb{E}}[a_{up} b_{uq}^T] \|_F + \| \mathbb{E}[a_{up}] \mathbb{E}[b_{uq}^T] - \hat{\mathbb{E}}[a_{up}] \hat{\mathbb{E}}[b_{uq}^T] \|_F.$$ (27)

Because $\mathbb{E}[\ell] = 0$ and $\mathbb{E}[c] = 0$, $\mathbb{E}[a_{up}] \mathbb{E}[b_{uq}^T]$ is nonzero only for the case $(a = b = c, p = q = 2)$. In this case, $\hat{\mathbb{E}}[U_p^T (a - \hat{\mathbb{E}}[a])(b - \hat{\mathbb{E}}[b])^T U_q] = \hat{\mathbb{E}}[U_q^T (c - \hat{\mathbb{E}}[c])(c - \hat{\mathbb{E}}[c])^T U_2]$ is an empirical covariance matrix. As shown in [26], such a matrix is unchanged when the origin is shifted by a fixed translation. Therefore we may assume that the origin has been shifted to the center of mass of the distribution and we may take $\mathbb{E}[c_{u2}]$ and $\mathbb{E}[c_{u2}^T]$ to be zero. Note that we may only do so in the context of this calculation, and in general $\mathbb{E}[c_{u2}]$ and $\mathbb{E}[c_{u2}^T]$ are nonzero. Then for all choices of $(a, b, p, q)$, we have $\mathbb{E}[a_{up}] \hat{\mathbb{E}}[b_{uq}^T] = 0$ and the right-hand side of (27) becomes

$$\| \mathbb{E}[a_{up} b_{uq}^T] - \hat{\mathbb{E}}[a_{up} b_{uq}^T] \|_F + \| \mathbb{E}[a_{up}] \hat{\mathbb{E}}[b_{uq}^T] \|_F.$$ (28)

We now use Theorem 2 to bound each of the three terms in (28). For this analysis, the random variable $Y$ in Theorem 2 takes one of the following two forms:

$$Y = a_{up} b_{uq}^T \quad \text{or} \quad Y = a_{up}.$$
for \( p, q = \{1, 2\} \). Thus there are two corresponding definitions for \( R \):

\[
R_{pq}^{ab} = \sup_{\text{supp}(P_a)} \sup_{\text{supp}(P_b)} \|a_{u_p} b_{u_q}^T\|_F \tag{29}
\]

\[
R_p^a = \sup_{\text{supp}(P_a)} \|a_{u_p}\|_F \tag{30}
\]

where \( a \) and \( b \) are sampled according to distributions \( P_a \) and \( P_b \), respectively. Directly applying Theorem 2 to each of the three terms in (28) and using a standard union bound argument yields

\[
\left\| \mathbb{E}[a_{u_p} b_{u_q}^T] - \tilde{\mathbb{E}}[a_{u_p} b_{u_q}^T] \right\|_F + \left\| \mathbb{E}[a_{u_p}] - \tilde{\mathbb{E}}[a_{u_p}] \right\|_F \left\| \mathbb{E}[b_{u_q}^T] - \tilde{\mathbb{E}}[b_{u_q}^T] \right\|_F \leq R_{pq}^{ab} \sqrt{N} \left(2 + \eta_{ab} \sqrt{2}\right) + R_p^a R_b^q N \left(2 + \eta_a \sqrt{2}\right) \left(2 + \eta_b \sqrt{2}\right) \tag{31}
\]

with probability greater than

\[
1 - e^{-\eta_{ab}^2} - e^{-\eta_a^2} - e^{-\eta_b^2} \tag{32}
\]

over the random sampling of \( a \) and \( b \). For the case that \( a = b \) we instead simply have the result holding with probability greater than

\[
1 - e^{-\eta_a^2} \tag{33}
\]

over the random sampling of \( a \). The probability constants may be chosen to ensure such an event holds with high probability. For example, in (32), letting \( \eta_{ab} = \eta_a = \eta_b = \eta \), we have probability greater than 0.9451 for \( \eta = 2 \) and greater than 0.9996 for \( \eta = 3 \).

Putting it all together, we have that

\[
\left\| \mathbb{E}[U_p^T(a - \mathbb{E}[a])(b - \mathbb{E}[b])^T U_q] \right\|_F - \left\| \tilde{\mathbb{E}}[U_p^T(a - \tilde{\mathbb{E}}[a])(b - \tilde{\mathbb{E}}[b])^T U_q] \right\|_F \leq R_{pq}^{ab} \sqrt{N} \left(2 + \eta_{ab} \sqrt{2}\right) + R_p^a R_b^q N \left(2 + \eta_a \sqrt{2}\right) \left(2 + \eta_b \sqrt{2}\right) \tag{34}
\]

and we may conclude that

\[
\left\| U_p^T \left( \frac{1}{N} \tilde{A} \tilde{B}^T \right) U_q \right\|_F \in [\mu - \Gamma, \mu + \Gamma], \tag{35}
\]
where $\mu = \left\| E[a_{up} b_{uq}^T] - E[a_{up}] E[b_{uq}^T] \right\|_F$

and $\Gamma = \frac{R_{pq}^a}{\sqrt{N}} \left( 2 + \eta_{ab} \sqrt{2} \right) + \frac{R_{pq}^b}{N} \left( 2 + \eta_a \sqrt{2} \right) \left( 2 + \eta_b \sqrt{2} \right),$

with probability greater than

$$\left\{ \begin{array}{ll} 1 - e^{-\eta_a^2} - e^{-\eta_b^2} & \text{for } a \neq b \\ 1 - e^{-\eta_a^2} - e^{-\eta_b^2} & \text{for } a = b \end{array} \right.$$ 

over the random sampling of $a$ and $b$.

Before computing the constants $R_{pq}^a$ and $R_{pq}^b$, we must ensure that either the suprema (29) and (30) exist or that finite bounds may be given in place of suprema. Noting that

\begin{equation} \tag{36} \left\| a_{up} b_{uq}^T \right\| \leq \left\| a_{up} \right\| \left\| b_{uq} \right\|, \end{equation}

it suffices to show that the projections $\ell_{up}$, $c_{up}$, and $e_{up}$ are bounded. The vectors $\ell$ and $c$ are functions of the coordinates of points drawn uniformly from $B^d(r)$. Therefore their entries are bounded, as are the norms of their projections. The entries of $e$, while unbounded in general, are Gaussian random variables and are therefore bounded over a set with large measure. Let $\Omega_e$ be the set in $\mathbb{R}^D$ for which both

\begin{equation} \tag{37} \left\| e_{u1} \right\| \leq \sigma \left( \sqrt{d} + \xi \sqrt{\frac{2}{d}} \right) \end{equation}

\begin{equation} \tag{38} \left\| e_{u2} \right\| \leq \sigma \left( \sqrt{D - d} + \xi \sqrt{\frac{2}{D - d}} \right) \end{equation}

hold. A formal construction of $\Omega_e$ is given in Appendix A and the measure $\gamma_D(\Omega_e)$ of this set is shown to be large,

$$\gamma_D(\Omega_e) > 1 - 2e^{-\xi^2}.$$ 

Therefore, on $\Omega_e$, we may state bounds for $R_{pq}^p$ ($p = 1, 2$). All results involving projections of the vector $e$ will be given over this set. We apply Theorem 2 by conditioning on $e \in \Omega_e$. If we consider the joint probability of the independent random variables $a$ and $e$, then we can estimate the probability
that the inner-product of $a_{up}$ and $e_{u_q}$ be large. This probability is given by:

$$\text{Prob} \left[ \left\| E[a_{up}e_{u_q}] - \hat{E}[a_{up}e_{u_q}] \right\|_F > \frac{R_{pq}^{ae}}{\sqrt{N}} (2 + \eta_{ae} \sqrt{2}) \right]$$

$$= \text{Prob} \left[ \left\| E[a_{up}e_{u_q}] - \hat{E}[a_{up}e_{u_q}] \right\|_F > \frac{R_{pq}^{ae}}{\sqrt{N}} (2 + \eta_{ae} \sqrt{2}) \right| e \in \Omega_e \right] \text{Prob} \left[ e \in \Omega_e \right]
+ \text{Prob} \left[ \left\| E[a_{up}e_{u_q}] - \hat{E}[a_{up}e_{u_q}] \right\|_F > \frac{R_{pq}^{ae}}{\sqrt{N}} (2 + \eta_{ae} \sqrt{2}) \right| e \in \Omega_e \right] \text{Prob} \left[ e \in \Omega_e \right]
\leq (e^{-\eta_{ae}^2})(1 - 2e^{-\xi_e^2}) + (1)(2e^{-\xi_e^2})
= e^{-\eta_{ae}^2} + 2e^{-\xi_e^2} - 2e^{-\eta_{ae}^2}e^{-\xi_e^2}
$$

Thus we have that

$$\left\| E[a_{up}e_{u_q}] - \hat{E}[a_{up}e_{u_q}] \right\|_F \leq \frac{R_{pq}^{ae}}{\sqrt{N}} (2 + \eta_{ae} \sqrt{2})$$

with probability greater than $1 - e^{-\eta_{ae}^2} - 2e^{-\xi_e^2} + 2e^{-\eta_{ae}^2}e^{-\xi_e^2}$ over the random sampling of $a$ and random realization of $e$, and an identical calculation holds when applying the theorem to $\left\| E[e_{u_p}] - \hat{E}[e_{u_p}] \right\|_F$. The probability given in (32) can be bounded by

$$1 - e^{-\eta_{ae}^2} - e^{-\eta_e^2} - e^{-\eta_a^2} - 2e^{-\xi_e^2} + 2e^{-\xi_e^2}(e^{-\eta_{ae}^2} + e^{-\eta_e^2})$$

(40)

and when $(a = b = e)$ the probability given in (33) can be bounded by

$$1 - e^{-\eta_{ae}^2} - e^{-\eta_e^2} - 2e^{-\xi_e^2} + 2e^{-\xi_e^2}(e^{-\eta_{ae}^2} + e^{-\eta_e^2})$$

(41)

where we have neglected the positive contribution of the product of exponentially small terms.

3.2.1. Suprema $R_{pq}^{ab}$ and $R_p^a$. We now compute bounds for the $R_p^a$ terms. Simple norm calculations give

$$\left\| \ell_{u_1} \right\|_F^2 = \sum_{i=1}^d \ell_i^2 \leq r_{max}^2 \left( \frac{N}{N_{max}} \right)^{\frac{a}{2}}.$$
\[ \|c_{u_2}\|_F^2 = \sum_{i=d+1}^{D} c_i^2 = \frac{1}{4} \sum_{i=d+1}^{D} \left( \kappa_1^{(i)} \ell_1^2 + \ldots + \kappa_d^{(i)} \ell_d^2 \right)^2 \leq \]

\[ \frac{r_{\max}^4}{4} \left( \frac{N}{N_{\max}} \right)^{\frac{4}{4}} \sum_{i=d+1}^{D} \left( \sum_{n=1}^{d} \kappa_n^{(i)} \right)^2 = \frac{K^2 r_{\max}^4}{4} \left( \frac{N}{N_{\max}} \right)^{\frac{4}{4}}, \]

where we have assumed all principal curvatures have the same sign. Combining with (37) and (38) yields bounds for \( R_{ab}^p \) terms, and (36) may be used to bound the \( R_{ab}^{pq} \) terms. The results are listed in Appendix C.

**Remark.** The calculation (42) for \( \|c_{u_2}\|_F \) requires that all principal curvatures have the same sign for the inequality to hold. This requirement will carry through as an assumption in the statement of Theorem 3 (Main Result 1). To avoid this requirement we must work with moments of \( \ell_i \) rather than with norms, specifically when computing \( LC^T \) terms. While not possible here, it is possible to do so as demonstrated in Main Result 2 (see Section 4.3.2). Despite this assumption, it will be seen in Section 5 that this current analysis does in fact provide meaningful results for principal curvatures of mixed signs (except for when \( K_i = 0 \)), indicating that tighter \( R_{ab}^{pq} \) bounds are possible. We note that Main Result 2 will require no such assumption and will hold for any value of \( K_i \).

3.2.2. **Expectations.** We are almost in position to define confidence intervals of the form (35), where \( \tilde{A} \) and \( \tilde{B} \) may be the centered matrices \( \tilde{L} \), \( \tilde{C} \), and \( \tilde{E} \). All that remains is to compute the true expectation term of equation (34):

\[ \left\| \mathbb{E}[U_p^T(a - \mathbb{E}[a])(b - \mathbb{E}[b])^T U_q] \right\|_F = \left\| \mathbb{E}[a_{u_p} b_{u_q}^T] - \mathbb{E}[a_{u_p}] \mathbb{E}[b_{u_q}^T] \right\|_F. \]

As the coordinates of \( \ell \) and \( c \) are functions of points sampled uniformly from \( B^d(r) \) and \( e \sim N(0, \sigma^2 I_D) \), the expectation terms are zero for \( \ell \ell^T \), \( \ell e^T \), and \( ce^T \). Only the pure curvature (\( cc^T \)) and pure noise (\( ee^T \)) terms may have nonzero expectations and their calculations are given in Appendices B, and C. We list here only the results.

**Pure Curvature Term:**

\[ \left\| \mathbb{E}[c_{u_2} c_{u_2}^T] - \mathbb{E}[c_{u_2}] \mathbb{E}[c_{u_2}^T] \right\|_F = \]

\[ \frac{r_{\max}^4}{2(d + 2)^2(d + 4)} \left( \frac{N}{N_{\max}} \right)^{\frac{4}{4}} \left( \sum_{i=d+1}^{D} \sum_{j=d+1}^{D} (d + 1) K_{nn}^{ij} - K_{mn}^{ij} \right)^2 \].
We note that later, for the purpose of interpretation, we will replace this exact expectation with an upper bound. Using that $E[\ell_m^2 \ell_n^2] < E[\ell_n^4]$ and $K_{nn}^i + K_{mn}^i = K_i K_j$, we bound

$$\|E[c_{u_2} c_{u_2}^T] - E[c_{u_2}] E[c_{u_2}^T]\|_F < K^2 \frac{(d + 1)}{2(d + 2)^2(d + 4)} r_{\max}^4 \left(\frac{N}{N_{\max}}\right)^{\frac{4}{d}}.$$  

The nonzero expectation (44) and its bound (45) grow with $N$ and in this way may be thought of as the source of the estimator bias (see the beginning discussion of Section 3).

**Pure Noise Terms:**

$$\|E[e_{p_2} e_{u_2}^T] - E[e_{p_2}] E[e_{u_2}^T]\|_F = \begin{cases} \sigma^2 \sqrt{d} & \text{if } (p,q) = (1,1), \\ \sigma^2 \sqrt{D-d} & \text{if } (p,q) = (2,2), \\ 0 & \text{if } (p,q) = (1,2). \end{cases}$$

Note that unlike the nonzero expectation of the pure curvature term, the nonzero expectations of the pure noise terms $(p=q)$ are constant and do not grow with $N$. Thus they represent a noise floor rather than a source of estimator bias.

3.2.3. **Norm Bounds.** We may now use the right-hand side of the confidence interval (35) to bound the size of the perturbation norms. When considering noise terms, recall that we must condition on $e \in \Omega_e$. To aid in interpretation, we recall the rescaled notation $r = r_{\max} (N/N_{\max})^{1/d}$. As each curvature term $c$ has norm roughly of size $Kr^2$, we expect $\|\frac{1}{N}CC^T\|_F$ to grow as $K^2 r^4$. Concentration of Gaussian measure indicates that the norm of the noise matrix will have size that depends on the square root of the projection dimension and the variance $\sigma$. All other terms are zero in expectation and thus we expect $1/\sqrt{N}$ decay. The linear-curvature, linear-noise, and curvature-noise matrices should have norm $K^3 r^3 / \sqrt{N}$, $\sigma r / \sqrt{N}$, and $K \sigma^2 r^2 / \sqrt{N}$, respectively. The leading order behavior of all norm bounds may be found in Table 1 and the full expressions with associated probabilities are given in Appendix C.

4. **Optimal Scale Selection and Subspace Recovery.** Our main result, a bound on the angle between the recovered and true tangent planes, is formulated in this section. First we use the triangle inequality to bound the norms appearing in Theorem 1. We then inject the perturbation norms computed in Section 3 and Appendix C to formulate the main result.
We have:

\[
\|U_1^T \Delta U_1\|_F \leq 2 \|U_1^T \frac{1}{N} L \tilde{E}^T U_1\|_F + \|U_1^T \frac{1}{N} \tilde{E} \tilde{E}^T U_1\|_F
\]

\[
\|U_2^T \Delta U_2\|_F \leq \left\| U_2^T \frac{1}{N} \tilde{C} \tilde{C}^T U_2 \right\|_F + 2 \left\| U_2^T \frac{1}{N} \tilde{C} \tilde{E}^T U_2 \right\|_F + \left\| U_1^T \frac{1}{N} \tilde{E} \tilde{E}^T U_2 \right\|_F
\]

\[
\|U_1^T \Delta U_2\|_F \leq \left\| U_1^T \frac{1}{N} \tilde{L} \tilde{C}^T U_2 \right\|_F + \|U_1^T \frac{1}{N} \tilde{L} \tilde{E}^T U_2\|_F + \|U_1^T \frac{1}{N} \tilde{E} \tilde{C}^T U_2\|_F + \left\| U_1^T \frac{1}{N} \tilde{E} \tilde{E}^T U_2 \right\|_F.
\]

As \( \Delta \) is a symmetric matrix, we also have that

\[
\left\| U_1^T \Delta U_2 \right\|_F = \left\| U_2^T \Delta U_1 \right\|_F.
\]

Using a standard union bound argument, the bounds for each term hold simultaneously with probability greater than

\[
1 - e^{-\eta^2} - 3e^{-\eta_c^2} - e^{-\eta_e^2} - 2e^{-\eta_{ce}^2} - 2e^{-\eta_{ce}^2} - e^{-\eta_c^2} - 2e^{-\eta_e^2} - 2e^{-\eta_{ce}^2}
\]

over the joint random selection of the sample points and random realization of the noise. We may pick a constant \( \eta \) and set

\[
\eta_{ce} = \eta_{ee} = \eta_{tc} = \eta_{tc} = \eta_{ce} = \eta_{tc} = \eta = \eta = \eta = \eta = \eta
\]

such that (46) becomes

\[
1 - 13e^{-\eta^2} - 2e^{-\xi^2}.
\]

Finally, recall from Theorem 1 that

\[
\delta = \lambda_d - \|U_1^T \Delta U_1\|_F - \|U_2^T \Delta U_2\|_F.
\]

In order to bound the size of \( \delta \) we must compute \( \lambda_d \), the \( d \)th eigenvalue of \( \frac{1}{N} \tilde{L} \tilde{L}^T \). This matrix is a centered covariance matrix and therefore its \( d \)th eigenvalue is the variance in the \( d \)th coordinate. From our moment calculations in Appendix D, we let

\[
\lambda_d = \text{Var}[\ell_d] = \frac{v_{\max}^2}{d + 2} \left( \frac{N}{N_{\max}} \right)^\frac{3}{2}.
\]

4.1. Main Result: Bounding the Angle Between Subspaces. We are now in position to apply Theorem 1 and state our main result. First, define the following constants:

\[
K = \left[ \sum_{i=d+1}^{D} \sum_{j=d+1}^{D} \left[ (d+1)K_{nn}^{ij} - K_{mn}^{ij} \right]^2 \right]^{\frac{1}{2}}.
\]
\[ p_1(\eta) = \left(2 + \eta \sqrt{2}\right) \left(1 + \frac{1}{\sqrt{N}}(2 + \eta \sqrt{2})\right), \]
\[ p_2(\xi_e) = \left(\sqrt{d + \xi_e \sqrt{\frac{2}{d}}}\right), \]
\[ p_3(\xi_e) = \left(\sqrt{D - d + \xi_e \sqrt{\frac{2}{D - d}}}\right), \]
\[ \zeta(K, \sigma, \eta, \xi_e) = \left[ K^2 r_{\max}^4 \left(\frac{N}{N_{\max}}\right)^{\frac{2}{3}} + \sigma p_3(\xi_e) K r_{\max}^2 \left(\frac{N}{N_{\max}}\right)^{\frac{2}{3}} + 2\sigma p_2(\xi_e) r_{\max} \left(\frac{N}{N_{\max}}\right)^{\frac{2}{3}} + \sigma^2 (p_2(\xi_e)^2 + p_3(\xi_e)^2) \right]. \]

Then we have our main result:

**Theorem 3.** (Main Result 1). Let the following conditions hold:

- (Condition 1) \( \delta = \lambda_d - \|U_1^T \Delta U_1\|_F - \|U_2^T \Delta U_2\|_F > 0, \)
- (Condition 2) \( \|U_1^T \Delta U_2\|_F < \frac{1}{2} \delta. \)

For each \( i = (d + 1), \ldots, D, \) let principal curvatures \( \kappa_n^{(i)} \) have the same sign, \( n = 1, \ldots, d. \) Then we have

\[ \|P - \hat{P}\|_F \leq \frac{2\sqrt{2} \frac{p_1(\eta)}{\sqrt{N}} \left[ \frac{K^2 r_{\max}^4}{(d+2)} + \frac{K r_{\max}^2}{2(d+2)^2(d+4)} - \frac{\sigma^2}{\sqrt{N}} \right] \zeta(K, \sigma, \eta, \xi_e)}{r_{\max}^2 (d+2) - \frac{K r_{\max}^4}{2(d+2)^2(d+4)} - \sigma^2 \left(\sqrt{d} + \sqrt{D - d}\right) - \frac{p_1(\eta)}{\sqrt{N}} \zeta(K, \sigma, \eta, \xi_e)} \]

with probability greater than \( 1 - 13e^{-\eta^2} - 2e^{-\xi_e^2} \) over the joint random selection of the sample points and random realization of the noise, where the rescaled notation \( r = r_{\max}(N/N_{\max})^{1/d} \) has been used.

**Proof.** Applying the norm bounds computed in Appendix C to Theorem 1 and choosing the probability constants as given in (47) yields the result.

\[ \Box \]

The optimal scale \( N^* \) may be selected as the \( N \) for which (51) is minimized. As will be analyzed in Section 4.2.1, (51) is either monotonically decreasing (for the curvature-free case), monotonically increasing (for the noise-free case), or decreasing at small scales and increasing at large scales (for the general case). We therefore expect a unique minimizer in all cases.
Note that the constants $\eta$ and $\xi_e$ need to be selected to ensure that this bound holds with high probability.

As discussed in Section 2, we may interpret $\delta^{-1}$ as the condition number for tangent plane recovery, and we may analyze it by bounding (49) using the bounds for $\|U_1^T \Delta U_1\|_F$ and $\|U_2^T \Delta U_2\|_F$. We note that the denominator in (51) is a lower bound on $\delta$ and we therefore analyze the condition number via this bound. From Main Result 1 (51), we see that when $\delta^{-1}$ is small, we may recover a tight approximation to the true tangent space. Likewise, when $\delta^{-1}$ becomes large, the angle between the computed and true subspaces becomes large. The notion of an angle loses meaning as $\delta^{-1}$ tends to infinity, and we are unable to recover an approximating subspace.

Condition 1 imposes that the spectrum corresponding to the linear subspace ($\lambda_d$) must be well separated from the spectra of the noise ($\|U_1 \Delta U_1\|_F$) and curvature ($\|U_2 \Delta U_2\|_F$) perturbations. In this way, condition 1 quantifies our requirement that there exists a scale such that the linear subspace is sufficiently decoupled from the effects of curvature and noise. When the spectra are not well separated, the angle between the subspaces becomes ill-defined. In this case, the approximating subspace contains an eigenvector corresponding to a direction orthogonal to the true tangent plane. In the language of [22], there is a crossover between the spectrum of the linear subspace and the spectrum of the perturbation, and we will observe a loss of tracking of the true tangent plane. Unlike the result of [22] where the crossover results from noise perturbation, we will demonstrate a crossover due to geometry in Section 5. As we will see, condition 1 indeed imposes a geometric requirement for tangent plane recovery.

With condition 1 imposing restrictions on the effects of curvature and noise, condition 2 may be interpreted as a control on sampling. This condition may be satisfied by increasing the sampling density, as such an increase allows $N$ to take on large values. Recall that $r = r_{\text{max}} (N/N_{\text{max}})^{1/d}$. While $(N/N_{\text{max}})$ terms are bounded by one, the numerator of (51) is composed of terms that behave as $1/\sqrt{N}$. Thus, provided that the denominator is well-conditioned, condition 2 may be satisfied by allowing for large enough $N$.

Before numerically demonstrating our main result, we give a practical interpretation of its conditions and implications. In doing so, we quantify the separation needed between the linear structure and the noise and curvature with a geometric uncertainty principle. Then in Section 4.3, we present evidence of a tighter main result through the Central Limit Theorem.
4.2. Interpreting the Bound. The bound (51) is difficult to interpret due to many complicated terms. For subspace tracking, the behavior of the bound as a function of scale is of more practical use than the bound’s actual value. The scale at which the bound reaches its minimum and the scale(s) at which it becomes ill-conditioned are the quantities of interest. Thus we now analyze our main result using more practical and interpretable, albeit less sharp, bounds. The following bound is not as sharp as Main Result 1 owing mainly to a less precise treatment of the principal curvatures. We replace equation (44) with (45) for the expectation of the pure curvature term, and we analyze only leading-order behavior of each term. Neglecting the probability-dependent constants and the contributions of the $\frac{1}{N}LE^T$ and $\frac{1}{N}CE^T$ terms, we may write our main result as:

Interpretable Main Result 1.

$$
\left\| P - \hat{P} \right\|_F \leq \frac{2\sqrt{2} \frac{1}{\sqrt{N}} \left[ \frac{K}{2} r^3 + \sigma^2 \sqrt{d(D-d)} \right]}{r^2 - K^2 r^4 \frac{(d+1)}{2(d+2)^2(d+4)} - \sigma^2 \left( \sqrt{d} + \sqrt{D-d} \right)}
$$

and we recall that $r = r_{\text{max}}(N/N_{\text{max}})^{1/d}$.

4.2.1. Revisiting the Noise-Curvature Trade-off Through the Condition Number. We may now use this more compact, yet less sharp, form (52) of our main result to provide some interpretation for the bound (51). Consider first the denominator. Assume that sampling is sufficiently dense such that $N$ may become large. Doing so ensures condition 2 is met and we may focus our attention on condition 1 (and our neglect of the $\frac{1}{N}LE^T$ and $\frac{1}{N}CE^T$ terms in the denominator of (51) is also justified). The denominator of (52) is of the form

$$
\delta = \frac{r^2}{d+2} - K^2 r^4 \frac{(d+1)}{2(d+2)^2(d+4)} - \sigma^2 \left( \sqrt{d} + \sqrt{D-d} \right).
$$

It is now easy to see that $\delta$ quantifies the separation between the linear subspace ($O(r^2)$) and the perturbation due to the curvature ($O(K^2 r^4)$) and the noise level ($\sigma^2(\sqrt{d} + \sqrt{D-d})$). To approximate the appropriate linear subspace, we must at least have that $\delta > 0$ as required by condition 1, and the approximation improves for larger $\delta$. We note the similarity of this condition to that of equation (2.10) in [22].

The noise-curvature trade-off is now readily apparent. The linear and curvature contributions, controlled by $r = r_{\text{max}}(N/N_{\text{max}})^{1/d}$, are small for
small values of $N$. Thus for $N$ small, the denominator \((53)\) is either negative or ill conditioned for most values of $\sigma$. This makes intuitive sense as we have not yet encountered much curvature but the linear structure has also not been explored. Therefore the noise dominates the early behavior of this bound and an approximating subspace may not be recovered from noise. As $N$ increases, the conditioning of the denominator improves, and the bound is controlled by the $1/\sqrt{N}$ behavior of the numerator. This again corresponds with our intuition, as the addition of more points serves to overcome the effects of noise as the linear structure is explored. Thus, the bound becomes tighter. Eventually, $N$ becomes large enough such that the curvature contribution approaches the size of the linear contribution, and $\delta^{-1}$ becomes large. The $1/\sqrt{N}$ term is overtaken by the ill conditioning of the denominator and the bound is forced to become large. The noise-curvature trade-off, seen analytically here in \((53)\), will be demonstrated numerically in Section 5.

4.2.2. Geometric Uncertainty Principle for Subspace Recovery. Imposing condition 1 on \((53)\) yields a geometric uncertainty principle quantifying the amount of curvature and noise we may tolerate. Solving for the range of scales such that $\delta > 0$, the following condition naturally arises. To recover an approximating subspace, we must have that:

**Geometric Uncertainty Principle.**

\[
K\sigma < \frac{(d + 4)}{2(d + 1)(\sqrt{d} + \sqrt{D - d})}.
\]

By preventing curvature and noise from simultaneously becoming large, this requirement ensures that the geometry of the data is not destroyed by noise. With high probability, the noise concentrates on a sphere with mean curvature $1/\sigma \sqrt{D}$. Intuitively, we expect to require that the curvature of the manifold be less than the curvature of this noise-ball. Recalling the definitions of $K_i$ and $K$ from equations \((21)\) and \((22)\), $K_i/d$ is the mean curvature in codimension $i$. The quadratic mean of the $(D - d)$ mean curvatures is then given by $K/d\sqrt{D - d}$ and we denote this normalized version of curvature as $\overline{K}$. Then \((54)\) requires that $\overline{K} < \mathcal{O}(1/\sigma d D^{\frac{3}{4}})$. Noting that $d \geq 1 > D^{-\frac{1}{4}}$, the uncertainty principle \((54)\) indeed may be interpreted as a requirement that the mean curvature of the manifold be less than that of the perturbing noise-ball. Figure 1 provides an illustration.

It is important to keep in mind that equation \((54)\) is computed using the compact bound \((52)\), and is thus meant for interpretation. For the precise
expression represented by (54), the derivation must start with the full bound (51).

4.3. Towards a Tighter Bound: Chasing the Constants. Thus far we have presented a rigorous analysis bounding the norm of each perturbation term. The analysis captures leading order behavior with high probability by utilizing Theorem 2, but does so at the cost of attaching large constants to each term. Theorem 2, a result derived from the bounded difference and Hoeffding inequalities [26], introduces constants based on suprema of functions of random variables taken over the support of their distributions. Accordingly, each perturbation term is shown to deviate from its expectation by factors larger than constant multiples of its variance.

In this section we use the Central Limit Theorem (CLT) to show that the variance of the perturbation terms controls the deviation from their expectations. Doing so yields tighter bounds for each term and a tighter overall main result. Further, by working with moments of the underlying random variables rather than norms, a more precise treatment of curvature terms is possible, allowing a relaxation of the assumption in Main Result 1 that all principal curvatures in codimension $i$ have the same sign. Despite the fact that our analysis is most often to be applied to sample sizes on the order of $N = 10^5$ or $10^6$, we must acknowledge that the sample means with which we work have a Gaussian distribution only in the limit as $N$ tends to infinity. This finite-sample analysis can be made rigorous through the use of Bernstein-type inequalities and concentration of measure (in fact such approaches yield only slightly larger constants). However, we proceed.
with a CLT-based analysis, treating convergence in distribution as equality, to provide evidence that a tight bound may be achieved and to motivate future analyses that may rigorously yield tighter constants.

4.3.1. Central Limit Theorem and Gaussian Tail Bounds. As previously seen, each entry of a matrix $\frac{1}{N} \tilde{A} \tilde{B}^T$ is a sample mean of $N$ i.i.d. random variables. The CLT and a Gaussian tail bound yield a confidence interval for such an entry. Using a union bound to simultaneously control all of the entries of this matrix, we may give an overall confidence interval for the value of its Frobenius norm. While such an analysis yields a tighter result than that using Theorem 2, it holds with lower probability due to the use of many union bounds.

Remark. It is important to note that the probability attached to this analysis corresponds to the result serving as an upper bound for the true subspace recovery error. The probability is increased by choosing large values for the constants appearing in the exponential terms. This in turn loosens the result so as to bound any random fluctuation of the true error from above. In many practical applications, we are most interested in tracking the recovery error regardless of whether we guarantee an upper bound. When the need for an upper bound is relaxed, we will demonstrate in Section 5 that the leading order behavior of the following analysis tightly tracks the trend of the true error curve.

The analysis proceeds as follows. An entry of matrix $\frac{1}{N} \tilde{A} \tilde{B}^T$ has the form

$$\left( \frac{1}{N} \tilde{A} \tilde{B}^T \right)_{i,j} = \frac{1}{N} \sum_{k=1}^{N} A_{i,k}B_{j,k} - \tilde{E}[A_i]\tilde{E}[B_j],$$

where $A$ and $B$ represent the matrices $L$, $C$, and $E$ from equation (7). We use the CLT to assert that for the i.i.d. random variables $(A_{i,k}B_{j,k})$ with mean $\mu$ and variance $\sigma^2$, $k = 1, \ldots, N$, and for large $N$,

$$\frac{1}{N} \sum_{k=1}^{N} A_{i,k}B_{j,k} \sim N\left( \mu, \frac{\sigma^2}{N} \right).$$

Let $Y \sim N (\mu, \sigma^2)$. Then we have

$$\text{Prob}\{|Y - \mu| \geq \epsilon\} \leq \exp \left\{ - \frac{\epsilon^2}{2\sigma^2} \right\}$$

and we may pick $\epsilon \sim O(\sigma)$ to ensure that $|Y - \mu| < \epsilon$ with high enough probability. Setting $\epsilon = \eta \sigma \sqrt{2}$ gives

$$Y \in [\mu - \eta \sigma \sqrt{2}, \mu + \eta \sigma \sqrt{2}]$$
with probability greater than $1 - e^{-\eta^2}$, and we have a confidence interval that depends on the variance of $Y$ and whose probability is controlled by the constant $\eta > 0$.

We compute the size of the entries in $\Delta$ and detail the norm bounds in Appendix E. The main result is now stated after defining some constants.

4.3.2. A Tighter Main Result. Just as we needed notation for the sake of the readability of Main Result 1, we now define new constants for Main Result 2:

$$CC_{ij} = \left\{ \frac{2 (d+1) K_{ij}^{mn} - K_{ij}^{nn}}{(d+2)(d+4)} + \frac{1}{\sqrt{N}} \left( \eta_{CC_1} K_{ij}^{mn} \sqrt{\frac{48(d+1)(4d+17)}{(d+4)^2(d+6)(d+8)}} + 4\eta_{CC_2} K_{ij}^{mn} \sqrt{\frac{(d^2+5d+3)}{(d+4)^2(d+6)(d+8)}} + \frac{4\eta_{KL} K_{ij}^j(d+1)}{(d+2)} \frac{d+1}{d+4} \right) \right\}.$$

$$LC_{ij} = \left[ K_j \left( \frac{\eta_{LC_1} \sqrt{3}}{\sqrt{(d+4)(d+6)}} + \frac{\eta_L}{(d+2)} \right) + \kappa_i^{(j)} \frac{\sqrt{3} \left( \eta_{LC_2} \sqrt{5} - \eta_{LC_1} \right)}{\sqrt{(d+4)(d+6)}} \right] + K_j \frac{2\eta_L \eta_{KL}}{\sqrt{d+2}} \sqrt{\frac{d+1}{d+4}},$$

$$CE_i = \left[ \frac{\eta_{CE}}{\sqrt{d+4}} \sqrt{3 K_{nn}^{ii} + K_{mn}^{ii}} + \frac{\eta_E}{\sqrt{d+2}} K_i \left( 1 + 2\eta_{KL} \frac{d+1}{\sqrt{d+4}} \right) \right],$$

$$\mathcal{E}(x) = \sigma^2 \sqrt{x} \left[ \sqrt{2 \left( \eta_{EE_1} \sqrt{2} + \eta_{EE_2} \sqrt{x-1} \right) + \frac{2}{\sqrt{x}} \eta_E^2 \left( 1 + \sqrt{x-1} \right) \right].$$

Additionally, we will need

$$K' = \left( \sum_{i=d+1}^{D} \sum_{j=d+1}^{D} CC_{ij}^2 \right)^{\frac{1}{2}}, \quad L' = \sqrt{d(D-d)} \left[ \eta_{LE} + \eta_L \eta_{LE} \frac{2}{\sqrt{N}} \right],$$

$$K'' = \left( \sum_{i=1}^{d} \sum_{j=d+1}^{D} LC_{ij}^2 \right)^{\frac{1}{2}}, \quad E' = \sqrt{d(D-d)} \left[ \eta_{EE_2} + \eta_E^2 \frac{2}{\sqrt{N}} \right].$$
\[
K''' = \left( \sum_{i=d+1}^{D} \mathcal{C}E_i^2 \right)^{\frac{1}{2}},
\]

and
\[
\zeta' = K''' \sigma \sqrt{\frac{2(D - d)}{d + 2}} r_{\max} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{2}{3}} + 2L' \sigma d \sqrt{\frac{2}{d + 2}} r_{\max} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{3}} + \mathcal{E}E(d) + \mathcal{E}E(D - d).
\]

After using the triangle inequality in the exact same manner as in the beginning of Section 4, combining the norm bounds (computed in Appendix E) with Theorem 1 yields a new main result. Assume that conditions 1 and 2 hold. Then using the rescaled notation \( r = r_{\max} (N/N_{\text{max}})^{1/d} \), we have:

Main Result 2.

\[
\| P - \hat{P} \|_F \leq \frac{2 \sqrt{2} \sqrt{\frac{1}{N}} \left[ \frac{K''' \sigma^2}{\sqrt{2(d + 2)}} + K''' \sigma r^2 \sqrt{\frac{d}{2(d + 2)}} + L' \sigma r \sqrt{\frac{2}{d + 2}} + 2 \sigma^2 E' \right]}{\frac{r^2}{(d + 2)} - \frac{K' \sigma^4}{4(d + 2)} - \sigma^2 \left( \sqrt{d} + \sqrt{D - d} \right) - \frac{1}{\sqrt{N}} \zeta'}
\]

with probability greater than
\[
1 - (D - d)^2 \left[ d e^{-\eta L c_{1}} + \frac{d(d - 1)}{2} e^{-\eta L c_{2}} \right] - D e^{-\eta L E_{1}} - \frac{D^2 - D}{2} e^{-\eta L E_{2}} - d(D - d) \left[ (d - 1) e^{-\eta L c_{1}} + e^{-\eta L c_{2}} \right] - d D e^{-\eta L E_{1}} - D(D - d) e^{-\eta L E_{2}} - d e^{-\eta \hat{L}} - e^{-\eta \hat{C}} - D e^{-\eta \hat{E}}
\]

over the joint random selection of the sample points and random realization of the noise.

The comments following Main Result 1 apply here as well. In particular, conditions 1 and 2 have the same interpretation and the denominator \( \delta \) controls the conditioning of the recovery problem. Further, we note that the leading order behavior of the perturbation norms has not changed. Main Result 2 exhibits the same behavior as Main Result 1, but provides a tighter tracking of the true subspace recovery error, as will be shown in Section 5.
Thus this analysis gives rise to the same interpretable bound as (52), up to multiplicative constants. The same geometric uncertainty principle applies as well.

Table 1 shows the leading order behavior for each perturbation norm. Note that $r_{\text{max}} \left( N / N_{\text{max}} \right)^{\frac{1}{2}}$ has been replaced by $r$ and only the leading order in $d$ is shown. This side-by-side comparison reveals the reasons why Main Result 2 is a tighter bound. As suprema terms are replaced by variance terms, the CLT result introduces powers of $1/d$ that reduce the size of many norms. Additionally, the approach of Main Result 2 allows for a more precise treatment of the principal curvatures, most importantly for the $\text{LC}^T$ term. This precise treatment allows Main Result 2 to relax the assumption that all principal curvatures in codimension $i$ have the same sign, as required by Main Result 1. Finally, notice that Theorem 2 introduces probability constants of the form $(2 + \eta \sqrt{2})$, whereas the CLT introduces probability constants of the form $\eta \sqrt{2}$. Thus the CLT yields tighter bounds.

4.4. Consistency with Previously Established Results. In [27], Singer and Wu study local PCA for tangent plane recovery in the absence of noise. The covariance matrix is decomposed following the assumption that for a given $r$, the number of points in a ball of radius $r$ is large, implying a model with variable density. Fixing a density allows us to translate their results to our model. Then the covariance matrix decomposition yields error terms corresponding to curvature of size $O(r^4)$ and finite-sample variance of sizes $O(r^2) / \sqrt{N}$, $O(r^3) / \sqrt{N}$, and $O(r^4) / \sqrt{N}$. Recalling that $\| \frac{1}{N} \text{CC}^T \|_F \sim O(r^4) + O(r^4) / \sqrt{N}$ and $\| \frac{1}{N} \text{LC}^T \|_F \sim O(r^3) / \sqrt{N}$, our analysis recovers the same leading order behavior as reported by Singer and Wu in the noise-free setting.

As previously discussed, Nadler presents a finite-sample PCA analysis in [22] assuming a linear model. Setting curvature terms in Main Results 1 and 2 to zero recovers Nadler’s leading order bound on the angle between the finite-sample eigenvector(s) and the true eigenvector(s). In our notation, Nadler reports that, to leading order, the angle is bounded by

$$\sin \theta_{\hat{U}_1 U_1} \lesssim \frac{\sigma}{\sqrt{\lambda_d}} \sqrt{\frac{D}{N}} + O(\sigma^2),$$

where $d$ is taken to be one. We now show that our results recover this leading order behavior. First, set all curvature terms to zero. Next, assume condition 1 holds such that the denominator $\delta$ is sufficiently well conditioned and we may neglect all terms other than $\lambda_d$. Using the more compact notation of $r$ in place of $r_{\text{max}} \left( N / N_{\text{max}} \right)^{\frac{1}{2}}$ and following the approach in [22] of retaining
Comparison of leading order perturbation terms for Main Result 1 (top) and Main Result 2 (bottom). Notationally, \( r_{\text{max}} (N/N_{\text{max}})^{1/4} \) has been replaced by \( r \) and only leading order \( d \) terms are shown.

<table>
<thead>
<tr>
<th>Norm</th>
<th>Main Result 1 (top) / Main Result 2 (bottom)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( | U_2^T \tilde{C} \tilde{C}^T U_2 |_F )</td>
<td>( \sigma^2 \sqrt{d} )</td>
</tr>
<tr>
<td></td>
<td>( \sigma^2 \sqrt{d} )</td>
</tr>
<tr>
<td>( | U_3^T \tilde{E} \tilde{E}^T U_3 |_F )</td>
<td>( \sigma^2 \sqrt{D - d} )</td>
</tr>
<tr>
<td></td>
<td>( \sigma^2 \sqrt{D - d} )</td>
</tr>
<tr>
<td>( | U_3^T \tilde{E} \tilde{E}^T U_3 |_F )</td>
<td>( \sigma^2 \sqrt{D - d} )</td>
</tr>
<tr>
<td></td>
<td>( \sigma^2 \sqrt{D - d} )</td>
</tr>
<tr>
<td>( | U_3^T \tilde{L} \tilde{C}^T U_3 |_F )</td>
<td>( \frac{1}{\sqrt{N}} \sqrt[N]{\frac{d}{d-1} K (2 + \eta_{\text{ce}} \sqrt{2})} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{\sqrt{N}} \sqrt[N]{\frac{d}{d-1} K (2 + \eta_{\text{ce}} \sqrt{2})} )</td>
</tr>
<tr>
<td>( | U_3^T \tilde{L} \tilde{E}^T U_3 |_F )</td>
<td>( \frac{\sigma}{\sqrt{N}} r \left( \sqrt{d} + \xi c \sqrt{\frac{d}{d-1}} \right) (2 + \eta_{\text{ce}} \sqrt{2}) )</td>
</tr>
<tr>
<td></td>
<td>( \frac{\sigma}{\sqrt{N}} r \left( \sqrt{d} + \xi c \sqrt{\frac{d}{d-1}} \right) (2 + \eta_{\text{ce}} \sqrt{2}) )</td>
</tr>
<tr>
<td>( | U_3^T \tilde{C} \tilde{E}^T U_3 |_F )</td>
<td>( \frac{\sigma}{\sqrt{N}} [\frac{2}{r} K \left( \sqrt{d} + \xi c \sqrt{\frac{d}{d-1}} \right) (2 + \eta_{\text{ce}} \sqrt{2})] )</td>
</tr>
<tr>
<td></td>
<td>( \frac{\sigma}{\sqrt{N}} [\frac{2}{r} K \left( \sqrt{d} + \xi c \sqrt{\frac{d}{d-1}} \right) (2 + \eta_{\text{ce}} \sqrt{2})] )</td>
</tr>
</tbody>
</table>

**Table 1**
only leading order terms and dropping probability constants, we have

\[
\sin \theta_{\hat{U}_1, U_1} \lesssim \frac{\sqrt{D-d}}{\sqrt{N}} \left( \sigma r + \sigma^2 \sqrt{d} \right) = \frac{\sigma (d+2) \sqrt{D-d}}{r} + O(\sigma^2).
\]

Taking \( d = 1 \) and noticing that \( \sqrt{\lambda_d} \sim r \) yields Nadler’s bound.

5. Numerical Results. In this section we demonstrate that the bounds of Main Results 1 and 2 accurately and efficiently track the true subspace recovery error and may therefore be used to obtain the optimal scale for tangent plane recovery. We then address the case of data sampled from a saddle (such that the principal curvatures are of mixed signs) that brings to light a particular difference between the two main results. We explain and numerically demonstrate the behavior of the true subspace recovery error at large scales and the corresponding lack of subspace tracking and connect this observation to the “crossover phenomenon” reported in [22]. Finally we demonstrate the accurate and efficient recovery of local curvature.

5.1. Subspace Tracking and Recovery. We generate a data set sampled from a 3-dimensional manifold embedded in \( \mathbb{R}^{20} \) according to the local model (2). The radius of the local model is set to \( r_{\text{max}} = 1 \) and \( N = 1.25 \times 10^6 \) points are uniformly sampled from the tangent plane. Note that \( 2^{20} \approx 10^6 \), and thus this choice of \( N \) represents reasonable sampling in \( \mathbb{R}^{20} \). Curvature and the standard deviation \( \sigma \) of the added Gaussian noise will be specified in each experiment.

We compare our bounds with the true tangent plane recovery error. The tangent plane at reference point \( x_0 \) is computed at each scale \( N \) via PCA of the \( N \) nearest neighbors of \( x_0 \). The true tangent plane recovery error \( \|P - \hat{P}\|_F \) is then computed at each scale. Note that computing the true error requires \( N \) SVDs. A “true bound” is computed by applying Theorem 1 after measuring each perturbation norm directly from the data. While no SVDs are required, this true bound utilizes information that is not practically available, and therefore represents the best possible bound that we can hope to achieve. We will compare the mean of the true error and mean of the true bound over 10 trials (with error bars indicating one standard deviation) to the following three curves:

1. Main Result 1 holding with probability 0.5 (magenta),
2. Main Result 2 holding with probability 0.5 (black), and
3. Main Result 2 with all probability constants set to 1 (green).
Table 2
Principal curvatures of the manifold for Figure 2-c.

<table>
<thead>
<tr>
<th>( \kappa_{i}^{(j)} )</th>
<th>( i = 1 )</th>
<th>( i = 2 )</th>
<th>( i = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j = 4, \ldots, 6 )</td>
<td>3.0000</td>
<td>1.5000</td>
<td>1.5000</td>
</tr>
<tr>
<td>( j = 7, \ldots, 20 )</td>
<td>1.6351</td>
<td>0.1351</td>
<td>0.1351</td>
</tr>
</tbody>
</table>

The third curve abandons any guarantee of providing an upper bound in favor of capturing the trend of the true error. This curve represents the case where we may not care to upper bound the error, but instead wish to track the height of the true error curve as closely as possible. We will refer to these curves as “Main Result 1,” “Main Result 2,” and “Main Result 2-trend.”

The results are displayed in figure 2. Panel (a) shows the noisy (\( \sigma = 0.01 \)) curvature-free (linear subspace) result. As the only perturbation is due to noise, we expect the error to decay as \( 1/\sqrt{N} \) as the scale increases. The curves are shown on a logarithmic scale (for the Y-axis) and decrease monotonically, indicating the expected decay. As expected, Main Result 2 (black) is tighter than Main Result 1 (magenta) and both accurately track the behavior of the true error (blue). Main Result 2-trend (green) tightly tracks the same behavior, in this case sitting on top of the true bound (red). Panel (b) shows the results for a noise-free manifold with principal curvatures given in Table 2 such that \( K = 12.6025 \). Notice that three of the codimensions experience high curvature while the others are flatter, giving a tube-like structure to the manifold. In this case, perturbation is due to curvature only and the error increases monotonically (ignoring the slight numerical instability at extremely small scales), as predicted in the discussion of Section 4.2.1. Eventually, a scale is reached at which there is too much curvature and the bounds blow up to infinity. This corresponds exactly to where the true error plateaus at its maximum value, representing the fact that the computed subspace is now orthogonal to the true tangent plane. This large scale behavior will be further explained in Section 5.3.

Figure 2-c shows the results for a noisy (\( \sigma = 0.01 \)) version of the manifold used in panel (b). Note that the true error is large at small scales due to noise and large at large scales due to curvature. At these scales the bounds are accordingly ill conditioned and track the behavior of the true error when well conditioned. Panel (d) shows the results for a manifold again with \( K = 12.6025 \), but with the principal curvatures equal in all codimensions (\( \kappa_{i}^{(j)} = 1.0189 \) for \( i = 1, \ldots, 3 \) and \( j = 4, \ldots, 20 \)), and noise (\( \sigma = 0.01 \)) is added. We observe the same general behavior as seen in panel (c), but both the true error and the bounds remain well conditioned at larger scales. This is explained by the fact that higher curvature is encountered at smaller scales.
for the manifold corresponding to panel (c) but is not encountered until larger scales in panel (d).

In all four plots, the bounds accurately track the behavior of the true error. In fact, the curves are shown to be parallel on a logarithmic scale, indicating that they differ only by multiplicative constants. Note also that the true bound (red) tightly tracks the true error (blue), providing evidence that the triangle inequalities used in computing the bounds are reasonably tight. As no matrix decompositions are needed to compute our bounds, we have efficiently tracked the tangent plane recovery error. The black dots in figure 2 indicate the minimum of each curve. In general we see agreement of the location at which the minima occur, indicating the scale that will yield
the optimal tangent plane approximation. We note that when the location of the bounds’ minima do not correspond with the minimum of the true error (such as in panel (d)), the discrepancy occurs at a range of scales for which the true error is quite flat. In fact, in panel (d), the difference between the error at the computed optimal scale and the error at the true optimal scale is on the order of $10^{-2}$. Thus the angle between the computed and true tangent planes will be less than half of a degree. For a large data set it is impractical to examine every scale and one would instead most likely use a coarse sampling of scales. The true optimal scale would almost surely be missed by such a coarse sampling scheme. Our analysis indicates that despite missing the true optimum, we may recover a scale that yields an approximation to within a fraction of a degree of the optimum.

5.2. Principal Curvatures of Mixed Signs (Saddle). As discussed in Section 3.2.1, a key difference between Main Results 1 and 2 is the ability of Main Result 2 to properly handle principal curvatures of mixed signs. Main Result 1 requires the assumption that all principal curvatures in codimension $i$ have the same sign and thus cannot properly track the tangent plane recovery error for points sampled from a saddle. This is demonstrated in figure 3-a, showing the results for a 2-dimensional noise-free saddle ($d = 2, D = 3$) with principal curvatures $\kappa_1^{(3)} = 3$, and $\kappa_2^{(3)} = -3$. While all other curves behave as expected, the curve corresponding to Main Result 1 is identically zero because $K = 3 - 3 = 0$. Main Result 2, through its use of $K_{ij}^{\text{nn}}$ and $K_{ij}^{\text{mn}}$, avoids this problem. Figure 3-b shows the results for a 2-dimensional noise-free saddle ($d = 2, D = 3$) with principal curvatures $\kappa_1^{(3)} = 4$, and $\kappa_2^{(3)} = -3$. Despite the fact that the assumption of Main Result 1 is violated (the principal curvatures are of mixed signs), the corresponding curve does in fact track the recovery error because $K = 4 - 3 = 1$ is not identically zero. The fact that the proper behavior is seen despite the violated assumption indicates that a tighter curvature analysis in Section 3.2.1 is possible.

5.3. Spectral Crossover at Large Scales. Here we examine the inability to track the proper subspace at large scales, which is clearly indicated by the ill conditioning of the bounds and the plateau of the true error at its maximum value in figures 2 and 3. We demonstrate that this is an effect of curvature.

In [22] it was shown that the PCA of a noisy linear subspace is prone to a “sudden loss of tracking” of the dominant eigenvector. This occurs when an eigenvalue corresponding to noise overtakes an eigenvalue corresponding to signal. In this setting, once the crossover has occurred, the dominant
Fig 3. [color online] A 2-dimensional saddle (noise free) is shown with (a) $K = 0$ and (b) $K = 1$. Note that Main Result 1 is identically zero in (a) but accurately tracks the true error in (b). See text for discussion.

eigenvector may point in any random direction. Consider now our geometric model and let the sample points be noise-free to demonstrate a similar phenomenon owing to geometry rather than noise. Recall that, in this setup, condition 1 requires there be sufficient separation between the spectrum of the linear structure and the spectrum of the curvature. Also recall that $\delta^{-1}$ is the corresponding condition number. When $\delta^{-1}$ becomes large, there is little separation of the spectra, and a curvature eigenvalue approaches a tangent plane eigenvalue. Once the curvature eigenvalue becomes larger than the tangent plane eigenvalue, the computed eigenspace contains a direction orthogonal to the true subspace. This is seen in figure 2-b and figure 3 where the bounds blow up to infinity and the true error plateaus at its maximum value indicating orthogonality. As the crossover is due to curvature, an eigenvector in a direction orthogonal to the true tangent plane is introduced into the top $d$ computed eigenvectors. Thus the computed and true tangent planes are orthogonal at large scales.

Numerical evidence of this phenomenon is given in figure 4. The eigenvalues (mean over 10 trials) corresponding to the saddle from figure 3-b ($d = 2$, $D = 3$, $\kappa^{(3)}_1 = 4$, $\kappa^{(3)}_2 = -3$) are plotted as a function of scale. At small scales, the two tangent plane eigenvalues (blue and red) dominate the curvature eigenvalue (green) and the subspace recovery error is well conditioned at small scales in figure 3-b. Notice that at roughly $N = 2500$, the curvature eigenvalue crosses the two tangent plane eigenvalues. After this scale, the largest (blue) eigenvalue corresponds to curvature but is now included in
the computed tangent plane. Thus the computed tangent plane contains a direction orthogonal to the true tangent plane. This is seen in figure 3-b as the true error (blue) plateaus at its maximum value. At this large scale the bounds become ill conditioned (or negative) as condition 1 ($\delta > 0$) is violated. As there is no noise in this example, the crossover phenomenon is similar to that reported in [22], but is the result of curvature at large scales rather than noise.

5.4. Recovering Neighborhood Curvature. The expectation of the curvature term in each codimension has the following form:

$$E[C_i] = \frac{K_i}{2} \frac{r_{max}^2}{(d + 2)} \left( \frac{N}{N_{max}} \right)^{\frac{d}{2}}. \tag{59}$$

Thus given data in the coordinate system aligned with the principal directions described in Section 1.2, we may track the trajectory of the center as a function of scale and compare it to (59) to recover an estimate of $K_i$ for $i = (d + 1), \ldots, D$. Table 3 shows that this procedure results in a very accurate recovery of the local curvature at low noise levels, and the recovery becomes worse as the noise increases and the limit of the geometric uncertainty principle (54) is approached. We note that accuracy improves as $N$ becomes large, as expected by the CLT. Note that using (59), we recover the individual $K_i$'s from which the overall $K$ is computed (by equation (22)) and is reported in the tables. While this method does recover each $K_i$, the individual principal curvatures $\kappa_n^{(i)}$ are not recovered. As it does not
<table>
<thead>
<tr>
<th>σ</th>
<th>K = 2</th>
<th>K = 10</th>
<th>K = 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ = 0</td>
<td>1.9989 ± 0.0038</td>
<td>10.0079 ± 0.0145</td>
<td>20.0047 ± 0.0476</td>
</tr>
<tr>
<td>σ = 0.005</td>
<td>2.0020 ± 0.0032</td>
<td>10.0096 ± 0.0251</td>
<td>19.9903 ± 0.0406</td>
</tr>
<tr>
<td>σ = 0.01</td>
<td>2.0005 ± 0.0048</td>
<td>9.9952 ± 0.0202</td>
<td>20.0051 ± 0.0478</td>
</tr>
<tr>
<td>σ = 0.025</td>
<td>1.9928 ± 0.0044</td>
<td>9.9516 ± 0.0279</td>
<td>19.9104 ± 0.0428</td>
</tr>
<tr>
<td>σ = 0.1</td>
<td>1.8877 ± 0.0705</td>
<td>8.8781 ± 0.0808</td>
<td>17.8829 ± 0.0949</td>
</tr>
</tbody>
</table>

Table 3

Estimation of curvature at different noise levels (d = 5, D = 20, N = 10^4). The mean and standard deviation are reported from 10 trials. The estimation is accurate for low levels of noise and loses accuracy as the noise level increases. Note that the individual K_i's are recovered from which the overall K is computed according to equation (22).

require matrix decompositions and only uses vector addition, this method is computationally efficient. If one is willing to perform N SVDs, this method combined with the analysis of [4] might yield the individual principal curvatures.

While it is unrealistic for data to be observed in the desired coordinate system aligned with the principal directions, tracking the trajectory of the center in each dimension yields the rotation necessary to transform to this coordinate system. Further, tracking the trajectory may yield a clean estimate of the reference point of the local model in the presence of noise. While the noise renders this trajectory unstable at small scales, it is very stable at scales above the noise level. Using the stability of the trajectory at large scales may allow us to extrapolate back and accurately recover the trajectory at small scales, yielding an estimate of the “denoised” reference point.

6. Algorithmic Considerations and Future Work. There are several algorithmic issues to be considered in implementing this analysis for optimal tangent plane recovery. Such considerations are topics of our current research and we give a brief discussion here.

6.1. Parameter Recovery. In any practical use of this analysis (and in keeping with its spirit), each of the parameters $d$, $r_{max}$, $K$, and $\sigma$ must be recovered from the data itself rather than estimated by an a priori fixed value.

d. There exist algorithms for estimating the (local) intrinsic dimensionality of a data set. The recent work in [4] presents a multiscale approach to estimate $d$ in a pointwise fashion. Performing an SVD at each scale, $d$ is determined by examining growth rate of the multiscale singular values. It would be interesting to investigate if this approach remains robust if only a coarse exploration of the scales is performed, as it may be possible to
reduce the computational cost through an SVD-update scheme. Another scale-based approach is presented in [30] and the problem was studied from a dynamical systems perspective in [9].

\( r_{\text{max}} \). The maximum radius for which the local model (2) is valid may be estimated by a multiscale partitioning of the data set. Partitioning from fine to coarse, regions that produce similar tangent plane estimates at the same scale may be merged. Such an approach is similar to the aggregation process in [17], hierarchical clustering [5], data partitioning to find affine subspaces (“flats”) [29], subspace arrangement for homogeneous data subsets [19], and spectral clustering [1].

\( K \). We have demonstrated our ability to recover \( K \) given data in the coordinate system described in Section 1.2. Additionally we have discussed how tracking the trajectory of the centering may yield both the rotation into the desired coordinate system as well as a clean estimate of the otherwise noisy reference point. The accuracy and stability of such a scheme remains to be tested and it will be interesting to investigate if this may be a path to a simpler recovery of the tangent plane.

It is worth mentioning that while the definitions of \( K \) and \( K_i \) used in this work arise naturally from the analysis, they are not the only possible definitions. One could define

\[
K_i = \left( \sum_{n=1}^{d} \left( \kappa_n^{(i)} \right)^2 \right)^{\frac{1}{2}}.
\]

This definition does not match the calculation for \( E[C_i] \) but has the advantage of handling negative principal curvatures in a more natural manner. Indeed, we have seen that Main Result 1 does not hold for the case of a saddle with \( K_i = 0 \), but in fact does hold for a saddle with \( K_i \neq 0 \), thereby indicating that this bound may hold in a more general context.

\( \sigma \). There exist many statistical methods for estimating the noise level present in a data set (see, for example, [3, 7]). In [4], the smallest multiscale singular values are used as an estimate for the noise level and a scale-dependent estimate of noise variance is suggested in [8] for curve-denoising.

The parameters of our analysis may not remain constant over the entire data set. It is possible, if not likely, to experience very different sampling densities, noise-levels, curvature and dimensionality as one explores different regions of a data set. This fact increases the need for careful parameter selection and emphasizes the importance of a local analysis. Initial experiments indicate that Main Results 1 and 2 are sensitive to changes in these parameters. For example, over/under estimating \( K \) or \( \sigma \) will result in ill
conditioning at smaller/larger scales than seen in the true error. In depth experimentation will be necessary to precisely quantify the robustness of the results to parameter perturbation.

6.2. Sampling. For a tractable analysis, assumptions about sampling must be made. In this work we have assumed uniform sampling in the tangent plane. This is merely one choice and we have conducted initial experiments uniformly sampling the manifold rather than the tangent plane. Results suggest that for a given radius, sampling the manifold yields a smaller curvature contribution than does sampling the tangent plane. While more rigorous analysis and experimentation is needed, it is clear that consideration must be given to the sampling assumptions of any practical algorithm.

6.3. From Tangent Plane Recovery to Data Parameterization. The tangent plane recovered by our approach may not provide the best approximation over the entire neighborhood from which it was derived. Depending on a user-defined error tolerance, a smaller or larger sized neighborhood may be parameterized by the local chart. If high accuracy is required, one might only parameterize a neighborhood of size \( N < N^* \) to ensure the accuracy requirement is met. Similarly, if an application requires only modest accuracy, one may be able to parameterize a larger neighborhood than that given by \( N^* \).

Finally, we may wish to use tangent planes recovered from different neighborhoods to construct a covering of a data set. There exist methods for aligning local charts into a global coordinate system (for example \([2, 25, 33]\), to name a few). Care should be taken to define neighborhoods such that a data set may be optimally covered.

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Supplement to “Optimal Tangent Plane Recovery From Noisy Manifold Samples”. Technical results and long calculations from the main text are presented as supplementary material. The supplement is organized as follows. Section A details the construction of the set \( \Omega_e \) over which projections of a Gaussian noise vector are bounded. Section B presents bounds on suprema terms and calculations for expectations used in Main Result 1. The norm bounds and associated probabilities for Main Result 1 are listed in Section C. Section D details moment calculations for coordinates uniformly sampled from the tangent plane. Finally, confidence intervals and
norm bounds for Main Result 2 are computed via the Central Limit Theorem in Section E.

APPENDIX A: THE SET $\Omega_E$

Here the set $\Omega_e$ over which projections of the vector $e$ have bounded norm is formally defined. Begin by recalling a standard result on the concentration of the Gaussian measure on $\mathbb{R}^N$. Given a vector $x \in \mathbb{R}^N$ such that each of its entries is from the Gaussian $\mathcal{N}(0, \sigma^2)$ distribution, consider the set

$$S = \{x \in \mathbb{R}^N : \sqrt{N} - \varepsilon \leq \|x\|/\sigma \leq \sqrt{N} + \varepsilon\}$$

and let $\gamma_N$ denote the Gaussian measure on $\mathbb{R}^N$. Then by the concentration of Gaussian measure [20],

$$\gamma_N(S) > 1 - 2e^{-\frac{Ne^2}{2}}.$$ 

This result states that the set of points in $\mathbb{R}^N$ that concentrate about the sphere of radius $\sigma\sqrt{N}$ has extremely large Gaussian measure.

The sets

$$\Omega_1 = \{x \in \mathbb{R}^d : \|x\| \leq \sigma(\sqrt{d} + \varepsilon_1)\}$$
$$\Omega_2 = \{x \in \mathbb{R}^{D-d} : \|x\| \leq \sigma(\sqrt{D-d} + \varepsilon_2)\}$$

have Gaussian measure

$$\gamma_d(\Omega_1) > 1 - e^{-\frac{d\varepsilon_1^2}{2}}$$
$$\gamma_{D-d}(\Omega_2) > 1 - e^{-\frac{(D-d)\varepsilon_2^2}{2}}.$$ 

We may define sets

$$\Omega_{U_1} = \{x \in \mathbb{R}^D : \|U_1^T x\| \leq \sigma(\sqrt{d} + \varepsilon_1)\}$$
$$\Omega_{U_2} = \{x \in \mathbb{R}^D : \|U_2^T x\| \leq \sigma(\sqrt{D-d} + \varepsilon_2)\}$$

such that $\Omega_{U_1}$ and $\Omega_{U_2}$ are the preimages of $\Omega_1$ and $\Omega_2$, respectively, in $\mathbb{R}^D$. The Gaussian measures of the sets $\Omega_1$ and $\Omega_2$ are the pushforwards of the Gaussian measure in $\mathbb{R}^D$ by the respective projections $U_1^T$ and $U_2^T$:

$$\gamma_D(\Omega_{U_1}) = \gamma_d(\Omega_1) \leq e^{-\frac{d\varepsilon_1^2}{2}}$$
$$\gamma_D(\Omega_{U_2}) = \gamma_{D-d}(\Omega_2) \leq e^{-\frac{(D-d)\varepsilon_2^2}{2}}.$$ 

where $\Omega$ denotes the complement of the set $\Omega$.

Finally, define
$$\Omega_e = \Omega U_1 \cap \Omega U_2$$

and a standard union bound argument yields
$$\gamma_D(\Omega_e) = \gamma_D(\Omega U_1 \cup \Omega U_2) = \gamma_D(\Omega U_1) + \gamma_D(\Omega U_2) \leq e^{-\frac{d\varepsilon_1^2}{2}} + e^{-\frac{(D-d)\varepsilon_2^2}{2}}.$$ 

Set $\varepsilon_1 \sqrt{d/2} = \varepsilon_2 \sqrt{(D-d)/2} = \xi$. Then both
$$\|U_1^* e\| \leq \sigma \left( \sqrt{d + \xi e \sqrt{2/d}} \right)$$
$$\|U_2^* e\| \leq \sigma \left( \sqrt{D-d + \xi e \sqrt{2/(D-d)}} \right)$$

hold on $\Omega_e$, and
$$\gamma_D(\Omega_e) > 1 - 2e^{-\xi^2}.$$
B.1. **Suprema $R_{ab}^{pq}$ and $R_{a}^{b}$.** Listed here are the suprema terms needed for the calculations leading to Main Result 1. The calculation is outlined in Section 3.2.1.

First we have $R_{a}^{b}$ terms:

\[
R_{1}^{1} \leq r_{\text{max}} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{3}},
\]
\[
R_{1}^{2} \leq \frac{1}{2} K r_{\text{max}}^{2} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{2}{3}},
\]
\[
R_{e}^{1} \leq \sigma \left( \sqrt{d} + \xi_{e} \sqrt{\frac{2}{d}} \right) \quad \text{on } \Omega_{e},
\]
\[
R_{e}^{2} \leq \sigma \left( \sqrt{D - d} + \xi_{e} \sqrt{\frac{2}{D - d}} \right) \quad \text{on } \Omega_{e}.
\]

Then using (36) we may bound the remaining $R_{ab}^{pq}$:

\[
R_{22}^{11} \leq \sigma^{2} \left( \sqrt{d} + \xi_{e} \sqrt{\frac{2}{d}} \right)^{2} \quad \text{on } \Omega_{e},
\]
\[
R_{22}^{22} \leq \sigma^{2} \left( \sqrt{D - d} + \xi_{e} \sqrt{\frac{2}{D - d}} \right)^{2} \quad \text{on } \Omega_{e},
\]
\[
R_{12}^{12} \leq \sigma^{2} \left( \sqrt{d} + \xi_{e} \sqrt{\frac{2}{d}} \right) \left( \sqrt{D - d} + \xi_{e} \sqrt{\frac{2}{D - d}} \right) \quad \text{on } \Omega_{e},
\]
\[
R_{11}^{11} \leq \sigma r_{\text{max}} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{3}} \left( \sqrt{d} + \xi_{e} \sqrt{\frac{2}{d}} \right) \quad \text{on } \Omega_{e},
\]
\[
R_{12}^{12} \leq \sigma r_{\text{max}} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{3}} \left( \sqrt{D - d} + \xi_{e} \sqrt{\frac{2}{D - d}} \right) \quad \text{on } \Omega_{e},
\]
\[
R_{21}^{21} \leq \frac{1}{2} K \sigma r_{\text{max}}^{2} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{2}{3}} \left( \sqrt{d} + \xi_{e} \sqrt{\frac{2}{d}} \right) \quad \text{on } \Omega_{e}.
\]
\[ R_{ce}^{22} \leq \frac{1}{2} K \sigma \rho_{\max}^2 \left( \frac{N}{N_{\max}} \right)^{\frac{1}{2}} \left( \sqrt{D - d} + \xi e \sqrt{\frac{2}{D - d}} \right) \] on \( \Omega_c \).

**B.2. Expectations.** Here we detail the calculation of the expectations from Section 3.2.2. Each term is of the form

\[ \| \mathbb{E}[U_p^T (a - \mathbb{E}[a]) (b - \mathbb{E}[b])^T U_q] \|_F = \| \mathbb{E}[a_{up} b_{uq}^T] - \mathbb{E}[a_{up}] \mathbb{E}[b_{uq}]^T \|_F. \]

As only pure curvature \((cc^T)\) and pure noise \((ee^T)\) terms have nonzero expectation, the calculations of all other terms are omitted.

**Pure Curvature Term:** The expectation of the pure curvature term is computed as follows. Consider first \((c_{u2}c_{u2})_{i,j} = c_i c_j\) for \(i, j = (d+1), \ldots, D\). Then

\[
\mathbb{E}[c_i c_j] = \frac{1}{4} \mathbb{E}[(\kappa_1^{(i)} \ell_1^2 + \cdots + \kappa_d^{(i)} \ell_d^2)(\kappa_1^{(j)} \ell_1^2 + \cdots + \kappa_d^{(j)} \ell_d^2)] = \frac{1}{4} K_{nn}^{ij} \mathbb{E}[\ell_n^4] + \frac{1}{4} K_{mn}^{ij} \mathbb{E}[\ell_m^2 \ell_n^2]
\]

\[
= \frac{r_{\max}^4}{4(d+2)(d+4)} \left( \frac{N}{N_{\max}} \right)^{\frac{3}{2}} [3K_{nn}^{ij} + K_{mn}^{ij}].
\]

Next consider \((c_{u2})_{i} = c_i\) for \(i = (d+1), \ldots, D\). Then

\[
\mathbb{E}[c_i] = \mathbb{E}[c_j] = \frac{r_{\max}^4}{4(d+2)^2} \left( \frac{N}{N_{\max}} \right)^{\frac{3}{2}} [K_{nn}^{ij} + K_{mn}^{ij}]
\]

and we have

\[
\| \mathbb{E}[c_{u2}c_{u2}^T] - \mathbb{E}[c_{u2}] \mathbb{E}[c_{u2}^T] \|_F = \frac{r_{\max}^4}{2(d+2)^2(d+4)} \left( \frac{N}{N_{\max}} \right)^{\frac{3}{2}} \left[ \sum_{i=d+1}^{D} \sum_{j=d+1}^{D} [(d+1)K_{nn}^{ij} - K_{mn}^{ij}]^2 \right]^{\frac{1}{2}}.
\]

**Pure Noise Terms:** Using that the entries of \(e\) are i.i.d. random variables from the \(\mathcal{N}(0, \sigma^2)\) distribution, we have
\[ \left\| \mathbb{E}[e_{up}^T u_q] - \mathbb{E}[e_{up}] \mathbb{E}[e_{u_q}^T] \right\|_F = \left\| \mathbb{E}[e_{up} e_{u_q}^T] \right\|_F = \]
\[
\begin{cases} 
\left( \sum_{i=1}^{d} \sum_{j=1}^{d} \mathbb{E}[e_i e_j]^2 \right)^{\frac{1}{2}} = \left( \sum_{i=1}^{d} \mathbb{E}[e_i^2]^2 + \sum_{i,j=1}^{d} \mathbb{E}[e_i e_j]^2 \right)^{\frac{1}{2}} = \sigma^2 \sqrt{d} & \text{if } (p, q) = (1, 1), \\
\left( \sum_{i=d+1}^{D} \sum_{j=d+1}^{D} \mathbb{E}[e_i e_j]^2 \right)^{\frac{1}{2}} = \left( \sum_{i=d+1}^{D} \mathbb{E}[e_i^2]^2 + \sum_{i,j=d+1}^{D} \mathbb{E}[e_i e_j]^2 \right)^{\frac{1}{2}} = \sigma^2 \sqrt{D - d} & \text{if } (p, q) = (2, 2), \\
\left( \sum_{i=1}^{d} \sum_{j=d+1}^{D} \mathbb{E}[e_i e_j]^2 \right)^{\frac{1}{2}} = \left( \sum_{i,j=1}^{d} \mathbb{E}[e_i e_j]^2 \right)^{\frac{1}{2}} = 0 & \text{if } (p, q) = (1, 2).
\end{cases}
\]

**APPENDIX C: NORM BOUNDS FOR MAIN RESULT 1**

The right-hand side of the confidence interval (35) is used to bound the size of the perturbation norms. When considering noise terms, recall that we must condition on \( e \in \Omega_e \). Recalling the rescaled notation \( r = r_{\text{max}} \left( N / N_{\text{max}} \right)^{1/d} \) is helpful for interpretation.

**Curvature**

\[
\left\| U_2^T \left( \frac{1}{N} \bar{C} \bar{C}^T \right) U_2 \right\|_F \leq \]
\[
= \frac{r_{\text{max}}^4}{2(d + 2)^2(d + 4)} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{4}{d}} \left[ \sum_{i=d+1}^{D} \sum_{j=d+1}^{D} \left[ (d + 1) K_{nn}^{ij} - K_{mn}^{ij} \right]^2 \right]^{\frac{1}{2}} + \frac{1}{\sqrt{N}} \frac{K^2}{4} r_{\text{max}}^4 \left( \frac{N}{N_{\text{max}}} \right)^{\frac{4}{d}} \left[ \left( 2 + \eta_{cc} \sqrt{2} \right) + \frac{1}{\sqrt{N}} \left( 2 + \eta_{c} \sqrt{2} \right)^2 \right]
\]

(64)

with probability at least \( 1 - e^{-\eta_{cc}^2} - e^{-\eta_{c}^2} \) over the random selection of the sample points.
Noise

\[ \left\| U_1^T \left( \frac{1}{N} \tilde{E} \tilde{E}^T \right) U_1 \right\|_F \leq \]

\[ \sigma^2 \sqrt{d} + \frac{1}{\sqrt{N}} \sigma^2 \left( \sqrt{d} + \xi_e \sqrt{\frac{2}{d}} \right)^2 \left( \left( 2 + \eta_{ee} \sqrt{2} \right) + \frac{1}{\sqrt{N}} \left( 2 + \eta_e \sqrt{2} \right)^2 \right) \]

with probability at least \( 1 - e^{-\eta_{ee}^2} - e^{-\eta_e^2} - 2e^{-\xi_e^2} \) over the random realization of the noise.

\[ \left\| U_2^T \left( \frac{1}{N} \tilde{E} \tilde{E}^T \right) U_2 \right\|_F \leq \]

\[ \sigma^2 \sqrt{D - d} + \frac{1}{\sqrt{N}} \sigma^2 \left( \sqrt{D - d} + \xi_e \sqrt{\frac{2}{D - d}} \right)^2 \]

\[ \times \left( \left( 2 + \eta_{ee} \sqrt{2} \right) + \frac{1}{\sqrt{N}} \left( 2 + \eta_e \sqrt{2} \right)^2 \right) \]

with probability at least \( 1 - e^{-\eta_{ee}^2} - e^{-\eta_e^2} - 2e^{-\xi_e^2} \) over the random realization of the noise.

\[ \left\| U_1^T \left( \frac{1}{N} \tilde{L} \tilde{C}^T \right) U_2 \right\|_F \leq \]

\[ \frac{1}{\sqrt{N}} \sigma^2 \left( \sqrt{d} + \xi_e \sqrt{\frac{2}{d}} \right) \left( \sqrt{D - d} + \xi_e \sqrt{\frac{2}{D - d}} \right) \]

\[ \times \left( \left( 2 + \eta_{ee} \sqrt{2} \right) + \frac{1}{\sqrt{N}} \left( 2 + \eta_e \sqrt{2} \right)^2 \right) \]

with probability at least \( 1 - e^{-\eta_{ee}^2} - e^{-\eta_e^2} - 2e^{-\xi_e^2} \) over the random realization of the noise.

Linear-Curvature Interaction

\[ \left\| U_1^T \left( \frac{1}{N} \tilde{L} \tilde{C}^T \right) U_2 \right\|_F \leq \]

\[ \frac{1}{\sqrt{N}} \frac{K}{2} \lambda_{max}^3 \left( \frac{N}{N_{max}} \right) \frac{3}{2} \left( \left( 2 + \eta_{ee} \sqrt{2} \right) + \frac{1}{\sqrt{N}} \left( 2 + \eta_e \sqrt{2} \right)^2 \right) \]

with probability at least \( 1 - e^{-\eta_{ee}^2} - e^{-\eta_e^2} - e^{-\xi_e^2} \) over the random selection of the sample points.
Linear-Noise Interaction

\[ \left\| U_1^T \left( \frac{1}{N} \tilde{L} \tilde{E}^T \right) U_1 \right\|_F \leq \frac{1}{\sqrt{N}} \sigma_{r_{max}} \left( \frac{N}{N_{max}} \right)^{\frac{3}{2}} \left( \sqrt{d + \xi \sqrt{\frac{2}{d}}} \right) \]
\[ \times \left[ \left( 2 + \eta \ell \sqrt{2} \right) + \frac{1}{\sqrt{N}} \left( 2 + \eta \ell \sqrt{2} \right) \left( 2 + \eta \ell \sqrt{2} \right) \right] \tag{69} \]

with probability at least \( 1 - e^{-\eta \ell e} - e^{-\eta \ell} - e^{-\eta \ell} - 2e^{-\xi_e^2} \) over the joint random selection of the sample points and random realization of the noise.

\[ \left\| U_1^T \left( \frac{1}{N} \tilde{L} \tilde{E}^T \right) U_2 \right\|_F \leq \frac{1}{\sqrt{N}} \sigma_{r_{max}} \left( \frac{N}{N_{max}} \right)^{\frac{3}{2}} \left( \sqrt{D - d + \xi \sqrt{\frac{2}{D - d}}} \right) \]
\[ \times \left[ \left( 2 + \eta \ell \sqrt{2} \right) + \frac{1}{\sqrt{N}} \left( 2 + \eta \ell \sqrt{2} \right) \left( 2 + \eta \ell \sqrt{2} \right) \right] \tag{70} \]

with probability at least \( 1 - e^{-\eta \ell e} - e^{-\eta \ell} - e^{-\eta \ell} - 2e^{-\xi_e^2} \) over the joint random selection of the sample points and random realization of the noise.

Curvature-Noise Interaction

\[ \left\| U_2^T \left( \frac{1}{N} \tilde{C} \tilde{E}^T \right) U_1 \right\|_F \leq \frac{1}{\sqrt{N}} \frac{K}{2} \sigma_{r_{max}}^2 \left( \frac{N}{N_{max}} \right)^{\frac{3}{2}} \left( \sqrt{D - d + \xi \ell \sqrt{\frac{2}{D - d}}} \right) \]
\[ \times \left[ \left( 2 + \eta \ell \sqrt{2} \right) + \frac{1}{\sqrt{N}} \left( 2 + \eta \ell \sqrt{2} \right) \left( 2 + \eta \ell \sqrt{2} \right) \right] \tag{71} \]

with probability at least \( 1 - e^{-\eta \ell e} - e^{-\eta \ell} - e^{-\eta \ell} - 2e^{-\xi_e^2} \) over the joint random selection of the sample points and random realization of the noise. Note that \( \left\| U_1^T \left( \frac{1}{N} EC^T \right) U_2 \right\|_F = \left\| U_2^T \left( \frac{1}{N} \tilde{C} \tilde{E}^T \right) U_1 \right\|_F \).

\[ \left\| U_2^T \left( \frac{1}{N} \tilde{C} \tilde{E}^T \right) U_2 \right\|_F \leq \frac{1}{\sqrt{N}} \frac{K}{2} \sigma_{r_{max}}^2 \left( \frac{N}{N_{max}} \right)^{\frac{3}{2}} \left( \sqrt{D - d + \xi \ell \sqrt{\frac{2}{D - d}}} \right) \]
\[ \times \left[ \left( 2 + \eta \ell \sqrt{2} \right) + \frac{1}{\sqrt{N}} \left( 2 + \eta \ell \sqrt{2} \right) \left( 2 + \eta \ell \sqrt{2} \right) \right] \tag{72} \]

with probability at least \( 1 - e^{-\eta \ell e} - e^{-\eta \ell} - e^{-\eta \ell} - 2e^{-\xi_e^2} \) over the joint random selection of the sample points and random realization of the noise.

**APPENDIX D: MOMENT CALCULATIONS**

The following moment calculations will be used in the confidence interval calculations. Let \( L_i \) be the random variable that returns the \( i \)th coordinate.
of a point from $B^d(r)$, randomly chosen according to a uniform distribution. Let $x = [x_1 \ x_2 \ \ldots \ x_d]$ and compute the following expectations with respect to $\mu$, the uniform measure on $B^d(r)$:

$$E[L_i] = \int_{B^d(r)} x_i^a \, d\mu(x)$$

$$= \frac{1}{\text{vol}(B^d(r))} \int_{-r}^{r} x_i^a \int_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_d} dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d \, dx_i$$

$$= \frac{1}{\text{vol}(B^d(r))} \int_{-r}^{r} x_i^a \text{vol}(B^{d-1}(\sqrt{r^2-x_i^2})) \, dx_i$$

$$= \frac{\text{vol}(B^{d-1}(1))}{\text{vol}(B^d(r))} \int_{-r}^{r} x_i^a (r^2-x_i^2)^{\frac{d-1}{2}} \, dx_i.$$

Similarly,

$$E[L_i L_j] = \frac{\text{vol}(B^{d-2}(1))}{\text{vol}(B^d(r))} \int_{-r}^{r} \int_{\sqrt{r^2-x_j^2}} x_i^a x_j^b (r^2-x_i^2)^{\frac{d-2}{2}} \, dx_j \, dx_i.$$

Then we compute the following moments:

$$\mathbb{E}[L_i] = 0 \quad \text{Var}[L_i] = \frac{r^2}{d+2}$$

$$\mathbb{E}[L_i L_j] = 0 \quad \text{Var}[L_i L_j] = \frac{r^4}{(d+2)(d+4)}$$

$$\mathbb{E}[L_i^2] = \frac{r^2}{d+2} \quad \text{Var}[L_i^2] = \frac{2(d+1)r^4}{(d+2)^2(d+4)}$$

$$\mathbb{E}[L_i^2 L_j^2] = 0 \quad \text{Var}[L_i^2 L_j^2] = \frac{(d+2)(d+4)(d+6)}{10r^6}$$

$$\mathbb{E}[L_i^3] = 0 \quad \text{Var}[L_i^3] = \frac{(d+2)(d+4)(d+6)}{2(3d+5)^2}$$

$$\mathbb{E}[L_i^4] = \frac{r^4}{(d+2)(d+4)} \quad \text{Var}[L_i^4] = \frac{8(d^2+5d+3)r^8}{(d+2)^2(d+4)^2(d+6)(d+8)}$$

$$\mathbb{E}[L_i^2 L_j^2] = \frac{r^4}{3(d+2)(d+4)} \quad \text{Var}[L_i^2 L_j^2] = \frac{(d+2)(d+4)^2(d+6)(d+8)}{24(d+1)(4d+17)^8}$$

APPENDIX E: CENTRAL LIMIT THEOREM CALCULATIONS FOR MAIN RESULT 2

Here we detail the Central Limit Theorem (CLT)-based calculations that are used for Main Result 2. In the following analysis we will write $\frac{1}{N} \sum_{k=1}^{N} Y_k \overset{d}{\to}$
meaning that the sum converges in distribution to the random variable \( Y \), and we will then indicate the distribution from which \( Y \) is drawn.

### E.1. Matrix Entries.

#### E.1.1. Centering

We first compute the entries of the matrices representing the centering terms \( \hat{\mathbb{E}}[L] \), \( \hat{\mathbb{E}}[C] \), and \( \hat{\mathbb{E}}[E] \).

- **Linear**

\[ \hat{\mathbb{E}}[L] = \frac{1}{N} \sum_{k=1}^{N} L_{i,k} \xrightarrow{d} Y \in [\mu - \Gamma, \mu + \Gamma] \]

\[ \mu = 0 \]

\[ \Gamma = \eta \frac{1}{\sqrt{N}} \sqrt{\frac{2}{d + 2}} r_{\text{max}} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{2}} \]

with probability greater than \( 1 - e^{-\eta^2/2} \), where \( Y \sim \mathcal{N}(\mathbb{E}[L], \frac{1}{N} \text{Var}[L]) \).

- **Curvature**

\[ \hat{\mathbb{E}}[C] = \frac{1}{N} \sum_{k=1}^{N} C_{i,k} \]

\[ = \frac{1}{2} \left[ \kappa_{1}^{(i)} \frac{1}{N} \sum_{k=1}^{N} L_{1,k}^2 + \cdots + \kappa_{d}^{(i)} \frac{1}{N} \sum_{k=1}^{N} L_{d,k}^2 \right] \]

\[ \xrightarrow{d} \frac{1}{2} \left[ \kappa_{1}^{(i)} Y_1 + \cdots + \kappa_{d}^{(i)} Y_d \right] \]

\[ \in [\mu - \Gamma, \mu + \Gamma] \]

\[ \mu = K_{i} \frac{r_{\text{max}}^2}{2(d + 2)} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{2}} \]

\[ \Gamma = K_{i} \frac{r_{\text{max}}^2}{2(d + 2)} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{2}} \left( \frac{2\eta C}{\sqrt{N}} \sqrt{\frac{d + 1}{d + 4}} \right) \]

with probability greater than \( 1 - de^{-\eta^2/2} \), where \( Y_j \sim \mathcal{N}(\mathbb{E}[L_i^2], \frac{1}{N} \text{Var}[L_i^2]) \) for \( j = 1, \ldots, d \).
• Noise

\[ \hat{E}[E_i] = \frac{1}{N} \sum_{k=1}^{N} E_{i,k} = Y \in \left[ -\eta E \frac{\sigma \sqrt{2}}{\sqrt{N}}, \eta E \frac{\sigma \sqrt{2}}{\sqrt{N}} \right] \]

with probability greater than \(1 - e^{-\eta^2 E^2 / N \sigma^2}\), where \(Y \sim N(0, \frac{1}{N} \sigma^2)\).

Combining these results, we will use the following centering terms:

• \( \hat{E}[C_i] \hat{E}[C_j] \in [\mu - \Gamma, \mu + \Gamma] \)

\[ \mu = \frac{K_i K_j}{4(d+2)^2 r_{\max}^4} \left( \frac{N}{N_{\max}} \right)^{\frac{4}{3}} \]

\[ \Gamma = \frac{K_i K_j}{4(d+2)^2 r_{\max}^4} \left( \frac{N}{N_{\max}} \right)^{\frac{4}{3}} \left[ \frac{4\eta L}{\sqrt{N}} \frac{d+1}{d+4} + \frac{4\eta^2}{N} \frac{(d+1)}{(d+4)} \right] \]

with probability \(1 - e^{-\eta^2 L^2} - de^{-\eta^2 C^2}\),

• \( \hat{E}[E_i] \hat{E}[E_j] \in \left[ -\eta^2 E \frac{2 \sigma^2}{N}, \eta^2 E \frac{2 \sigma^2}{N} \right] \)

with probability \(1 - 2e^{-\eta^2 E^2} (i \neq j)\),

\(1 - e^{-\eta^2 E^2} (i = j)\),

• \( \hat{E}[L_i] \hat{E}[C_j] \in [\mu - \Gamma, \mu + \Gamma] \)

\[ \mu = 0 \]

\[ \Gamma = \eta L \frac{1}{\sqrt{N}} \frac{K_j}{\sqrt{2(d+2)^2} r_{\max}^3} \left( \frac{N}{N_{\max}} \right)^{\frac{3}{2}} \left[ 1 + 2\eta L \frac{d+1}{\sqrt{N}} \frac{1}{d+4} \right] \]

with probability \(1 - e^{-\eta^2 L^2} - de^{-\eta^2 C^2}\),

• \( \hat{E}[L_i] \hat{E}[E_j] \in [\mu - \Gamma, \mu + \Gamma] \)

\[ \mu = 0 \]

\[ \Gamma = \eta L \eta E \frac{1}{N} \frac{2\sigma}{\sqrt{d+2}} r_{\max} \left( \frac{N}{N_{\max}} \right)^{\frac{1}{2}} \]

with probability \(1 - e^{-\eta^2 L^2} - e^{-\eta^2 E^2}\),
• $\hat{E}[C_i] \hat{E}[E_j] \in [\mu - \Gamma, \mu + \Gamma]$

\begin{equation}
\mu = 0
\end{equation}

\begin{equation}
\Gamma = \eta_E \frac{1}{\sqrt{N}} \frac{\sigma K_i}{\sqrt{2(d+2)}} r_{\text{max}}^2 \left( \frac{N}{N_{\text{max}}} \right)^{\frac{2}{3}} \left[ 1 + \frac{2\eta_C}{\sqrt{N}} \sqrt{\frac{d+1}{d+4}} \right]
\end{equation}

with probability $> 1 - de^{-\eta_C^2} - e^{-\eta_E^2}$.

E.1.2. Curvature. The entries of the pure curvature term $CC^T$ are computed as follows. Note that the curvature term is the only one for which the entries grow with $N$.

\begin{equation}
\frac{1}{N} \sum_{k=1}^{N} C_{i,k} C_{j,k}
\end{equation}

\[= \frac{1}{N} \sum_{k=1}^{N} \frac{1}{2} \left( \kappa_1^{(i)} L_{1,k}^2 + \cdots + \kappa_d^{(i)} L_{d,k}^2 \right) \frac{1}{2} \left( \kappa_1^{(j)} L_{1,k}^2 + \cdots + \kappa_d^{(j)} L_{d,k}^2 \right) \]

\[= \frac{1}{4} \sum_{n=1}^{d} \kappa_n^{(i)} \kappa_n^{(j)} \frac{1}{N} \sum_{k=1}^{N} L_{n,k}^4 + \frac{1}{4} \sum_{m,n=1}^{d} \kappa_m^{(i)} \kappa_n^{(j)} \frac{1}{N} \sum_{k=1}^{N} L_{m,k}^2 L_{n,k}^2 \]

\[\overset{d \rightarrow 1}{\rightarrow} \frac{1}{4} \sum_{n=1}^{d} \kappa_n^{(i)} \kappa_n^{(j)} Z_n + \frac{1}{4} \sum_{m,n=1}^{d} \kappa_m^{(i)} \kappa_n^{(j)} Y_{mn} \]

\[\in [\mu - \Gamma, \mu + \Gamma] \]

\begin{equation}
\mu = \frac{1}{4} \frac{r_{\text{max}}^4}{(d+2)(d+4)} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{2}{3}} \left[ 3K_{ij}^{mn} + K_{mn}^{ij} \right]
\end{equation}

\begin{equation}
\Gamma = \frac{1}{\sqrt{N}} \frac{2\sqrt{2}(d+2)(d+4)}{2\sqrt{2}(d+2)(d+4)} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{2}{3}} \times \eta_{CC1} K_{ij}^{mn} \left( 24(d+1)(4d+17) \right) + \eta_{CC2} K_{mn}^{ij} \left( 8(d^2+5d+3) \right)
\end{equation}

with probability greater than $1 - de^{-\eta_C^2} - \frac{d(d-1)}{2} e^{-\eta_E^2}$, where $Z_n \sim \mathcal{N}(\mathbb{E}[L_i^4], \frac{1}{N} \text{Var}[L_i^4])$ and $Y_{mn} \sim \mathcal{N}(\mathbb{E}[L_i^2 L_j^2], \frac{1}{N} \text{Var}[L_i^2 L_j^2])$, for $m, n = 1, \ldots, d$. 

OPTIMAL TANGENT PLANE RECOVERY
Subtracting \( \hat{E}[C_i]\hat{E}[C_j] \) from (88),

\[
\begin{equation}
\left( \frac{1}{N} \tilde{C} \tilde{C}^T \right)_{i,j} \in [\mu - \Gamma, \mu + \Gamma]
\end{equation}
\]

\[
\mu = \frac{1}{2} \frac{r_{\text{max}}^4}{(d+2)^2(d+4)} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{d}{2}} [(d+1)K_{nn}^{ij} - K_{nn}^{ij}]
\]

\[
\Gamma = \frac{1}{\sqrt{N}} \frac{r_{\text{max}}^4}{4(d+2)} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{d}{2}} \left( \eta_{CC_1} K_{nn}^{ij} \sqrt{\frac{48(d+1)(4d+17)}{(d+4)^2(d+6)(d+8)} + 4\eta_{CC_2} K_{nn}^{ij} \sqrt{\frac{(d^2 + 5d + 3)}{(d+4)^2(d+6)(d+8)}}} \right.
\]

\[
\left. + \frac{4\eta_C K_i K_j}{(d+2)} \sqrt{\frac{d+1}{d+4}} + \frac{4\eta_{C_2} K_i K_j}{\sqrt{N}} \frac{(d+1)}{(d+2)(d+4)} \right)
\]

with probability greater than \( 1 - de^{-\eta_{CC_1}^2} - \frac{d(d-1)}{2} e^{-\eta_{CC_2}^2} - de^{-\eta_C^2} \). Note that only the lower right \((D - d) \times (D - d)\) entries are nonzero.

E.1.3. Noise. A diagonal entry of the pure noise matrix \( EE^T \) is the square of the norm of a vector in \( \mathbb{R}^N \) of Gaussian random variables. We could use the concentration of Gaussian measure to bound this norm (see Section A), but obtain a slightly tighter result using the CLT. Note that this norm neither grows nor decays with \( N \), and its leading order term \( \sigma^2 \) represents a noise-floor. An off-diagonal entry is the inner-product between two such vectors, and we therefore expect it to be small. Using a Bernstein-type inequality [20] to bound such entries yields the same leading behavior but with higher order terms, whereas using the CLT does not. Properties of the Wishart distribution could also be used [21]. Note that the off-diagonal terms tend to zero as \( N \) grows.

- Diagonal entry \((i = j)\)

\[
\begin{equation}
\frac{1}{N} \sum_{k=1}^{N} E_{i,k} E_{i,k} = \frac{\sigma^2}{N} \sum_{k=1}^{N} Y_k \xrightarrow{d} \sigma^2 Z
\end{equation}
\]

\[
\in \left[ \sigma^2 \left( 1 - \eta_{EE_1} \frac{2}{\sqrt{N}} \right), \sigma^2 \left( 1 + \eta_{EE_1} \frac{2}{\sqrt{N}} \right) \right]
\]
with probability greater than $1 - e^{-\eta_E^2}$, where $Y_k \sim \chi^2(1)$ so that $\mathbb{E}[Y_k] = 1$, $\text{Var}[Y_k] = 2$, and $Z \sim \mathcal{N}(\mathbb{E}[Y_k], \frac{1}{N}\text{Var}[Y_k])$. Subtracting $\hat{\mathbb{E}}[E_i]\hat{\mathbb{E}}[E_j]$ yields

\begin{equation}
\left( \frac{1}{N} \tilde{E} \tilde{E}^T \right)_{i,i} \in \left[ \sigma^2 \left( 1 - \frac{2}{\sqrt{N}} \left( \eta_{EE_1} + \frac{1}{\sqrt{N}} \eta_E^2 \right) \right), \sigma^2 \left( 1 + \frac{2}{\sqrt{N}} \left( \eta_{EE_1} + \frac{1}{\sqrt{N}} \eta_E^2 \right) \right) \right]
\end{equation}

with probability greater than $1 - e^{-\eta_E^2} - \frac{2}{e^{\eta_E^2}}$

- Off-diagonal entry ($i \neq j$)

\begin{equation}
\frac{1}{N} \sum_{k=1}^{N} E_{i,k}E_{j,k} \xrightarrow{d} Y \in \left[ -\eta_{EE_2} \frac{\sigma^2 \sqrt{2}}{\sqrt{N}}, \eta_{EE_2} \frac{\sigma^2 \sqrt{2}}{\sqrt{N}} \right]
\end{equation}

with probability greater than $1 - e^{-\eta^2_{EE_2}}$, where $Y \sim \mathcal{N}(\mathbb{E}[E_iE_j], \frac{1}{N}\text{Var}[E_iE_j])$. Subtracting $\hat{\mathbb{E}}[E_i]\hat{\mathbb{E}}[E_j]$ yields

\begin{equation}
\left( \frac{1}{N} \tilde{E} \tilde{E}^T \right)_{i,j} \in \left[ -\frac{\sigma^2 \sqrt{2}}{\sqrt{N}} \left( \eta_{EE_2} + \frac{1}{N} \eta_E^2 \right), \frac{\sigma^2 \sqrt{2}}{\sqrt{N}} \left( \eta_{EE_2} + \frac{1}{N} \eta_E^2 \right) \right]
\end{equation}

with probability greater than $1 - e^{-\eta^2_{EE_2}} - 2e^{-\eta_E^2}$. 
E.1.4. Linear-Curvature Interaction. The entries of the linear-curvature term are computed as follows.

\[ \frac{1}{N} \sum_{k=1}^{N} L_{i,k} C_{j,k} \]

\[ = \frac{1}{N} \sum_{k=1}^{N} L_{i,k} \left( \frac{1}{2} \left( \kappa_{1}^{(j)} L_{1,k}^{2} + \cdots + \kappa_{d}^{(j)} L_{d,k}^{2} \right) \right) \]

\[ = \frac{1}{2} \left[ \kappa_{1}^{(j)} \frac{1}{N} \sum_{k=1}^{N} L_{i,k} L_{1,k}^{2} + \cdots + \kappa_{d}^{(j)} \frac{1}{N} \sum_{k=1}^{N} L_{i,k} L_{d,k}^{2} \right] \]

\[ \frac{d}{2} \left[ \sum_{n=1 \atop n \neq i}^{d} \kappa_{n}^{(j)} Y_{n} + \kappa_{i}^{(j)} Z \right] \]

\[ \in [\mu - \Gamma, \mu + \Gamma] \]

\[ \mu = 0 \]

\[ \Gamma = \frac{1}{\sqrt{N}} \sqrt{\frac{3}{2(d + 2)(d + 4)(d + 6)}} \left( \frac{N}{N_{\max}} \right)^{\frac{3}{d}} \]

\[ \times \left[ \eta_{LC1} \sum_{n=1 \atop n \neq i}^{d} \kappa_{n}^{(j)} + \eta_{LC2} \sqrt{5} \kappa_{i}^{(j)} \right] \]

with probability greater than \( 1 - (d - 1)e^{-\eta_{LC1}} - e^{-\eta_{LC2}} \), where \( Z \sim \mathcal{N}\left( \frac{\mathbb{E}[L_{i}^{3}]}{\frac{1}{N} \text{Var}[L_{i}^{3}]} \right) \), and \( Y_{n} \sim \mathcal{N}\left( \frac{\mathbb{E}[L_{i} L_{j}^{2}]}{\frac{1}{N} \text{Var}[L_{i} L_{j}^{2}]} \right) \), for \( n = 1, \ldots, i - 1, i + 1, \ldots, d \), and Subtracting \( \hat{\mathbb{E}}[L_{i} \hat{C}_{j}] \),

\[ \left( \frac{1}{N} \hat{L}_{j} \hat{C}_{T} \right)_{i,j} \in [\mu - \Gamma, \mu + \Gamma] \]

\[ \mu = 0 \]

\[ \Gamma = \frac{1}{\sqrt{N}} \sqrt{\frac{r_{\max}^{3}}{2(d + 2)}} \left( \frac{N}{N_{\max}} \right)^{\frac{3}{d}} \left[ K_{j} \left( \frac{\eta_{LC1} \sqrt{3}}{\sqrt{(d + 4)(d + 6)}} + \frac{\eta_{L}}{(d + 2)} \right) \right] \]

\[ + \kappa_{i}^{(j)} \frac{\sqrt{3} (\eta_{LC2} \sqrt{5} - \eta_{LC1})}{\sqrt{(d + 4)(d + 6)}} + K_{j} \frac{2 \eta_{LC} \eta_{C}}{(d + 2) \sqrt{d + 4}} \]

\[ \text{max} \left[ \frac{d + 1}{\sqrt{N} (d + 2) \sqrt{d + 4}} \right] \]
with probability greater than $1 - (d - 1)e^{-\eta_L E} - e^{-\eta_L L} - e^{-\eta_E L} - de^{-\eta_E L} - e^{-\eta_L E} - e^{-\eta_L L} - e^{-\eta_C E} - e^{-\eta_L C} - e^{-\eta_L C} - e^{-\eta_E C}$. 

E.1.5. Linear-Noise Interaction. An entry of the linear-noise matrix may be shown to be a Lipschitz function of Gaussian variables on a set with large measure. One may show that on this set, such a function concentrates tightly about its expectation (see [20]). Using the CLT to compute the Lipschitz constant yields the same leading order behavior as directly applying the CLT to the entries, but results in higher order terms as well. Thus we proceed with the usual CLT calculation.

$$
\frac{1}{N} \sum_{k=1}^{N} L_{i,k} E_{j,k} \xrightarrow{d} Y \in \left[-\eta L E \frac{\sigma}{\sqrt{N}} \sqrt{\frac{2}{d+2}} r_{\max} \left(\frac{N}{N_{\max}}\right)^{\frac{1}{2}}, \eta L E \frac{\sigma}{\sqrt{N}} \sqrt{\frac{2}{d+2}} r_{\max} \left(\frac{N}{N_{\max}}\right)^{\frac{1}{2}}\right]
$$

with probability greater than $1 - e^{-\eta_L E}$, where $Y \sim \mathcal{N}(\mathbb{E}[L_{i}E_{j}], \frac{1}{N} \text{Var}[L_{i}E_{j}])$. Subtracting $\mathbb{E}[L_{i}]\mathbb{E}[E_{j}]$,

$$
\left(\frac{1}{N} \tilde{L} \tilde{E}^{T}\right)_{i,j} \in [\mu - \Gamma, \mu + \Gamma]
$$

(105) $\mu = 0$

$$
\Gamma = \frac{\sigma}{\sqrt{N}} \sqrt{\frac{2}{d+2}} r_{\max} \left(\frac{N}{N_{\max}}\right)^{\frac{1}{2}} \left[\eta L E + \eta L \eta E \sqrt{\frac{2}{N}}\right]
$$

(107) with probability greater than $1 - e^{-\eta_L E} - e^{-\eta_L L} - e^{-\eta_E L} - e^{-\eta_E C} - e^{-\eta_L C} - e^{-\eta_L C} - e^{-\eta_L C} - e^{-\eta_E L} - e^{-\eta_E L} - e^{-\eta_E C}$. 

E.1.6. Curvature-Noise Interaction. The entries of the curvature-noise matrix may be shown to be Lipschitz functions over a large set and the same comment holds as in the linear-noise case. Directly applying the CLT to the entries of this matrix, we have

$$
\frac{1}{N} \sum_{k=1}^{N} C_{i} E_{j} \xrightarrow{d} Y \in [\mu - \Gamma, \mu + \Gamma]
$$

(108) $\mu = 0$

$$
\Gamma = \eta C E \frac{1}{\sqrt{N}} \frac{\sigma r_{\max}^{2}}{\sqrt{2(d+2)(d+4)}} \left(\frac{N}{N_{\max}}\right)^{\frac{1}{2}} \sqrt{3k_{ii}^{* n} - k_{ii}^{* n}}
$$

(110)
with probability with probability greater than \(1 - e^{-\eta^2_{CE}}\), where \(Y \sim \mathcal{N}(\mathbb{E}[C_iE_j], \frac{1}{N} \text{Var}[C_iE_j])\). Subtracting \(\hat{\mathbb{E}}[C_i] \bar{\mathbb{E}}[E_j]\),

\[(111) \quad \left(\frac{1}{N} \bar{C} \bar{E}^T\right)_{i,j} \in [\mu - \Gamma, \mu + \Gamma]
\]

\[(112) \quad \mu = 0
\]

\[(113) \quad \Gamma = \frac{\sigma}{\sqrt{N}} \frac{r^2_{\text{max}}}{\sqrt{2(d+2)}} \left(\frac{N}{N_{\text{max}}}\right)^{\frac{3}{2}} \times \left[\frac{\eta_{CE}}{\sqrt{d+4}} \sqrt{3K_{nn}^i - K_{nn}^j} + \frac{\eta_{E}}{\sqrt{d+2}} K_i \left(1 + \frac{2\eta_C}{\sqrt{N}} \sqrt{d+1}\right)\right]
\]

with probability greater than \(1 - e^{-\eta^2_{CE}} - de^{-\eta^2_C} - e^{-\eta^2_E}\).

**E.2. Norm Bounds.** Recall the following constants, previously defined in Section 4.3 and restated here for clarity:

\[CC_{ij} = \left\{\frac{2}{(d+2)(d+4)} \left[(d+1)K_{nn}^i - K_{nn}^j\right] + \frac{1}{\sqrt{N}} \left(\eta_{CC_1} K_{nn}^i \sqrt{48(d+1)(4d+17) \over (d+4)^2(d+6)(d+8)} + 4\eta_{CC_2} K_{nn}^i \sqrt{(d^2+5d+3) \over (d+4)^2(d+6)(d+8)} + 4\eta_{E} K_{nn}^i \sqrt{d+1} \over (d+2) \sqrt{d+4} \right.\right\},\]

\[LC_{ij} = \left[K_j \left(\frac{\eta_{LC_1} \sqrt{5}}{\sqrt{(d+4)(d+6)}} + \frac{\eta_{L}}{d+2}\right) + \kappa_{ij} \sqrt{3} \left(\eta_{LC_2} \sqrt{5} - \eta_{LC_1}\right) \over \sqrt{(d+4)(d+6)}\right] + K_j \left(\frac{2\eta_{E}}{\sqrt{(d+2)} \sqrt{d+4}} \right),\]

\[CE_i = \left[\frac{\eta_{CE}}{\sqrt{d+4}} \sqrt{3K_{nn}^i + K_{nn}^j} + \frac{\eta_{E}}{\sqrt{d+2}} K_i \left(1 + \frac{2\eta_C}{\sqrt{N}} \sqrt{d+1}\right)\right],\]

\[\mathcal{E}(x) = \sigma^2 \sqrt{x} \left[\sqrt{2} \left(\eta_{EE_1} \sqrt{2} + \eta_{EE_2} \sqrt{x-1}\right) + \frac{2}{\sqrt{x}} \eta^2_{E} (1 + \sqrt{x-1})\right],\]
We now use the confidence intervals computed above to bound each perturbation norm.

- **Curvature**

\[
\left\| U_2^T \left( \frac{1}{N} \tilde{C} \tilde{C}^T \right) U_2 \right\|_F \leq \frac{1}{4} \left( \frac{N}{N_{\max}} \right)^{\frac{4}{3}} \left( \sum_{i=d+1}^{D} \sum_{j=d+1}^{D} \tilde{C}_i \tilde{C}_j \right)^{\frac{1}{2}}
\]

with probability greater than
\[
1 - (D - d)^2 \left[ d e^{-\eta^2_{\tilde{C}C_1}} + \frac{d(d-1)}{2} e^{-\eta^2_{\tilde{C}C_2}} \right] - d e^{-\eta^2_{\tilde{C}}}.\]

- **Noise**

\[
\left\| U_1^T \left( \frac{1}{N} \tilde{E} \tilde{E}^T \right) U_1 \right\|_F \leq \sigma^2 \sqrt{d} \sqrt{N} \left[ \eta_{\tilde{E}E_1} \sqrt{2} + \eta_{\tilde{E}E_2} \sqrt{D - d - 1} \right] + 2 \eta^2_{\tilde{E}} \left( 1 + \sqrt{d - 1} \right)
\]

with probability greater than \(1 - d e^{-\eta^2_{\tilde{E}E_1}} - \frac{d(d-1)}{2} e^{-\eta^2_{\tilde{E}E_2}} - d e^{-\eta^2_{\tilde{E}}}\).

\[
\left\| U_2^T \left( \frac{1}{N} \tilde{E} \tilde{E}^T \right) U_2 \right\|_F \leq \sigma^2 \sqrt{D - d} \sqrt{2 N} \left[ \eta_{\tilde{E}E_2} + \eta^2_{\tilde{E}} \frac{\sqrt{2}}{\sqrt{N}} \right]
\]

with probability greater than
\[
1 - (D - d) e^{-\eta^2_{\tilde{E}E_1}} - \frac{(D - d)(D - d - 1)}{2} e^{-\eta^2_{\tilde{E}E_2}} - (D - d) e^{-\eta^2_{\til{E}}}.\]
• Linear-Curvature Interaction

\[(118)\]
\[\left\| U_1^T \left( \frac{1}{N} \tilde{L} \tilde{C}^T \right) U_2 \right\|_F \leq \frac{1}{\sqrt{N}} r_{\text{max}}^3 \left( \frac{N}{N_{\text{max}}} \right)^{\frac{3}{2}} \left( \sum_{i=1}^d \sum_{j=d+1}^D \mathcal{L}C_{ij}^2 \right)^{\frac{1}{2}} \]

with probability greater than
\[1 - d(D - d) \left[ (d - 1)e^{-\eta_L c_1} + e^{-\eta_L c_2} \right] - de^{-\eta_L} - de^{-\eta_C}.\]

• Linear-Noise Interaction

\[(119)\]
\[\left\| U_1^T \left( \frac{1}{N} \tilde{L} \tilde{E}^T \right) U_1 \right\|_F \leq \frac{d \sigma}{\sqrt{N}} \sqrt{\frac{2}{d + 2}} r_{\text{max}} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{2}} \left[ \eta_{\text{LE}} + \eta_L \eta_E \frac{\sqrt{2}}{\sqrt{N}} \right] \]

with probability greater than \(1 - d^2 e^{-\eta_{\text{LE}}} - de^{-\eta_L} - de^{-\eta_E}\).

\[(120)\]
\[\left\| U_1^T \left( \frac{1}{N} \tilde{L} \tilde{E}^T \right) U_2 \right\|_F \leq \sqrt{d(D - d)} \frac{\sigma}{\sqrt{N}} \sqrt{\frac{2}{d + 2}} r_{\text{max}} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{2}} \left[ \eta_{\text{LE}} + \eta_L \eta_E \frac{\sqrt{2}}{\sqrt{N}} \right] \]

with probability greater than \(1 - d^2 e^{-\eta_{\text{LE}}} - de^{-\eta_L} - de^{-\eta_E}\).
• Curvature-Noise Interaction

\[ \left\| U_2^T \left( \frac{1}{N} \tilde{C} \tilde{E}^T \right) U_1 \right\|_F \leq \frac{\sigma}{\sqrt{N}} \sqrt{\frac{d}{2(d + 2)}} r_{\text{max}} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{2}} \left( \sum_{i=d+1}^{D} \mathcal{C} \mathcal{E}^2_i \right)^{\frac{1}{2}} \]

with probability at least \( 1 - d(D - d)e^{-\eta_2^2} - de^{-\eta_1^2} - de^{-\eta_E^2} \).

\[ \left\| U_2^T \left( \frac{1}{N} \tilde{C} \tilde{E}^T \right) U_2 \right\|_F \leq \frac{\sigma}{\sqrt{N}} \sqrt{\frac{D - d}{2(d + 2)}} r_{\text{max}} \left( \frac{N}{N_{\text{max}}} \right)^{\frac{1}{2}} \left( \sum_{i=d+1}^{D} \mathcal{C} \mathcal{E}^2_i \right)^{\frac{1}{2}} \]

with probability greater than

\( 1 - (D - d)^2 e^{-\eta_2^2} - de^{-\eta_1^2} - (D - d)e^{-\eta_E^2} \).

The norm bounds are combined to yield Main Result 2 (equation (58)) and a union bound is used to establish its associated probability.

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