On QMA Protocols with Two Short Quantum Proofs

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Abstract. This paper gives a QMA (Quantum Merlin-Arthur) protocol for 3-SAT with two logarithmic-size quantum proofs (that are not entangled with each other) such that the gap between the completeness and the soundness is \(\Omega(\frac{1}{\text{polylog}(n)})\). This improves the best completeness/soundness gaps known for NP-complete problems in this setting.

1 Introduction

The quantum complexity class QMA [9, 10, 16] is a quantum analogue of the complexity class NP (or of the class MA). That is, a decision problem is in QMA if there is a polynomial-time quantum algorithm \(V\) (called the verifier) that satisfies the following two properties: (completeness) \(V\) accepts any yes-instance with probability \(\geq a\) by the help of a quantum state (called a quantum proof); (soundness) \(V\) accepts any no-instance with probability \(\leq b\) (< \(a\)) whatever quantum state is provided. Bounding from below the gap between completeness and soundness \(a - b\) by a positive constant (or an inverse-polynomial) is enough since efficient gap amplification is possible (see, e.g., [9]).

Several variants of QMA, whose classical counterparts are uninteresting, have been introduced in the literature. One variant is the case where the verifier receives multiple quantum proofs that are unentangled with one another, which was first considered by Kobayashi, Matsumoto, and Yamakami [11]. Unexpectedly from the classical case, multiple quantum proofs may be more helpful than one proof since the verifier can use the fact that these proofs are not entangled to improve the soundness. In fact, Liu, Christandl, and Verstraete [12] found a problem that can be verified in quantum polynomial time using multiple quantum proofs but is not known to be in QMA. Recently, Harrow and Montanaro [8] showed that two quantum proofs are enough to obtain the full power of multiple quantum proofs by proving that efficient gap amplification is possible (note that it was shown before that the number of quantum proofs can be reduced to two if and only if efficient gap amplification is possible [11]). Another variant is the case where the verifier receives only a logarithmic-size quantum proof. Marriott and Watrous [13] showed that, similarly to the classical case, a logarithmic-size quantum proof is useless, that is, such a variant of QMA collapses to BQP by proving that efficient gap amplification, where the proof must be kept to be logarithmic-size, is possible.

A combination of the above two variants (multiple quantum proofs with logarithmic length) was first studied by Blier and Tapp [3]. They showed that an NP-complete problem such as the 3-coloring problem can be verified in quantum polynomial time only using two quantum proofs with logarithmic length, while the gap between completeness and soundness is an inverse-polynomial (note that it is unknown whether efficient gap amplification is possible). Moreover, Aaronson et al. [14] showed that 3-SAT can be efficiently verified with a constant completeness/soundness gap using \(O(\sqrt{n\text{polylog}}(n))\) quantum proofs, each proof being logarithmic length. These results thus give new evidences that multiple quantum proofs may be helpful.

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This paper focuses on how much the completeness/soundness gap can be improved in QMA protocols using two quantum proofs with logarithmic length for NP-complete problems. The gap obtained by Blier and Tapp was $\Omega(\frac{1}{n})$. After that, Beigi [2] improved the gap to $\Omega(\frac{1}{n^{3+\epsilon}})$ for 3-SAT, where $\epsilon > 0$ is any constant. In the present work we further improve the gap to $\Omega(\frac{1}{\text{polylog}(n)})$ for 3-SAT.

Independently of us, Chiesa and Forbes [6] also improved the completeness/soundness gap of QMA protocols with two logarithmic-size quantum proofs. They showed that the gap of the Blier-Tapp protocol can be improved to $\Omega(\frac{1}{n})$ by tightening the analysis. (In fact, two of the authors obtained the same conclusion before the present work [14].) The reason why our gap is better is simple: we combine the Blier-Tapp protocol with Dinur’s PCP reduction [7]. However, we then need a complicated case-study analysis different from the one of [3], while the analysis in [6, 14] basically follows [3].

## 2 Preliminaries

In this section, we present technical tools that are used to obtain our result. All of the tools have already been used previously [3, 5] for studying QMA protocols using multiple quantum proofs with logarithmic length but we state them for self-containment.

The first group of our tools, which was used in [3], consists of the distance between (pure) quantum states, the distance between probability distributions, the relation between their distances, and a basic fact on the swap test [4].

**Definition 1** (Quantum distance) $D(|\Psi\rangle, |\Phi\rangle) := \sqrt{1 - |\langle \Psi | \Phi \rangle|^2}$. 

**Definition 2** (Classical distance) Let $P = \{p_1, \ldots, p_k\}$ and $Q = \{q_1, \ldots, q_k\}$ be two probability distributions. Then, $D(P, Q) := \frac{1}{2} \sum_{i=1}^{k} |p_i - q_i|$. 

**Theorem 1** (Relationship between the quantum and classical notions of distance [15]) Let $M$ be a von Neumann measurement. Let $P$ and $Q$ be the distributions of outcomes when performing $M$ on $|\Psi\rangle$ and $|\Phi\rangle$, respectively. Then, $D(|\Psi\rangle, |\Phi\rangle) \geq D(P, Q)$.

**Theorem 2** (Swap test [4]) When performing the swap test on $|\Psi\rangle$ and $|\Phi\rangle$, the probability that the test outputs NO (which means the two states are not equal) is $\frac{1}{2} - \frac{|\langle \Psi | \Phi \rangle|^2}{2}$. 

The second group of our tools is from Dinur’s PCP theorem [7], which was used in [5]. We present necessary terminologies and Dinur’s PCP reduction, following the description given in [5].

**Definition 3** (Constraint graph) A constraint graph $G = (V(G), E(G))$ is an undirected graph (possibly with self-loops) along with a set $\Sigma$ of “colors” and mappings $R_e : \Sigma \times \Sigma \to \{0, 1\}$ for each edge $e = (v, u) \in E(G)$ (called the constraint to $e$). A mapping $\tau : V(G) \to \Sigma$ (called a coloring) satisfies the constraint $R_e$ if $R_e(\tau(v), \tau(u)) = 1$ for an edge $e = (v, u) \in E(G)$. The graph $G$ is said to be satisfiable if there is a coloring $\tau$ that satisfies all constraints, while $G$ is said to be $(1 - \eta)$-unsatisfiable if for all colorings $\tau$, the fraction of constraints satisfied by $\tau$ is at most $1 - \eta$.

**Theorem 3** [7] There exists a mapping $T$ from 3-SAT instances to constraint graphs with the following properties.

- **(Completeness)** If $\phi$ is a satisfiable formula, $T(\phi)$ is a satisfiable constraint graph.
- **(Soundness)** There exists an absolute constant $\eta > 0$ such that if $\phi$ is unsatisfiable formula, $T(\phi)$ is $(1 - \eta)$-unsatisfiable.
- **(Size-Efficiency)** If $\phi$ has $m$ clauses, then $|V(T(\phi))| = n = O(\text{polylog}(m))$ and also $|E(T(\phi))| = O(\text{polylog}(m))$. 

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• (Alphabet Size) \(|\Sigma| = K > 1\) is a constant independent of \(m\).

• (Regularity) \(T(\phi)\) is \(d\)-regular (with self-loops), where \(d\) is a constant independent of \(m\).

3 Our Result

We first recall the formal definition of the quantum complexity class \(\text{QMA}_{\log}(2, a, b)\), which is the set of languages that can be verified in quantum polynomial time using two logarithmic-size quantum proofs. In what follow, let \(\mathcal{H}_n = \text{span}\{|0\rangle, |1\rangle, \ldots, |n-1\rangle\} \).

Definition 4 A language \(L\) is in \(\text{QMA}_{\log}(2, a, b)\) if there exists a polynomial-time quantum algorithm \(V\) (verifier) and a constant \(c\) such that for any \(n\) and any instance \(x\) of length \(n\) the following two conditions hold:

(Completeness) If \(x \in L\), there exists a state \(|\Psi\rangle \otimes |\Phi\rangle \in \left(\mathcal{H}_2^{c\log(n)}\right)^{\otimes 2}\) (two quantum proofs) such that \(V\) accepts with probability at least \(a\).

(Soundness) If \(x \notin L\), then for all states \(|\Psi\rangle \otimes |\Phi\rangle \in \left(\mathcal{H}_2^{c\log(n)}\right)^{\otimes 2}\), the probability that \(V\) accepts is at most \(b\).

Our result is the following theorem where a 3-SAT instance has \(n\) clauses.

Theorem 4 3-SAT is in \(\text{QMA}_{\log}(2, 1, 1 - \Omega(\frac{1}{n \text{polylog}(n)})\).

There are a few remarks about this theorem. First, our result keeps perfect completeness similarly to the Blier-Tapp’s result (3) (and the recent improvement of the gap to \(\Omega(\frac{1}{n})\) [3, 14]). Second, our protocol is applicable to other NP-complete problems for which Theorem 3 holds (e.g., the 3-coloring problem).

To prove Theorem 4, it suffices to show the following theorem thanks to Theorem 3.

Theorem 5 There is a QMA protocol with two logarithmic-size quantum proofs such that for any constraint graph \(G = (V(G), E(G))\) obtained from 3-SAT instances by the mapping of Theorem 3 (where \(n = |V(G)|\)):

(Completeness) If \(G\) is satisfiable, then there exist two logarithmic-size quantum proofs \(|\Psi\rangle\) and \(|\Phi\rangle\) such the verifier accepts with probability 1.

(Soundness) If \(G\) is \((1 - \eta)\)-unsatisfiable, the verifier accepts with probability at most \(1 - \Omega(\frac{1}{n})\) for any two logarithmic-size quantum proofs \(|\Psi\rangle\) and \(|\Phi\rangle\).

In the next section, we prove Theorem 5. The verifier’s protocol is described in Section 4.1. Section 4.2 discusses its completeness, and Section 4.3 discusses its soundness, which is our main technical part.

4 Proof of Theorem 5

4.1 Protocol

As mentioned before, our protocol is obtained by incorporating Dinur’s PCP reduction into the Blier-Tapp protocol. The protocol of the verifier is as follows. Recall that \(n\) stands for the number of vertices of a given constraint graph \(G = (V(G), E(G))\), and \(K\) is the alphabet size. We denote the Fourier transform on \(\mathcal{H}_k\) by \(F_k\).
Verifier’s protocol for instance $G$

Suppose that $|\Psi\rangle$ and $|\Phi\rangle$ on $\mathcal{H}_n \otimes \mathcal{H}_K$ are given to the verifier as the two quantum proofs. The verifier then performs, with equal probability, one of the following three tests on $\mathcal{H}_n \otimes \mathcal{H}_K$. If he does not reject, then he accepts. We call the first part of $\mathcal{H}_n \otimes \mathcal{H}_K$ the vertex register and the second part of $\mathcal{H}_n \otimes \mathcal{H}_K$ the color register.

- **Test 1 (Swap test).** Perform the swap test $[4]$ on $|\Psi\rangle$ and $|\Phi\rangle$, and reject if the test outputs NO.
- **Test 2 (Consistency test).** Measure the two states $|\Psi\rangle$ and $|\Phi\rangle$ in the computational basis, yielding the outcomes $(i, j)$ and $(i', j')$, respectively. Then, do as follows:
  a) If $i = i'$, verify that $j = j'$. Reject if $j \neq j'$.
  b) If $i \neq i'$ and $(i, i') \in E(G)$, verify that $R(i, i')(j, j') = 1$. Reject if $R(i, i')(j, j') = 0$.
- **Test 3 (Uniformity test).** For both $|\Psi\rangle$ and $|\Phi\rangle$, do as follows: The Fourier transform $F_K$ is applied on the color register, which is then measured in the computational basis. If the outcome is 0, the inverse Fourier transform $F_n^*$ is applied on the vertex register, which is then measured in the computational basis. Reject if the second outcome is not 0.

4.2 Completeness

The following theorem shows that our protocol has perfect completeness.

**Proposition 1** If $G$ is satisfiable, then there exist two quantum proofs $|\Psi\rangle$ and $|\Phi\rangle$ such that the verifier accepts with probability 1.

*Proof.* Take $|\Psi\rangle = |\Phi\rangle = \frac{1}{\sqrt{n}} \sum_i |i\rangle |\tau(i)\rangle$ where $\tau$ is a coloring that satisfies all constraints. Since $|\Psi\rangle = |\Phi\rangle$, the verifier accepts with probability 1 in Test 1. Because $\tau$ satisfies the constraint $R_e$ for any edge $e \in E(G)$, the verifier accepts with probability 1 in Test 2. Finally, we analyze Test 3. The Fourier transform $F_K$ is performed on a color register, and

\[
(I \otimes F_K) \frac{1}{\sqrt{n}} \sum_i |i\rangle |\tau(i)\rangle = \frac{1}{\sqrt{n}} \sum_i |i\rangle \frac{1}{\sqrt{K}} \sum_k \exp \left( \frac{2\pi \sqrt{-1} \tau(i)k}{K} \right) |k\rangle.
\]

So, if the outcome of the measurement of the color register is 0, the state of the vertex register is $\frac{1}{\sqrt{n}} \sum_i |i\rangle = F_n |0\rangle$. Therefore, the verifier accepts with probability 1 in Test 3. \hfill $\square$

4.3 Soundness

What remains to show is the soundness of our protocol.

**Proposition 2** If $G$ is $(1 - \eta)$-unsatisfiable, the verifier rejects with probability at least $\Omega \left( \frac{1}{K} \right)$ for any two quantum proofs $|\Psi\rangle$ and $|\Phi\rangle$.

For the proof, we first describe general forms of two quantum proofs. Because the two proofs are not entangled, they can be written separately as

\[
|\Psi\rangle = \sum_{i=0}^{n-1} \alpha_i |i\rangle \sum_{j=0}^{K-1} \beta_{i,j} |j\rangle, \quad |\Phi\rangle = \sum_{i=0}^{n-1} \alpha_i' |i\rangle \sum_{j=0}^{K-1} \beta_{i,j}' |j\rangle,
\]

where $\sum_i |\alpha_i|^2 = 1$ and $\sum_j |\beta_{i,j}|^2 = 1$ for any $i$, and likewise for $|\Phi\rangle$. Next we give several lemmas.

**Lemma 1** For every $i$, there exists at least one $j$ such that $|\beta_{i,j}|^2 \geq \frac{1}{K}$. (Likewise for $\beta_{i,j}'$.)

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Proof. By contradiction. Suppose that $|\beta_{i,j}|^2 < \frac{1}{K}$ for every $j$. Then,

$$\sum_j |\beta_{i,j}|^2 < \frac{1}{K} \times K = 1.$$ 

This contradicts the condition $\sum_j |\beta_{i,j}|^2 = 1$. \hfill \Box

**Lemma 2** If $\left| \sum_j \beta_{i,j} \right|^2 < \frac{1}{1000K}$, then there are at least two $j$’s such that $|\beta_{i,j}|^2 \geq \frac{1}{K^2}$.

Proof. By Lemma 1 we know that there exists an index $j_0$ such that $|\beta_{i,j_0}|^2 \geq \frac{1}{K} \geq \frac{1}{K^2}$. We work by contradiction and suppose that $|\beta_{i,j}|^2 \leq \frac{1}{K^2}$ for all the indexes $j \neq j_0$. Note that this implies that

$$\left| \sum_{j \neq j_0} \beta_{i,j} \right| \leq \sum_{j \neq j_0} |\beta_{i,j}| \leq (K - 1) \times \frac{1}{K^2} \leq \frac{1}{K}.$$ 

Using the fact that the inequality $|a - b| \geq ||a| - |b||$ holds for any complex numbers $a$ and $b$, we obtain:

$$\left| \sum_j \beta_{i,j} \right|^2 = \beta_{i,j_0} + \sum_{j \neq j_0} \beta_{i,j}^2 \geq \left( |\beta_{i,j_0}| - \sum_{j \neq j_0} |\beta_{i,j}| \right)^2.$$ 

Since $|\beta_{i,j_0}| - \left| \sum_{j \neq j_0} \beta_{i,j} \right| \geq \frac{1}{\sqrt{K}} - \frac{1}{K} \geq 0$ and $K > 1$, we conclude that

$$\left| \sum_j \beta_{i,j} \right|^2 \geq \left( \frac{1 - \frac{1}{\sqrt{K}}}{K} \right)^2 \geq \frac{1}{100K},$$

which contradicts the assumption of the lemma. \hfill \Box

The following lemma follows from Lemma 1 in a straightforward way, but we prefer to state it explicitly for later reference.

**Lemma 3** For every $i$, there exists at least one $j$ such that $K|\beta_{i,j}|^2 \geq |\beta'_{i,j}|^2$ and $|\beta_{i,j}|^2 \geq \frac{1}{K}$. (For later proof, we denote such $j$ by $j_i$.)

Proof. By Lemma 1 there exists $j$ such that $|\beta_{i,j}|^2 \geq \frac{1}{K}$. Moreover, for the same $j$, $K|\beta_{i,j}|^2 \geq |\beta'_{i,j}|^2$ (since $K|\beta_{i,j}|^2 \geq K \times \frac{1}{K} = 1 \geq |\beta'_{i,j}|^2$). \hfill \Box

Now we are ready to prove Proposition 2.

**Proof of Proposition 2** We introduce subsets of $\{0, 1, \ldots, n-1\}$ (the set of possible $i$’s),

$$A = \left\{ i \mid |\alpha_i|^2 < \frac{1}{5000K^4n} \right\}, \quad B = \left\{ i \mid \left| \sum_j \beta_{i,j} \right|^2 < \frac{1}{100K} \right\}, \quad A' = \left\{ i \mid |\alpha'_i|^2 < \frac{1}{10000K^4n} \right\},$$

and

$$C = \left\{ i \in \overline{A} \cap \overline{A'} \mid ArgMax_j |\beta_{i,j}|^2 \neq ArgMax_j |\beta'_{i,j}|^2 \right\},$$

where $ArgMax_j |\beta_{i,j}|^2$ represents the $j$ that maximizes $|\beta_{i,j}|^2$ for fixed $i$ (when multiple such $j$’s exist, the smallest one is taken). Consider the four disjoint sets $A$, $\overline{A} \cap A'$, $\overline{A} \cap \overline{A'} \cap B$ and $\overline{A} \cap \overline{A'} \cap \overline{B}$, which partition the set $\{0, 1, \ldots, n-1\}$. We have $\sum_{i \in \overline{A}} |\alpha_i|^2 \geq 0.99$ since

$$\sum_{i \in A} |\alpha_i|^2 = 1 - \sum_{i \in A} |\alpha_i|^2 \geq 1 - \frac{1}{5000K^4n} \times n = 1 - \frac{1}{5000K^4} \geq 0.99.$$
Thus at least one of the three sums $\sum_{i \in \overline{A} \cap A'} |\alpha_i|^2$, $\sum_{i \in \overline{A} \cap B} |\alpha_i|^2$ and $\sum_{i \in \overline{A} \cap B'} |\alpha_i|^2$ is larger than 0.3. Now we analyze the following six cases.

1. $\sum_{i \in \overline{A} \cap A'} |\alpha_i|^2 \geq 0.3. : \text{case 1}$
2. $\sum_{i \in \overline{A} \cap B} |\alpha_i|^2 \geq 0.3. : \text{case 2}$
3. $\sum_{i \in \overline{A} \cap B'} |\alpha_i|^2 \geq 0.3.$
   (a) $|A| \geq 0.05\eta n. : \text{case 3}$
   (b) $|A| < 0.05\eta n.$
      i. $|A'| \geq 0.15\eta n. : \text{case 4}$
      ii. $|A'| < 0.15\eta n.$
         A. $|C| \geq 0.01\eta n. : \text{case 5}$
         B. $|C| < 0.01\eta n. : \text{case 6}$

It suffices to show that the rejecting probability is $\Omega\left(\frac{1}{n}\right)$ in every case.

**Case 1.** In this case, we show that the rejecting probability at Test 1 is $\Omega(1)$. By Theorem 2 it suffices to show that $D(|\Psi>, |\Phi>) = \Omega(1)$.

Note that by Eq.(1), $P = \{ |\alpha_i\beta_{i,j}|^2 \}$ and $Q = \{ |\alpha'_i\beta'_{i,j}|^2 \}$ are the probability distributions obtained by measuring $|\Psi>$ and $|\Phi>$ in the computational basis, respectively. By Theorem 1,

$$D(|\Psi>, |\Phi>) \geq D(P, Q) = \frac{1}{2} \sum_{i,j} \left| |\alpha_i\beta_{i,j}|^2 - |\alpha'_i\beta'_{i,j}|^2 \right|. \quad (2)$$

By Lemma 3 for any $i \in \overline{A} \cap A'$,

$$|\alpha_i\beta_{i,j[i]}|^2 - |\alpha'_i\beta'_{i,j[i]}|^2 \geq \left( \frac{1}{K} |\alpha_i|^2 - |\alpha'_i|^2 \right) |\beta'_{i,j[i]}|^2$$

$$\geq \left( 1 - \frac{1}{5000K^4n} - \frac{1}{10000K^4n} \right) |\beta'_{i,j[i]}|^2$$

$$\geq 0$$

(recall that $j[i]$ is the index defined in Lemma 3). Thus the right-hand side of Eq.(2) is at least

$$\frac{1}{2} \sum_{i \in \overline{A} \cap A'} \left( |\alpha_i|^2 |\beta_{i,j[i]}|^2 - \frac{1}{10000K^4n} |\beta_{i,j[i]}|^2 \right) = \frac{1}{2} \sum_{i \in \overline{A} \cap A'} \left( |\alpha_i|^2 |\beta_{i,j[i]}|^2 - \frac{1}{10000K^3n} \right). \quad (3)$$

Moreover, by Lemma 3 the right-hand side of Eq.(3) is at least

$$\frac{1}{2} \sum_{i \in \overline{A} \cap A'} \left( |\alpha_i|^2 |\beta_{i,j[i]}|^2 - \frac{1}{10000K^3n} \right) = \frac{1}{2} \sum_{i \in \overline{A} \cap A'} \left( |\alpha_i|^2 - \frac{1}{10000K^3n} \right)$$

$$\geq \frac{1}{2K} \left( \sum_{i \in \overline{A} \cap A'} |\alpha_i|^2 - \sum_{i \in \overline{A} \cap A'} \frac{1}{10000K^3n} \right).$$

Finally, we can see that the right-hand side of the above inequality is at least

$$\frac{1}{2K} \left( 0.3 - \frac{1}{10000K^3n} \right) = \Omega(1)$$
by the condition of case 1. This completes the analysis of case 1.

**Case 2.** In this case, we show that the rejecting probability \( \Omega(n) \) is guaranteed by Test 2a).

Fix \( i \in A \cap A' \cap B \). Then, by Lemma 1, there exists an index \( j' \) such that \( |\beta_{i,j'}|^2 \geq \frac{1}{K^4} \). Since \( i \) is in \( B \), by Lemma 2 there exists an index \( j \) such that \( j \neq j' \) and \( |\beta_{i,j}|^2 \geq \frac{1}{K^4} \). Since \( |\alpha_i|^2 \geq \frac{1}{10000K^n} \), the probability of measuring \((i, j), (i, j')\) from two proofs is

\[
|\alpha_i|^2 |\beta_{i,j}|^2 |\alpha_i'|^2 |\beta_{i,j'}|^2 \geq |\alpha_i|^2 \frac{1}{K^4} \frac{1}{10000K^4n} \frac{1}{K} = \frac{1}{10000K^9n} |\alpha_i|^2.
\]

Therefore, the rejecting probability at Test 2a) is at least

\[
\sum_{i \in A \cap A' \cap B} \frac{1}{10000K^9n} |\alpha_i|^2 \geq \frac{1}{10000K^9n} \times 0.3 = \Omega \left( \frac{1}{n} \right),
\]

where the inequality is obtained by the condition of case 2.

**Case 3.** In this case, we show that the rejecting probability \( \Omega(1) \) is guaranteed by Test 3. For this purpose, we analyze the probability \( P_{t_3c} \) that 0 is obtained from the color register of \( |\Psi\rangle \) and the probability \( P_{t_3v} \) that 0 is not obtained from the vertex register after measuring the color register. In what follows, let \( \beta_i = \sum_j \beta_{i,j} \).

First, we consider \( P_{t_3c} \). The state after performing the Fourier transform \( F_K \) on the color register of \( |\Psi\rangle \) is

\[
(I \otimes F_K) \sum_i |\alpha_i| |i\rangle \sum_j |\beta_{i,j}| |j\rangle = \frac{1}{\sqrt{K}} \sum_i |\alpha_i| |i\rangle \sum_j |\beta_{i,j}| \sum_k \exp \left( \frac{2\pi \sqrt{-1} j k}{K} \right) |k\rangle.
\]

Therefore, we have

\[
P_{t_3c} = \frac{1}{K} \sum_i |\alpha_i|^2 |\beta_i|^2
\geq \frac{1}{K} \sum_{i \in A \cap A' \cap B} |\alpha_i|^2 \frac{1}{100K}
\geq \frac{1}{100K^2} \sum_{i \in A \cap A' \cap B} |\alpha_i|^2
\geq \frac{1}{100K^2} \times 0.3
= \Omega (1),
\]

where the first inequality comes from the definition of the set \( B \) and the third inequality comes from the condition of case 3.

Next, we consider \( P_{t_3v} \). Let \( L = \sqrt{\sum_k |\alpha_k|^2 |\beta_k|^2} \) (\( \geq \sqrt{\frac{3}{10000K^2}} \)) by the above analysis of \( P_{t_3c} \).

The state after measuring 0 from the color register is (omitting the color register):

\[
\frac{1}{\sqrt{K}} \sqrt{\sum_k |\alpha_k|^2 |\beta_k|^2} \sum_i |\alpha_i| |i\rangle \sum_j |\beta_{i,j}| = \frac{1}{L} \sum_i |\alpha_i| |i\rangle.
\]

Let \( |X\rangle = \frac{1}{L} \sum_i |\alpha_i| |i\rangle \). After performing the inverse Fourier transform \( F_n^\dagger \) on \( |X\rangle \), the probability of measuring 0 from it in the computational basis is the square of the inner product between \( F_n^\dagger |X\rangle \)
Lemma 3, and the definitions of $A$ and $C$. Thus we have

$$P_{3\omega} = 1 - |\langle X | F_n | 0 \rangle|^2 = 1 - \left| \sum_i \alpha_i \beta_i / \sqrt{nL} \right|^2 \geq \sum_i \left( \frac{1}{n} - \frac{|\alpha_i|^2 |\beta_i|^2}{nL^2} \right). \quad (4)$$

Note that the right-hand side of Eq. (4) is at least

$$\frac{1}{n} \sum_{i \in A} \left( 1 - \frac{1000K}{3} \frac{1}{5000K^3n} K^2 \right) = \frac{1}{n} \sum_{i \in A} \left( 1 - \frac{1}{15n} \right) \quad (5)$$

since $|\beta_i|^2 \leq K^2$, $\frac{1}{L^2} \leq \frac{1000K}{3}$ and $|\alpha_i|^2 \leq \frac{1}{5000K^3n}$ for $i \in A$. Since $|A| \geq 0.05\eta n$ by the condition of case 3, the right-hand side of Eq. (5) is at least

$$\frac{1}{n} \left( 1 - \frac{1}{15n} \right) \times 0.05\eta n \geq \frac{7\eta}{150}.$$

Thus, we obtain

$$P_{3\omega} = 1 - |\langle X | F_n | 0 \rangle|^2 \geq \frac{7\eta}{150} = \Omega(1).$$

Finally, the rejecting probability at Test 3 is at least $P_{3\omega} = \Omega(1)$.

**Case 4.** This is similar to case 1. We show that the rejecting probability at Test 1 is $\Omega(1)$.

By Theorem 2, it suffices to show that $D(\psi, \phi) = \Omega(1)$. Similarly to case 1, by Theorem 2, Lemma 3, and the definitions of $A$ and $A'$ we have

$$D(\psi, \phi) \geq \frac{1}{2} \sum_{i,j} \left| |\alpha_i \beta_{i,j}|^2 - |\alpha'_{i,j} \beta'_{i,j}|^2 \right|$$

$$\geq \frac{1}{2} \sum_{i \in \mathcal{A} \cap \mathcal{A}'} \frac{1}{5000K^3n} |\beta_{i,j}|^2 - \frac{1}{10000K^4n} |\beta'_{i,j}|^2.$$

By Lemma 3 we can see that (similarly to case 1, also) this value is at least

$$\frac{1}{2K} \sum_{i \in \mathcal{A} \cap \mathcal{A}'} \left( \frac{1}{5000K^3n} - \frac{1}{10000K^4n} \right) \geq \frac{1}{2K} \left( \frac{1}{5000K^3n} - \frac{1}{10000K^4n} \right) \times 0.1\eta n = \Omega(1).$$

Note that the inequality comes from $|\mathcal{A} \cap \mathcal{A}'| \geq 0.1\eta n$, which is obtained by $|A| < 0.05\eta n$ and $|A'| \geq 0.15\eta n$ (the condition of case 4).

**Case 5.** In this case, we show that the rejecting probability $\Omega\left(\frac{1}{C}\right)$ is guaranteed by Test 2a).

Fix $i \in C$. Let $c = \text{ArgMax}_{j} |\beta_{i,j}|^2$ and $c' = \text{ArgMax}_{j} |\beta'_{i,j}|^2$. Note that $c \neq c'$ by the definition of $C$. By Lemma 3, $|\beta_{i,c}|^2 \geq \frac{1}{K}$ and $|\beta'_{i,c'}|^2 \geq \frac{1}{K}$. Since $|\alpha_i|^2 \geq \frac{1}{5000K^3n}$ and $|\alpha'_i|^2 \geq \frac{1}{10000K^4n}$, the probability of measuring $(i,c),(i,c')$ from $\psi$ and $\phi$ is

$$|\alpha_i|^2 |\beta_{i,c}|^2 |\alpha'_i|^2 |\beta'_{i,c'}|^2 \geq \frac{1}{5000K^3n} \frac{1}{K} \frac{1}{10000K^4n} \frac{1}{K} = \frac{1}{5 \times 10^7 K^9 n^2}.$$

Therefore, the rejecting probability at Test 2a) is at least

$$\sum_{i \in C} |\alpha_i|^2 |\beta_{i,c}|^2 |\alpha'_i|^2 |\beta'_{i,c'}|^2 \geq \frac{1}{5 \times 10^7 K^9 n^2} \times 0.01\eta n = \Omega\left(\frac{1}{n}\right),$$

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where the inequality comes from the condition of case 5.

**Case 6.** In this case, we show that the rejecting probability $\Omega(\frac{1}{n})$ is guaranteed by Test 2b).

For the analysis, we first introduce the set $C' = \{ i \in \overline{A} \cap \overline{A'} \mid \text{ArgMax}_{j} |\beta_{i,j}|^2 = \text{ArgMax}_{j} |\beta'_{i,j}|^2 \}$ and give a lower bound of $|C'|$. Since $|A| < 0.05\eta n$ and $|A'| < 0.15\eta n$,

$$|\overline{A} \cap \overline{A'}| = n - |A \cup A'| \geq n - 0.2\eta n.$$  

Noting that $|C| < 0.01\eta n$,

$$|C'| = |\overline{A} \cap \overline{A'}| - |C| > n - 0.2\eta n - 0.01\eta n = n - 0.21\eta n.$$  

Next we consider $\left| \{(i, i') \in E(G) \mid i \in C' \text{ and } i' \in C'\} \right|$. Since $\left| \{(i \mid i \notin C') \right| = n - |C'| < n - (n - 0.21\eta n) = 0.21\eta n$ and the degree of every vertex is $d$,

$$\left| \{(i, i') \in E(G) \mid i \notin C' \text{ or } i' \notin C'\} \right| \leq \left| \{(i, i') \in E(G) \mid i \notin C'\} \right| + \left| \{(i, i') \in E(G) \mid i' \notin C'\} \right|$$

$$< 0.21\eta n \times d + 0.21\eta n \times d$$

$$= 0.42\eta dn.$$  

Thus we have

$$\left| \{(i, i') \in E(G) \mid i \in C' \text{ and } i' \in C'\} \right| = |E(G)| - \left| \{(i, i') \in E(G) \mid i \notin C' \text{ or } i' \notin C'\} \right|$$

$$> |E(G)| - 0.42\eta dn.$$  

Now we can regard $\tau(i) := \text{ArgMax}_{j} |\beta_{i,j}|^2$ as the coloring for vertex $i$. Since the graph $G$ is $(1-\eta)$-unsatisfiable, for this coloring the number of edges that do not satisfy the constraint is at least $\eta|E(G)|$. We evaluate the probability of getting such an edge from two quantum proofs in Test 2. Since the degree of every vertex is $d$, $|E(G)| = \frac{dn}{2}$. Thus the set $\{(i, i') \in E(G) \mid i \in C' \text{ and } i' \in C'\}$ contains at least

$$\left| \{(i, i') \in E(G) \mid i \in C' \text{ and } i' \in C'\} \right| - (1-\eta)|E(G)|$$

$$> |E(G)| - 0.42\eta dn - (1-\eta)|E(G)|$$

$$= 0.08\eta dn$$

edges that do not satisfy the constraint. By Lemma 1, the probability of getting each of these edges is

$$|\alpha_i|^2 |\beta_{i, \tau(i)}|^2 |\alpha'_{i'}|^2 |\beta'_{i', \tau(i')}|^2 \geq \frac{1}{5000K^2n} \frac{1}{K} \frac{1}{10000K^4n} \frac{1}{K}$$

$$= \frac{1}{5 \times 10^7 K^6 n^2}.$$  

Therefore, the rejecting probability at Test 2b) is at least

$$\frac{1}{5 \times 10^7 K^6 n^2} \times 0.08\eta dn = \Omega \left( \frac{1}{n} \right).$$

Now the proof of Proposition is completed. □
5 Concluding Remarks

We have shown that there is a QMA protocol for 3-SAT with two quantum proofs of logarithmic length such that the gap between completeness and soundness is $\Omega(\frac{1}{\text{polylog}(n)})$, which improves the previous works [3, 2, 5, 6, 14]. It seems to be difficult to improve our current gap as long as a protocol similar to the Blier-Tapp one is used. Moreover, we notice that all the tests of our protocol are necessary. For example, we cannot delete Test 1 (Swap test) since, without it, perfect cheating becomes possible (i.e., there are quantum proofs such that the verifier accepts a no-instance with probability 1). This is different from the case of [5] where it was shown that the swap test can be eliminated while still obtaining the same conclusion as in [1] (namely, that 3-SAT can be verified in quantum polynomial time using $O(\sqrt{n}\text{polylog}(n))$ quantum proofs with logarithmic length).

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References


