An almost second order uniformly convergent method for parabolic singularly perturbed reaction-diffusion systems

C. Clavero
J.L. Gracia
F. Lisbona

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AN ALMOST SECOND ORDER UNIFORMLY CONVERGENT METHOD FOR PARABOLIC SINGULARLY PERTURBED REACTION-DIFFUSION SYSTEMS

C. CLAVERO, J.L. GRACIA AND F. LISBONA

Abstract. In this work we consider a parabolic system of two linear singularly perturbed equations of reaction-diffusion type coupled in the reaction terms. The small values of the diffusion parameters, in general, cause that the solution has boundary layers at the ends of the spatial domain. To obtain an efficient approximation of the solution we propose a numerical method combining the Crank-Nicolson method joint to the central finite difference scheme defined on a piecewise uniform Shishkin mesh. The resulting method is uniformly convergent of second order in time and almost second order in space, if the discretization parameters satisfy a non restrictive relation. We display some numerical experiments showing the order of uniform convergence theoretically proved. These numerical results also indicate that the relation between the discretization parameters is not necessary in practice.

Key words. singular perturbation, reaction-diffusion problems, uniform convergence, coupled system, Shishkin mesh, high order.

AMS subject classifications. 65M06, 65N06, 65N12

1. Introduction. In this paper we consider a type of parabolic singularly perturbed problems given by

$$\begin{cases}
L_{x,\varepsilon} u \equiv \frac{\partial \bar{u}}{\partial t} + L_{x,\varepsilon} \bar{u} = \tilde{f}, & (x, t) \in Q = \Omega \times (0, T] = (0, 1) \times (0, T], \\
\bar{u}(0, t) = 0, \quad \bar{u}(1, t) = 0, & \forall t \in [0, T], \\
\bar{u}(x, 0) = 0, & \forall x \in \bar{\Omega},
\end{cases}
$$

(1.1)

where the spatial differential operator is defined by

$$L_{x,\varepsilon} \equiv \begin{pmatrix}
-\varepsilon_1 \frac{\partial^2}{\partial x^2} \\
-\varepsilon_2 \frac{\partial^2}{\partial x^2}
\end{pmatrix} + A, \quad A = \begin{pmatrix}
a_{11}(x) & a_{12}(x) \\
a_{21}(x) & a_{22}(x)
\end{pmatrix}.
$$

We denote by $\Gamma_0 = \{(x, 0) \mid x \in \Omega\}$, $\Gamma_1 = \{(x, t) \mid x = 0, 1, \ t \in [0, T]\}$, $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\varepsilon = (\varepsilon_1, \varepsilon_2)^T$, with $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$, the vectorial singular perturbation parameter. We assume that the right hand side function $\tilde{f}(x, t) = (f_1(x, t), f_2(x, t))^T$ and the reaction matrix $A$ have enough smooth components. Also we suppose that the following positivity condition on the matrix reaction $A$ is satisfied:

$$\begin{align*}
a_{i,1} + a_{i,2} &\geq \alpha > 0, \quad a_{ii} > 0, \quad i = 1, 2, \\
a_{ij} &\leq 0 \text{ if } i \neq j.
\end{align*}
$$

(1.2) (1.3)

If (1.2) is not satisfied, we could consider the transformation $\bar{v}(x, t) = \bar{u}(x, t)e^{-\alpha_0 t}$ with $\alpha_0 > 0$ sufficiently large, and therefore in the new problem (1.2) holds. Finally we assume that sufficient compatibility conditions among the data of the differential
equation hold in order that the exact solution \( \vec{u} \in C^{4,3} (\bar{Q}) \). In particular, in this work we will assume the following compatibility conditions

\[
\frac{\partial^{k+k_0} \vec{f}}{\partial x^k \partial t^{k_0}} (0,0) = \frac{\partial^{k+k_0} \vec{f}}{\partial x^k \partial t^{k_0}} (1,0) = \vec{0}, \quad 0 \leq k + 2k_0 \leq 4.
\]

(1.4)

Linear coupled systems of type (1.1) appears in the modelization of the flow in fractured porous media, concretely in the double diffusion model of Barenblatt (see [1]). Another process giving similar problems to (1.1) are the model for turbulent interactions of waves and currents (see [15]) or the diffusion process in electroanalytic chemistry (see [14]).

It is known that, in general, the exact solution of problem (1.1) has a multiscale character (see [14]). Then, to find good approximations of the solution for any value of the diffusion parameters \( \varepsilon_1 \) and \( \varepsilon_2 \), it is necessary to dispose of efficient numerical methods (uniformly convergent methods). Recently in some papers uniformly convergent numerical methods were analyzed (see [6, 9, 10, 11, 12]). These methods were defined on piecewise uniform Shishkin meshes, and they were used to solve both singularly perturbed elliptic or parabolic linear systems. In the analysis of the convergence of the numerical methods, three different cases were considered, depending on the ratio between the two singular perturbation parameters \( \varepsilon_1 \) and \( \varepsilon_2 \):

1. \( \varepsilon_1 = \varepsilon, \varepsilon_2 = 1 \)
2. \( \varepsilon_1 = \varepsilon_2 = \varepsilon \)
3. \( \varepsilon_1, \varepsilon_2 \) arbitrary

In [6] a decomposition of the exact solution of problem (1.1) into its regular and singular components was given for any ratio between \( \varepsilon_1 \) and \( \varepsilon_2 \). Moreover, they were proven bounds for the derivatives of both components; from these bounds it follows theirs asymptotic behaviour with respect to \( \varepsilon_1 \) and \( \varepsilon_2 \). In that work, also a first order in time and almost second order in space uniformly convergent method was obtained.

In practice it is important to dispose of high order convergent schemes to find good approximations with a low computational cost. Some papers in the context of the numerical integration of singularly perturbed problems follow this direction (see [3, 5, 7]). So far we do not know any paper proving uniform order of convergence bigger than one in both in time and space for a method used to solve (1.1). To increase the order of convergence in time, here we use the Crank-Nicolson method; note that the totally discrete scheme obtained by using the Crank-Nicolson method and the central finite difference scheme, does not satisfy the discrete maximum principle except if the restrictive and unpractical condition \( \Delta t \leq C(N^{-1} \ln N)^2 \) is imposed. In this paper we follow Clavero et al. [2] to save this difficulty. The analysis of the convergence is done by defining some specific auxiliary problems, which permits to prove appropriate bounds for the local error of the Crank-Nicolson scheme. After that we will prove the uniform convergence of the central finite difference scheme used to discretize in space these auxiliary problems. Finally, using a recursive argument and the uniform stability of the totally discrete operator (given below), we will deduce the uniform convergence of the scheme. Here we will assume such uniform stability, supported by the numerical evidence showed in the performed experiments (see the numerical results section). In all cases, the results show that the spectral radius of the totally discrete operator is strictly less than one, independently of the diffusion parameters \( \varepsilon_1 \) and \( \varepsilon_2 \) and also of the discretization parameters \( N \) and \( \Delta t \).

In [8] the author establishes the maximum norm stability for some non inverse monotone discrete operators, using the Euler method to discretize in time a parabolic
reaction-diffusion equation, proving first order uniform convergence in time. Nevertheless, the extension of such ideas to higher order methods (including the Crank-Nicolson scheme) is at the moment an open question.

The paper is structured as follows. In Section 2 we establish the asymptotic behaviour of the solution of (1.1) and its partial derivatives. In Section 3 we analyze the local error of the Crank-Nicolson discretization and we show the asymptotic behaviour of the exact solution of the semidiscrete problems resulting after the time discretization process. In Section 4 we construct the central finite differences scheme, defined on an appropriate piecewise uniform Shishkin mesh, to discretize in space; we prove that if the discretization parameters $N$ and $\Delta t$ are related by $N^{-q} \leq C \Delta t$, which is not a very restrictive condition in practice, then the totally discrete method is an almost second order uniformly convergent method. Nevertheless, we believe that this relation is only necessary from a theoretical point of view; the reason is that in all numerical experiments that we have performed, we have never needed that the relation between $N$ and $\Delta t$ holds. Finally, in Section 5 we display some numerical experiments corroborating in practice the improvement in the order of uniform convergence of the numerical method.

Below we denote by $\bar{v} \leq \bar{w}$ if $v_1 \leq w_i$, $i = 1, 2$, $|\bar{v}| = (|v_1|, |v_2|)^T$, $\bar{C} = (C, C)^T$, where $C$ is a positive constant, $\|f\|_H$ is the maximum norm of $f$ on the closed set $H$ and $\|\bar{f}\|_H = \max\{\|f_1\|_H, \|f_2\|_H\}$. Henceforth, $C$ denotes a generic positive constant independent of the diffusion parameters $\varepsilon_1$ and $\varepsilon_2$, and also of the discretization parameters $N$ and $\Delta t$; sometimes we use a subscripted $C$ with the same purpose.

2. Asymptotic behaviour of the solution. In this section we extend the analysis given in [6], showing the asymptotic behaviour of the exact solution of the system (1.1). The proofs of the results basically use the maximum principle (see [13]) and differentiation of the partial differential equation (1.1) with respect to time and space variables.

**Lemma 2.1** (Maximum principle). If $\bar{\psi} \geq \bar{0}$ on $\Gamma$ and $L\varepsilon \bar{\psi} \geq \bar{0}$ in $Q$, then $\bar{\psi} \geq \bar{0}$ for all $(x, t) \in \bar{Q}$.

**Corollary 2.2** (Comparison principle). If $|\bar{\psi}| \leq \bar{\varphi}$ on $\Gamma$ and $|L\varepsilon \bar{\psi}| \leq L\varepsilon \bar{\varphi}$ in $Q$, then $|\bar{\psi}| \leq \bar{\varphi}$ for all $(x, t) \in \bar{Q}$.

**Lemma 2.3.** The solution of problem (1.1) satisfies

$$\left\{ \begin{array}{ll} L\varepsilon \bar{v} = \bar{f}, & \text{in } Q, \\ \bar{v}(x, 0) = \bar{0}, & \text{on } \Gamma_0, \quad \bar{v} = \bar{\varphi}, & \text{on } \Gamma_1, \end{array} \right.$$  

where $\bar{\varphi}$ satisfies the initial value problem

$$\bar{\varphi}_t + A\bar{\varphi} = \bar{f}, \quad (x, t) \in \{0, 1\} \times (0, T], \quad \bar{\varphi}(x, 0) = \bar{0}, \quad x \in \{0, 1\},$$

and the singular component $\bar{w}$ is the solution of

$$L\varepsilon \bar{w} = \bar{0}, \quad \text{in } Q, \quad \bar{w} = \bar{u} - \bar{v}, \quad \text{on } \Gamma.$$

Note that the right hand side of problem (2.1) satisfies the conditions (1.4) and also that $\bar{\varphi}(x, 0) = \bar{z}_t(x, 0) = \bar{z}_{ttt}(x, 0) = 0$, $x = 0, 1$. Then, we have that $\bar{v} \in C^4,3(\bar{Q})$ and therefore $\bar{w} \in C^4,3(\bar{Q})$.  

Lemma 2.4. The regular component \( \vec{v} = (v_1, v_2)^T \) satisfies

\[
\left\| \frac{\partial^k \vec{v} \times \vec{v}}{\partial x^k} \right\|_Q \leq C, \quad 0 \leq k \leq 3, \quad \left\| \frac{\partial^k \vec{v} \times \vec{v}}{\partial x^k} \right\|_Q \leq C, \quad k = 1, 2, \tag{2.4}
\]

Proof. We only give the main ideas of the proof for the crossed derivatives \( \vec{v}_{xxt} \) and \( \vec{v}_{tttx} \). From (2.1) and (2.2) we have that \( \vec{v}_{xx} = \vec{0} \) on \( \Gamma_1 \); hence \( \vec{v}_{xx} = \vec{0} \) on \( \Gamma_1 \). Using that \( \vec{v}(x,0) = \vec{0} \) on \( \Gamma_0 \) and differentiating (2.1) twice w.r.t. \( x \) we have that \( \vec{v}_{xxt} = f_{xx} \) on \( \Gamma_0 \). Then, differentiating now (2.1) twice w.r.t. \( x \) and once w.r.t. \( t \), it follows that

\[
\left\| \vec{v}_{xxtt}(x,0) \right\|_\Omega = \left\| -L_{x,x}\vec{f}_{xx}(x,0) + f_{xxt}(x,0) - 2A_x(x,0)\vec{v}_{xx}(x,0) - A_{xx}(x,0)\vec{v}_t(x,0) \right\|_\Omega \leq C,
\]

where \( A_x = (a_{ij}') \) and \( A_{xx} = (a_{ij}^{''}) \). Differentiating (2.1) twice w.r.t. \( x \) and twice w.r.t. \( t \), we can obtain

\[
(2.5) \quad \left\| \frac{L_{x}\vec{v}_{xxtt}}{Q} \right\| = \left\| (f_{xxtt} - 2A_x\vec{v}_{xt} - A_{xx}\vec{v}_t) \right\| \leq \bar{C} + \bar{C}_1 \left\| \vec{v}_{xxtt} \right\|_Q,
\]

where \( C_1 = 2 \max\{|a_{ij}'|\} \). The comparison principle applied on the barrier function \( \vec{v} = (1 + t)(\bar{C} + \bar{C}_1 \left\| \vec{v}_{xxtt} \right\|_Q) \) proves

\[
(2.6) \quad \left\| \vec{v}_{xxtt} \right\|_Q \leq (1 + T)(C + C_1 \left\| \vec{v}_{xxtt} \right\|_Q).
\]

Similarly to [11], Lemma 3, we can apply the mean value theorem on the interval \([a, a + C_2] \subset [0, 1]\), where \( C_2 = \min\{1, 1/(2C_1(1 + T))\} \) and \( a \geq 0 \), obtaining

\[
(2.7) \quad \left\| \vec{v}_{xxtt} \right\|_Q \leq C + \frac{\left\| \vec{v}_{xxtt} \right\|_Q}{2C_1}.
\]

Then, the result follows from (2.6) and (2.7). \( \square \)

Below we use the auxiliary function \( B_\gamma(x) = e^{-x\sqrt{\alpha/\gamma}} + e^{-(1-x)\sqrt{\alpha/\gamma}} \), where \( \gamma \) is an arbitrary positive constant and \( \alpha \) was defined in (1.2).

Lemma 2.5. The singular component \( \vec{w} = (w_1, w_2)^T \) satisfies

\[
(2.8) \quad \left\| \frac{\partial^k \vec{w}}{\partial x^k} \right\| \leq B_{\varepsilon_2}(x)\bar{C}, \quad \forall (x,t) \in \bar{Q}, \quad 0 \leq k \leq 3.
\]

\[
(2.9) \quad \left| w_1(x) \right| \leq C\varepsilon_2(x), \quad \left| w_2(x) \right| \leq C\varepsilon_2(x),
\]

\[
(2.10) \quad \left| \frac{\partial w_1}{\partial x} \right| \leq C(\varepsilon_1^{1/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2}B_{\varepsilon_2}(x)), \quad \left| \frac{\partial^2 w_2}{\partial x^2} \right| \leq C\varepsilon_2^{-1/2}B_{\varepsilon_2}(x),
\]

\[
(2.11) \quad \left| \frac{\partial^2 w_1}{\partial x^2} \right| \leq C(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2}B_{\varepsilon_2}(x)), \quad \left| \frac{\partial^3 w_2}{\partial x^3} \right| \leq C\varepsilon_2^{-1/2}B_{\varepsilon_2}(x),
\]

\[
(2.12) \quad \left| \frac{\partial^3 w_1}{\partial x^3} \right| \leq C(\varepsilon_1^{-3/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2}B_{\varepsilon_2}(x)),
\]
(2.13) \[ \frac{\partial^3 w_2}{\partial x^3} \leq C\varepsilon_2^{-1}(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2}B_{\varepsilon_2}(x)), \]

(2.14) \[ \frac{\partial^4 w_1}{\partial x^4} \leq C(\varepsilon_1^{-2}B_{\varepsilon_1}(x) + \varepsilon_2^{-2}B_{\varepsilon_2}(x)), \]

(2.15) \[ \frac{\partial^4 w_2}{\partial x^4} \leq C\varepsilon_2^{-1}(\varepsilon_1^{-1}B_{\varepsilon_1}(x) + \varepsilon_2^{-1}B_{\varepsilon_2}(x)). \]

**Lemma 2.6.** The singular component \( \bar{w} = (w_1, w_2)^T \) satisfies

(2.16) \[ \frac{\partial^2 w_i}{\partial x^2 \partial t} \leq C(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2}B_{\varepsilon_2}(x)), \quad i = 1, 2, \]

(2.17) \[ \frac{\partial^3 w_i}{\partial x^3 \partial t} \leq C(\varepsilon_1^{-1}B_{\varepsilon_1}(x) + \varepsilon_2^{-1}B_{\varepsilon_2}(x)), \quad i = 1, 2, \]

(2.18) \[ \frac{\partial^4 w_1}{\partial x^4 \partial t} \leq C(\varepsilon_1^{-3/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2}B_{\varepsilon_2}(x)), \]

(2.19) \[ \frac{\partial^4 w_2}{\partial x^4 \partial t} \leq C\varepsilon_2^{-1}(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2}B_{\varepsilon_2}(x)). \]

**Proof.** Bounds (2.16) and (2.17) are obtained by differentiating (2.3) once and twice w.r.t. \( x \) respectively. On the other hand, differentiating (2.3) twice w.r.t. \( t \), we deduce

\[ \left| \frac{\partial^2 w_i}{\partial x^2 \partial t^2} \right| \leq C\varepsilon_1^{-1}B_{\varepsilon_2}(x), \quad i = 1, 2, \]

and hence, using the mean value theorem it can be proved that

\[ \left| \frac{\partial^3 w_i}{\partial x \partial t^2} \right| \leq C\varepsilon_1^{-1/2}B_{\varepsilon_2}(x), \quad i = 1, 2. \]

Differentiating (2.3) once w.r.t. \( x \) and once w.r.t. \( t \), we can obtain (2.19), but for the first component we only prove the crude bound

(2.20) \[ \left| \frac{\partial^4 w_1}{\partial x^4 \partial t} \right| \leq C\varepsilon_1^{-3/2}. \]

Although (2.20) is not the required bound, it will be useful in the boundary points \((0, t)\) and \((1, t)\) with \( t \in [0, 1] \). To improve bound (2.20), we define the following auxiliary problem

\[ \begin{cases} 
\frac{\partial}{\partial t} \left( \frac{\partial^4 w_1}{\partial x^4 \partial t} \right) - \varepsilon_1 \frac{\partial^2}{\partial x^2} \left( \frac{\partial^4 w_1}{\partial x^4 \partial t} \right) + a_{11} \frac{\partial^4 w_1}{\partial x^4 \partial t} = g(x, t), \quad (x, t) \in Q, \\
\frac{\partial^4 w_1}{\partial x^4 \partial t} \quad \text{given on } \Gamma,
\end{cases} \]

where

\[ g(x, t) = -\frac{\partial^4 (a_{12} w_2)}{\partial x^2 \partial t} - \frac{\partial^3 a_{11} \partial w_1}{\partial x^3 \partial t} - 3 \frac{\partial^2 a_{11} \partial^2 w_1}{\partial x^2 \partial t} - 3 \frac{\partial a_{11} \partial^3 w_1}{\partial x \partial x^2 \partial t}. \]
Using that
\[ \frac{\partial^4 w_1}{\partial x^4 \partial t}(x,0) = 0, \quad x \in [0,1], \quad \left| \frac{\partial^4 w_1}{\partial x^3 \partial t}(x,t) \right| \leq C \varepsilon_1^{-3/2}, \quad t = 0,1, \]
\[ |g(x,0)| \leq C (\varepsilon_1^{-1/2} (\varepsilon_1^{-1/2} + \varepsilon_2^{-1}) B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2} B_{\varepsilon_2}(x)), \]
the maximum principle proves the result. \( \Box \)

**3. The time semidiscretization: the Crank-Nicolson scheme.** In \([0,T]\) we consider a uniform mesh \( \bar{\Omega} = \left\{ k \Delta t, \quad 0 \leq k \leq M, \quad \Delta t = T/M \right\} \). On this mesh the Crank–Nicolson scheme is given by
\[ \bar{u}^0 = \bar{u}(x,0) = \bar{0}, \]
\[ \left\{ \begin{array}{l}
(I + \frac{\Delta t}{2} L_{x,\varepsilon}) \bar{u}^{n+1} = \frac{\Delta t}{2} (\bar{f}^n + \bar{f}^{n+1}) + \left( I - \frac{\Delta t}{2} L_{x,\varepsilon} \right) \bar{u}^n, \quad n = 0,1,\ldots,M-1, \\
\bar{u}^{n+1}(0) = 0, \quad \bar{u}^{n+1}(1) = 0, \end{array} \right. \]
(3.1)
where \( \bar{f}^n = \bar{f}(x,t_n), \quad n = 0,1,\ldots,M-1 \). To study the local error of this method, we consider the following auxiliary problem
\[ \left\{ \begin{array}{l}
(I + \frac{\Delta t}{2} L_{x,\varepsilon}) \hat{u}^{n+1} = \frac{\Delta t}{2} (\hat{f}^n + \hat{f}^{n+1}) + \left( I - \frac{\Delta t}{2} L_{x,\varepsilon} \right) \hat{u}(x,t_n), \\
\hat{u}^{n+1}(0) = 0, \quad \hat{u}^{n+1}(1) = 0. \end{array} \right. \]
(3.2)

**Lemma 3.1.** The local error associated to (3.1), defined as \( \bar{e}^{n+1}(x) = \bar{u}(x,t_{n+1}) - \hat{u}^{n+1}(x) \), satisfies
\[ |\bar{e}^{n+1}(x)| \leq C (\Delta t)^3, \quad x \in \bar{\Omega}. \]

**Proof.** Following Clavero et al. (2005a) we have
\[ \frac{\bar{u}(x,t_{n+1}) - \bar{u}(x,t_n)}{\Delta t} = \bar{u}_t(x,t_n + \Delta t/2) + \mathcal{O}((\Delta t)^2) \bar{1} = -L_{x,\varepsilon} \bar{u}(x,t_n + \Delta t/2) + \bar{f}(x,t_n + \Delta t/2) + \mathcal{O}((\Delta t)^2) \bar{1}. \]
Differentiating (1.1) twice w.r.t. \( t \), we obtain \( |L_{x,\varepsilon} \bar{u}_{tt}| = |\bar{f}_{tt} - \bar{u}_{tt}| \leq \bar{C} \) and therefore
\[ L_{x,\varepsilon} \bar{u}(x,t_n + \Delta t/2) = L_{x,\varepsilon} \frac{\bar{u}(x,t_{n+1}) + \bar{u}(x,t_n)}{2} + \mathcal{O}((\Delta t)^2) \bar{1}. \]
(3.3)
Also, it holds
\[ \bar{f}(x,t_n + \Delta t/2) = \frac{\bar{f}(x,t_{n+1}) + \bar{f}(x,t_n)}{2} + \mathcal{O}((\Delta t)^2) \bar{1}. \]
(3.4)
From (3.3) and (3.4), it is straightforward to show that the local error is the solution of the problem
\[ \left( I + \frac{\Delta t}{2} L_{x,\varepsilon} \right) \bar{e}^{n+1} = \mathcal{O}((\Delta t)^3) \bar{1}, \quad \bar{e}^{n+1}(0) = \bar{e}^{n+1}(1) = 0. \]
(3.5)
Similarly to [11], it is straightforward to prove that the differential operator \( (I + (\Delta t/2) L_{x,\varepsilon}) \) satisfies a maximum principle; using this maximum principle in problem (3.5) the result follows. \( \Box \)
For the posterior analysis of the spatial discretization we need a more precise information about the asymptotic behaviour of the exact solution of the semidiscrete problems \((3.2)\) and their derivatives with respect to the variable \(x\). For that, we decompose \(\hat{\vec{w}}^{n+1} = \hat{\vec{v}}^{n+1} + \hat{\vec{w}}^{n+1}\), which are respectively the solution of problems

\[
\begin{align*}
\left\{ \begin{array}{l}
(I + \frac{\Delta t}{2} L_{x,\bar{\varepsilon}}) \hat{\vec{v}}^{n+1}(x) &= \frac{\Delta t}{2} (\bar{\vec{f}}^n + \bar{\vec{f}}^{n+1}) + \left(I - \frac{\Delta t}{2} L_{x,\bar{\varepsilon}}\right) \bar{\vec{v}}(x, t_n), \; x \in (0, 1), \\
(I + \frac{\Delta t}{2} A) \hat{\vec{w}}^{n+1}(x) &= \frac{\Delta t}{2} (\bar{\vec{f}}^n + \bar{\vec{f}}^{n+1}) + \left(I - \frac{\Delta t}{2} A\right) \bar{\vec{v}}(x, t_n), \; x = 0, 1,
\end{array} \right.
\end{align*}
\]

(3.6)

and

\[
\begin{align*}
\left\{ \begin{array}{l}
(I + \frac{\Delta t}{2} L_{x,\bar{\varepsilon}}) \hat{\vec{w}}^{n+1} &= \left(I - \frac{\Delta t}{2} L_{x,\bar{\varepsilon}}\right) \bar{\vec{v}}(x, t_n), \; x \in (0, 1), \\
\hat{\vec{w}}^{n+1}(0) &= \hat{\vec{w}}^{n+1}(0) - \hat{\vec{w}}^{n+1}(0), \; \hat{\vec{w}}^{n+1}(1) = \hat{\vec{w}}^{n+1}(1) - \hat{\vec{w}}^{n+1}(1),
\end{array} \right.
\end{align*}
\]

(3.7)

where \(\bar{\vec{v}}\) is the regular component, solution of \((2.1)-(2.2)\), and \(\bar{\vec{w}}\) is the singular component, solution of \((2.3)\). The rest of this section gives some technical results related with the behaviour, with respect to the diffusion parameters, of the derivatives of the regular, \(\hat{\vec{v}}^{n+1}\), and the singular, \(\hat{\vec{w}}^{n+1}\), components.

**Remark 1.** Note that from \(\bar{\vec{v}}_{xx}(x, t) = 0\), \(x = 0, 1\), trivially it follows \(\hat{\vec{v}}_{xx}(x) = 0\), \(x = 0, 1\).

**Remark 2.** In addition, \(\hat{\vec{v}}^{n+1}(x), \; x = 0, 1\), are the approximations given by the Crank-Nicolson method to solve the initial value problem

\[
\bar{\vec{v}}_t + A\bar{\vec{v}} = \bar{\vec{f}}, \quad (x, t) \in \{0, 1\} \times (t_n, t_{n+1}], \quad \bar{\vec{v}}(x, t_n), \; x \in \{0, 1\} \text{ known.}
\]

(3.8)

Then, it follows that

\[
|\bar{\vec{v}}(x, t_{n+1}) - \hat{\vec{v}}^{n+1}(x)| \leq C(\Delta t)^3, \quad x \in \{0, 1\}.
\]

(3.9)

Therefore, from (3.9), Lemma 3.1 and the triangular inequality we have

\[
|\bar{\vec{w}}(x, t_{n+1}) - \hat{\vec{w}}^{n+1}(x)| \leq C(\Delta t)^3, \quad x \in \{0, 1\}.
\]

**Lemma 3.2.** The local errors associated to the regular, \(\hat{\vec{v}}^{n+1}\), and the singular, \(\hat{\vec{w}}^{n+1}\), components satisfy

\[
|\bar{\vec{v}}(x, t_{n+1}) - \hat{\vec{v}}^{n+1}(x)| \leq C(\Delta t)^3, \quad |\bar{\vec{w}}(x, t_{n+1}) - \hat{\vec{w}}^{n+1}(x)| \leq C(\Delta t)^3 \mathcal{B}_{\varepsilon_2}(x), \quad x \in \bar{\Omega}.
\]

**Proof.** Taking into account Remark 2, the proof is completely analogous to this one of Lemma 3.1. □

To find precise bounds of the derivatives of \(\hat{\vec{v}}^{n+1}\), we need the following technical result.

**Lemma 3.3.** It holds that

\[
\left| \frac{\bar{\vec{v}}_{xx}(x, t_{n+1}) - \hat{\vec{v}}^{n+1}_{xx}(x)}{\Delta t} \right| \leq C.
\]
Proof. First, using that \( \|\tilde{v}_{xxt}\|_Q \leq C \), we have

\[
\frac{\tilde{v}_{xx} (x, t_n+1) - \tilde{v}_{xx} (x, t_n)}{\Delta t} = \tilde{v}_{xxt} (x, t_n + \Delta t/2) + \mathcal{O}(\Delta t) \tilde{t} = -L_{x, x} \tilde{v}_{xx} (x, t_n + \Delta t/2) + f_{xx} (x, t_n + \Delta t/2) - 2A_x \tilde{v}_x (x, t_n + \Delta t/2) - A_{xx} \tilde{v} (x, t_n + \Delta t/2) + \mathcal{O}(\Delta t) \tilde{t}.
\]

(3.10)

In second place, differentiating the equation (2.1) twice w.r.t. \( x \) and once w.r.t. \( t \), we obtain

\[
|L_{x, x} \tilde{v}_{xx} (x, t)| = |\tilde{f}_{xx} - \tilde{v}_{ttxx} - 2A_x \tilde{v}_t| \leq \tilde{C},
\]

and therefore we have

\[
(3.11) \quad L_{x, x} \tilde{v}_{xx} (x, t_n + \Delta t/2) = L_{x, x} \tilde{v}_{xx} (x, t_n + 1) + \tilde{v}_{xx} (x, t_n) + \mathcal{O}(\Delta t) \tilde{t}.
\]

Then, defining the problem

\[
(I + (\Delta t/2) L_{x, x}) (\tilde{v}_{xx}, t_n, t_n+1) - \tilde{v}_{xx} (x) = \tilde{g} (x), \quad \tilde{v}_{xx} (x, t_n+1) - \tilde{v}_{xx} (x) = 0, \quad x = 0, 1,
\]

from (3.10) and (3.11) the right hand side can be bounded by

\[
|\tilde{g} (x)| = |\Delta t (f_{xx} + \tilde{f}_{xx} + \Delta t/2 (x) + \frac{\Delta t}{2} 2A_x (\tilde{v}_{xx} (x) + \tilde{v}_x (x, t_n) - 2\tilde{v}_x (x, t_n + \Delta t/2)) + \mathcal{O}(\Delta t) \tilde{t}) = \mathcal{O}(\Delta t) \tilde{t}.
\]

Then, using the maximum principle for \((I + (\Delta t/2) L_{x, x}) \) the result follows. \( \square \)

PROPOSITION 3.4. The regular component \( \tilde{v}_{n+1} = (\tilde{v}_{1n+1}, \tilde{v}_{2n+1})^T \) satisfies

\[
(3.12) \quad \left\| \frac{d^k \tilde{v}_{n+1}}{dx^k} \right\|_{\tilde{t}_0} \leq C, \quad 0 \leq k \leq 2, \quad \left\| \frac{d^k \tilde{v}_{n+1}}{dx^k} \right\|_{\tilde{t}_0} \leq C \epsilon_i^{1-k/2}, \quad 3 \leq k \leq 4, \quad i = 1, 2.
\]

Proof. Using that \( |\partial^k v_i / \partial x^k| \leq C, \quad 0 \leq k \leq 2, \quad |\partial^k v_i / \partial x^k| \leq C \epsilon_i^{1-k/2}, \quad 3 \leq k \leq 4, \quad i = 1, 2, \) and that \( \tilde{v}_{xx} (x) = 0, \quad x = 0, 1 \) (see Remark 1), we can reproduce the proof given in [11] for the regular component, obtaining

\[
\left\| \frac{d^k \tilde{v}_{n+1}}{dx^k} \right\|_{\tilde{t}_0} \leq C, \quad 0 \leq k \leq 2.
\]

For higher derivatives, first we differentiate (3.6) twice w.r.t. \( x \), obtaining

\[
\left\{ \begin{array}{l}
L_{x, x} \tilde{v}_{xx} (x) = (f^+_{xx} + f^-_{xx}) + \frac{2}{\Delta t} (\tilde{v}_{xx} (x, t_n) - \tilde{v}_{xx} (x)) - 2A_x \tilde{v}_x (x, t_n) + A_{xx} (\tilde{v}_{xx} (x) + \tilde{v}_x (x, t_n)), \quad x \in (0, 1), \\
\tilde{v}_{xx} (0) = \tilde{v}_{xx} (1) = \tilde{0}.
\end{array} \right.
\]

(3.13)

Using Lemmas 2.4 and 3.3 we know that the right hand side of problem (3.13) is parameter uniform bounded and therefore the result follows using a similar argument to this one of [11]. \( \square \)
Similarly to the regular component, before finding appropriated bounds for the derivatives of \( \tilde{w}^{n+1} \), we prove the following technical result.

**Lemma 3.5.** It holds that

\[
\begin{align*}
\left| \frac{\tilde{w}_x^{n+1}(x) - \tilde{w}_x(x, t_n)}{\Delta t} \right| &\le CB_{\varepsilon_2}(x) \left( \frac{\varepsilon_1^{-1/2}}{\varepsilon_2^{1/2}} \right), \\
\left| \frac{\tilde{w}_{xx}^{n+1}(x) - \tilde{w}_{xx}(x, t_n)}{\Delta t} \right| &\le CB_{\varepsilon_2}(x) \left( \frac{\varepsilon_1^{-1}}{\varepsilon_2^{-1}} \right).
\end{align*}
\]

*Proof.* We consider the function

\[
(3.14) \quad \varphi_1^{n+1}(x) = \frac{\tilde{w}^{n+1}(x) - \tilde{w}(x, t_n)}{\Delta t}.
\]

First, from Lemmas 2.5 and 3.2 we have

\[
(3.15) |\varphi_1^{n+1}(x)| = \left| \frac{\tilde{w}^{n+1}(x) - \tilde{w}(x, t_{n+1})}{\Delta t} + \frac{\tilde{w}(x, t_{n+1}) - \tilde{w}(x, t_n)}{\Delta t} \right| \le B_{\varepsilon_2}(x)C^\varepsilon.
\]

The function \( \varphi_1^{n+1}(x) \) is the solution of the following boundary value problem:

\[
L_{x, \varepsilon} \varphi_1^{n+1}(x) = \varphi_2^{n+1}(x), \quad \varphi_1^{n+1}(0), \varphi_1^{n+1}(1) \quad \text{given},
\]

where \( \varphi_2^{n+1}(x) \) is adequately chosen. Then, from (3.7) we can deduce that

\[
|\tilde{w}_x^{n+1}(x)| = |L_{x, \varepsilon} \tilde{w}(x, t_n)| = |L_{x, \varepsilon} \tilde{w}_t(x, t_n)| = |\tilde{w}_tt(x, t_n)| \le B_{\varepsilon_2}(x)C^\varepsilon;
\]

for the interior mesh points. On the boundary, using (3.7), (3.16), a continuity argument and Remark 2, we have

\[
\varphi_2^{n+1}(x) = -\frac{2}{\Delta t} (\varphi_1^{n+1}(x) + L_{x, \varepsilon} \tilde{w}(x, t_n)) = \frac{2}{\Delta t} \left( \frac{\tilde{w}(x, t_n) - \tilde{w}_t^{n+1}(x)}{\Delta t} + \tilde{w}_t(x, t_n) \right) = \frac{2}{\Delta t} \left( \left| \frac{\tilde{w}(x, t_n) - \tilde{w}_t(x, t_n+1)}{\Delta t} \right| + \tilde{w}_t(x, t_n) \right) = O(\Delta t)\tilde{I} + \frac{2}{\Delta t} \left( \frac{\tilde{w}(x, t_n) - \tilde{w}(x, t_{n+1})}{\Delta t} \right), \quad x = 0, 1.
\]

Hence \( |\varphi_2^{n+1}(x)| \le \tilde{C} \) for \( x = 0, 1 \) and from the maximum principle for \( (I + \frac{\Delta t}{2} L_{x, \varepsilon}) \)

it follows that \( |\varphi_2^{n+1}(x)| \le CB_{\varepsilon_2}(x), \quad x \in \tilde{\Omega} \). Using this bound in (3.16) and (3.15),

we deduce that

\[
(3.17) \quad |d^2 \varphi_1^{n+1}/dx^2| \le C\varepsilon_2^{-1}B_{\varepsilon_2}(x), \quad |d^2 \varphi_1^{n+1}/dx^2| \le C\varepsilon_2^{-1}B_{\varepsilon_2}(x), \quad x \in \tilde{\Omega},
\]

where we denote \( \varphi_1^{n+1}(x) = ([\varphi_1^{n+1}(x)]_1, [\varphi_1^{n+1}(x)]_2)^T \). From (3.17) and using the mean value theorem (see [11]), we can obtain

\[
(3.18) \quad |d\varphi_1^{n+1}/dx|_1 \le C\varepsilon_2^{-1/2}B_{\varepsilon_2}(x), \quad |d\varphi_1^{n+1}/dx|_2 \le C\varepsilon_2^{-1/2}B_{\varepsilon_2}(x), \quad x \in \tilde{\Omega}.
\]

\[\square\]
Proposition 3.6. The singular component $\tilde{\mathbf{w}}^{n+1} = (\tilde{w}_1^{n+1}, \tilde{w}_2^{n+1})^T$ satisfies

\begin{equation}
|\tilde{w}_1^{n+1}(x)| \leq CB_\varepsilon(x), \quad |\tilde{w}_2^{n+1}(x)| \leq CB_\varepsilon(x),
\end{equation}

\begin{equation}
\left| \frac{d\tilde{w}_1^{n+1}}{dx} \right| \leq C(\varepsilon^{-1/2}B_\varepsilon(x) + \varepsilon^{-1/2}B_\varepsilon(x)), \quad \left| \frac{d\tilde{w}_2^{n+1}}{dx} \right| \leq C\varepsilon^{-1/2}B_\varepsilon(x),
\end{equation}

\begin{equation}
\left| \frac{d^2\tilde{w}_1^{n+1}}{dx^2} \right| \leq C(\varepsilon^{-1}B_\varepsilon(x) + \varepsilon^{-1}B_\varepsilon(x)), \quad \left| \frac{d^2\tilde{w}_2^{n+1}}{dx^2} \right| \leq C\varepsilon^{-1}B_\varepsilon(x).
\end{equation}

Proof. Note that Lemma 2.5 proves

$$|(I + \frac{\Delta t}{2} L_{x,\varepsilon}\tilde{\mathbf{w}})\tilde{\mathbf{w}}^{n+1}(x)| = |(I - \frac{\Delta t}{2} L_{x,\varepsilon})\tilde{w}(x, t_n)| \leq B_\varepsilon(x)\tilde{C}.$$}

Defining the barrier function $\tilde{\psi} = B_\varepsilon(x)\tilde{C}$, it holds

$$(I + \frac{\Delta t}{2} L_{x,\varepsilon})\tilde{\psi} = \tilde{\psi} + CB_\varepsilon(x)\frac{\Delta t}{2}(-\alpha/\varepsilon_2)\varepsilon + A\tilde{C} \geq \tilde{\psi},$$

and using again the maximum principle for $(I + \frac{\Delta t}{2} L_{x,\varepsilon})$, we obtain $|\tilde{\mathbf{w}}^{n+1}(x)| \leq \tilde{\psi}(x)$.

To prove estimates for higher order derivatives, we define the following auxiliary problem

$$\begin{cases}
(I + \frac{\Delta t}{2} L_{x,\varepsilon})\phi_1^{n+1} = -L_{x,\varepsilon}\tilde{w}(x, t_n), \\
\phi_1(0) = \frac{\tilde{w}^{n+1}(0) - \tilde{w}(0, t_n)}{\Delta t}, \quad \phi_1(1) = \frac{\tilde{w}^{n+1}(1) - \tilde{w}(1, t_n)}{\Delta t},
\end{cases}$$

whose solution is given in (3.14). Using that $|L_{x,\varepsilon}\tilde{w}(x, t_n)| = |\tilde{w}_t(x, t_n)| \leq B_\varepsilon\tilde{C}$ and $|\phi_1^{n+1}(0)| \leq \tilde{C}$, $|\phi_1^{n+1}(1)| \leq \tilde{C}$, again the maximum principle for $(I + \frac{\Delta t}{2} L_{x,\varepsilon})$ proves that $|\tilde{\mathbf{w}}^{n+1}(x)| \leq B_\varepsilon(x)\tilde{C}$.

Next, we write the problem (3.7)-(3.7) as follows:

$$L_{x,\varepsilon}\tilde{w}^{n+1}(x) = -2\phi_1^{n+1}(x) - L_{x,\varepsilon}\tilde{w}(x, t_n), \quad \tilde{w}^{n+1}(0), \quad \tilde{w}^{n+1}(1) \text{ given.}$$

From $|\phi_1^{n+1}(x)| \leq B_\varepsilon(x)\tilde{C}$, $|L_{x,\varepsilon}\tilde{w}(x, t_n)| = |\tilde{w}_t(x, t_n)| \leq B_\varepsilon(x)\tilde{C}$ and $|\tilde{\mathbf{w}}^{n+1}(x)| \leq B_\varepsilon(x)\tilde{C}$, we obtain that

\begin{equation}
\left| \frac{d^2\tilde{w}_1^{n+1}(x)}{dx^2} \right| \leq C\varepsilon_1^{-1}B_\varepsilon(x), \quad \left| \frac{d^2\tilde{w}_2^{n+1}(x)}{dx^2} \right| \leq C\varepsilon_2^{-1}B_\varepsilon(x).
\end{equation}

Again following the ideas of [11] it is possible to prove that

\begin{equation}
\left| \frac{d\tilde{w}_1^{n+1}(x)}{dx} \right| \leq C\varepsilon_1^{-1/2}B_\varepsilon(x), \quad \left| \frac{d\tilde{w}_2^{n+1}(x)}{dx} \right| \leq C\varepsilon_2^{-1/2}B_\varepsilon(x).
\end{equation}

From (3.22) and (3.23) it follows the required result for the first and the second derivative of $w_2$. Nevertheless, for the component $w_1$ we need to improve the obtained
bounds. Then, differentiating the first equation of (3.7) w.r.t. $x$, we obtain the problem

\[
\begin{align*}
\left\{ \begin{array}{l}
(I + \frac{\Delta t}{2} L_{x,\varepsilon_{1}})\varphi_{3}^{n+1}(x) &= -L_{x,\varepsilon_{1}} \frac{d\varphi_{3}^{n}}{dx}(x,t_{n}) - \frac{1}{2} \frac{d^{2}}{dx^{2}} \left[a_{12}(x)(w_{2}(x,t_{n}) + \hat{w}_{2}^{n+1}(x))\right] - \\
- \frac{1}{2} a_{11}'(x)(w_{1}(x,t_{n}) + \hat{w}_{1}^{n+1}(x)),
\end{array} \right.
\varphi_{3}^{n+1}(0) = -\frac{d\varphi_{3}^{n}(0) - d\varphi_{3}^{n}(0,t_{n})}{\Delta t}, \quad \varphi_{3}^{n+1}(1) = -\frac{d\varphi_{3}^{n+1}(1) - d\varphi_{3}^{n+1}(1,t_{n})}{\Delta t},
\end{align*}
\]

where $L_{x,\varepsilon_{1}} z \equiv -\varepsilon_{1} z'' + a_{11} z$, whose solution is

\[
(3.24) \quad \varphi_{3}^{n+1}(x) = \frac{d\varphi_{3}^{n+1}(x) - d\varphi_{3}^{n+1}(x,t_{n})}{\Delta t}.
\]

From Lemma 2.5, the bounds given in Lemma 3.5 applied on the boundary, the estimates (3.19)-(3.21) and the maximum principle, we obtain $|\varphi_{3}^{n+1}(x)| \leq C(\varepsilon_{1}^{-1/2} B_{\varepsilon_{1}}(x) + \varepsilon_{2}^{-1/2} B_{\varepsilon_{2}}(x))$. Now we define the problem

\[
\begin{align*}
\left\{ \begin{array}{l}
L_{x,\varepsilon_{1}} \hat{w}_{1}^{n+1}(x) = h(x) = -2\varphi_{3}^{n+1}(x) - L_{x,\varepsilon_{1}} \frac{d\varphi_{3}^{n}}{dx}(x,t_{n}) - \frac{d}{dx} \left[a_{12}(x)(w_{2}(x,t_{n}) + \hat{w}_{2}^{n+1}(x))\right] \\
-a_{11}'(x)(w_{1}(x,t_{n}) + \hat{w}_{1}^{n+1}(x)),
\end{array} \right.
\frac{d\hat{w}_{1}^{n+1}(0)}{dx}, \quad \frac{\hat{w}_{1}^{n+1}(1)}{dx} \quad \text{given}.
\end{align*}
\]

It is straightforward to prove that the right hand side $|h(x)| \leq C(\varepsilon_{1}^{-1/2} B_{\varepsilon_{1}}(x) + \varepsilon_{2}^{-1/2} B_{\varepsilon_{2}}(x))$, which together with the crude bounds (3.23) for the boundary conditions, permit us to deduce

\[
\left| \frac{d\varphi_{3}^{n+1}}{dx}(x,t_{n}) \right| \leq C(\varepsilon_{1}^{-1/2} B_{\varepsilon_{1}}(x) + \varepsilon_{2}^{-1/2} B_{\varepsilon_{2}}(x)).
\]

We use similar ideas for the second derivative. First, we define the auxiliary problem

\[
\begin{align*}
\left\{ \begin{array}{l}
(I + \frac{\Delta t}{2} L_{x,\varepsilon_{1}})\varphi_{4}^{n+1}(x) &= -L_{x,\varepsilon_{1}} \frac{d^{2}\varphi_{4}^{n}}{dx^{2}}(x,t_{n}) - \frac{1}{2} \frac{d^{2}}{dx^{2}} \left[a_{12}(x)(w_{2}(x,t_{n}) + \hat{w}_{2}^{n+1}(x))\right] - \\
- \frac{1}{2} a_{11}'(x)(w_{1}(x,t_{n}) + \hat{w}_{1}^{n+1}(x)) - a_{11}'(x) \frac{d\varphi_{3}^{n}}{dx}(x,t_{n}) + \frac{d\hat{w}_{1}^{n+1}}{dx}(x,t_{n}),
\end{array} \right.
\varphi_{4}^{n+1}(0) = -\frac{d\varphi_{4}^{n}(0) - d\varphi_{4}^{n}(0,t_{n})}{\Delta t}, \quad \varphi_{4}^{n+1}(1) = -\frac{d\varphi_{4}^{n+1}(1) - d\varphi_{4}^{n+1}(1,t_{n})}{\Delta t},
\end{align*}
\]

whose solution is

\[
\varphi_{4}^{n+1}(x) = \frac{d\varphi_{4}^{n+1}(x) - d\varphi_{4}^{n+1}(x,t_{n})}{\Delta t}.
\]

We can prove that $|\varphi_{4}^{n+1}(x)| \leq C(\varepsilon_{1}^{-1} B_{\varepsilon_{1}}(x) + \varepsilon_{2}^{-1} B_{\varepsilon_{2}}(x))$. Next, we define the boundary value problem

\[
\begin{align*}
L_{x,\varepsilon_{1}} \frac{d^{2}\varphi_{4}^{n+1}}{dx^{2}}(x) &= -2\varphi_{4}^{n+1}(x) - L_{x,\varepsilon_{1}} \frac{d^{2}\varphi_{4}^{n}}{dx^{2}}(x,t_{n}) - \frac{d^{2}}{dx^{2}} \left[a_{12}(x)(w_{2}(x,t_{n}) + \hat{w}_{2}^{n+1}(x))\right] - \\
-a_{11}''(x)(w_{1}(x,t_{n}) + \hat{w}_{1}^{n+1}(x)) - 2a_{11}'(x) \frac{d\varphi_{3}^{n}}{dx}(x,t_{n}) + \hat{w}_{1}^{n+1}(x),
\end{align*}
\]

(3.25) \quad \text{given,}
\]
and then we obtain \( \left| \frac{d^2 \hat{w}_1^{n+1}}{dx^2}(x) \right| \leq C(\varepsilon_1^{-1}B_{\varepsilon_1}(x) + \varepsilon_2^{-1}B_{\varepsilon_2}(x)) \). \( \square \)

**Proposition 3.7.** The singular component \( \hat{w}_2^{n+1} = (\hat{w}_1^{n+1}, \hat{w}_2^{n+1})^T \) satisfies

\[
\begin{align*}
\left| \frac{d^3 \hat{w}_1^{n+1}}{dx^3} \right| &\leq C(\varepsilon_1^{-3/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2}B_{\varepsilon_2}(x)), \\
\left| \frac{d^3 \hat{w}_2^{n+1}}{dx^3} \right| &\leq C\varepsilon_2^{-1}(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2}B_{\varepsilon_2}(x)), \\
\left| \frac{d^4 \hat{w}_1^{n+1}}{dx^4} \right| &\leq C\varepsilon_1^{-1}(\varepsilon_1^{-1}B_{\varepsilon_1}(x) + \varepsilon_2^{-1}B_{\varepsilon_2}(x)), \\
\left| \frac{d^4 \hat{w}_2^{n+1}}{dx^4} \right| &\leq C\varepsilon_2^{-1}(\varepsilon_1^{-1}B_{\varepsilon_1}(x) + \varepsilon_2^{-1}B_{\varepsilon_2}(x)).
\end{align*}
\]

*Proof.* From (3.25) it directly follows

\[
\left| \frac{d^3 \hat{w}_1^{n+1}}{dx^3} \right| \leq C\varepsilon_1^{-1}(\varepsilon_1^{-1}B_{\varepsilon_1}(x) + \varepsilon_2^{-1}B_{\varepsilon_2}(x)), \quad \left| \frac{d^4 \hat{w}_2^{n+1}}{dx^4} \right| \leq C\varepsilon_2^{-1}(\varepsilon_1^{-1}B_{\varepsilon_1}(x) + \varepsilon_2^{-1}B_{\varepsilon_2}(x)).
\]

Using the argument given in [11], we deduce (3.27) and

\[
\left| \frac{d^3 \hat{w}_2^{n+1}}{dx^3} \right| \leq C\varepsilon_2^{-1}(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2}B_{\varepsilon_2}(x)),
\]

which is not the required bound (3.27). It will be obtained by setting the problem

\[
\begin{align*}
L^1_{x,\varepsilon_1} \varphi_3^{n+1}(x) = p(x) &\quad x \in (0, 1), \\
\varphi_3^{n+1}(0), \varphi_3^{n+1}(1) &\quad \text{given},
\end{align*}
\]

which will be used to deduce bounds for \( d^2 \varphi_3^{n+1}(x)/dx^2 \) where function \( \varphi_3^{n+1}(x) \) has been defined in (3.24). First, we obtain appropriate bounds for \( p \) using the maximum principle. With this purpose, we note that

\[
(I + \frac{\Delta t}{2} L^1_{x,\varepsilon_1} p) = L^1_{x,\varepsilon_1} (I + \frac{\Delta t}{2} L^1_{x,\varepsilon_1}) \varphi_3^{n+1}(x) = \]

\[
= L^1_{x,\varepsilon_1} \left( \frac{\partial w_1}{\partial x \partial t} + \frac{1}{2} \left( a'_{11}(w_1 - \hat{w}_1^{n+1}) + \frac{\partial}{\partial x} (a_{12}(w_2 - \hat{w}_2^{n+1})) \right) \right).
\]

All functions appearing in the last term have been already bounded and therefore we can easily conclude

\[
| (I + \frac{\Delta t}{2} L^1_{x,\varepsilon_1}) p | \leq C \varepsilon_1^{-1/2}.
\]

Using a continuity argument, the function \( p \) satisfies on the boundary

\[
p(x) = L^1_{x,\varepsilon_1} \varphi_3^{n+1}(x) = \frac{2}{\Delta t} \left( \frac{\partial^2 w_1}{\partial x \partial t} + \frac{\partial}{\partial x} \left( \frac{\partial w_1}{\partial x}(x, t_n) - \frac{\partial}{\partial x}(x, t_n+1) \right) \right)
\]

\[
+ a'_{12} \frac{w_2(x, t_n) - \hat{w}_2^{n+1}(x)}{\Delta t} + a_{12} \frac{\partial w_2}{\partial x}(x, t_n) - \frac{d\hat{w}_2^{n+1}}{dx}(x) \right) \frac{\partial}{\partial x} \right) + a'_{11} \frac{w_1(x, t_n) - \hat{w}_1^{n+1}(x)}{\Delta t}.
\]
Lemmas 2.6, 3.2 and 3.5 prove $|p(x)| \leq C\varepsilon_1^{-1/2}$, $x = 0, 1$. Then, the maximum principle proves

$$|p(x)| \leq C\varepsilon_1^{-1/2}, \quad x \in [0, 1],$$

and therefore, from problem (3.31), we obtain

$$\left| \frac{\partial^2 \varphi_3^{n+1}}{\partial x^2} \right| \leq C\varepsilon_1^{-3/2}.$$

Differentiating equations (2.3) and (3.7) three times w.r.t. $x$, we have the following problem

$$\begin{cases}
L_{x, \varepsilon_1}^1 \frac{\partial^3 w_1}{\partial x^3} (\hat{w}_1^{n+1} - w_1) = -L_{x, \varepsilon_1}^1 \frac{d^3 w_1}{dx^3} - \frac{1}{2} \frac{\partial^3}{\partial x^3} (a_{12}(\hat{w}_2^{n+1} + w_2)) \\
- \frac{1}{2} \left( a_{11}''(\hat{w}_1^{n+1} + w_1) + 3a_{11}' \frac{\partial}{\partial x} (\hat{w}_1^{n+1} + w_1) + 3a_{11}' \frac{\partial^2}{\partial x^2} (\hat{w}_1^{n+1} + w_1) \right), \quad x \in (0, 1),
\end{cases}$$

Taking into account that

$$\left| L_{x, \varepsilon_1}^1 \frac{\partial^3 w_1}{\partial x^3} \right| = \left| \frac{\partial^4 w_1}{\partial x^4} + a_{11}'' w_1 + 3a_{11}' \frac{\partial w_1}{\partial x} + 3a_{11}' \frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^3}{\partial x^3} (a_{12} w_2) \right| \leq C(\varepsilon_1^{-3/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2} B_{\varepsilon_2}(x)),$$

the maximum principle proves

$$\left| \frac{\partial^3}{\partial x^3} \left( \frac{\hat{w}_1^{n+1} - w_1}{\Delta t} \right) \right| \leq C(\varepsilon_1^{-3/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2} B_{\varepsilon_2}(x)), \quad x \in [0, 1].$$

Finally, we consider the problem

$$\begin{cases}
L_{x, \varepsilon_1}^1 \frac{d^3 \hat{w}_1}{dx^3} = -2 \frac{\partial^3}{\partial x^3} \left( \frac{\hat{w}_1^{n+1} - w_1}{\Delta t} \right) - L_{x, \varepsilon_1}^1 \frac{\partial^3 w_1}{\partial x^3} - \frac{\partial^3}{\partial x^3} (a_{12}(\hat{w}_2^{n+1} + w_2)) \\
- \left( a_{11}''(\hat{w}_1^{n+1} + w_1) + 3a_{11}' \frac{\partial}{\partial x} (\hat{w}_1^{n+1} + w_1) + 3a_{11}' \frac{\partial^2}{\partial x^2} (\hat{w}_1^{n+1} + w_1) \right), \quad x \in (0, 1),
\end{cases}$$

Using in the boundary the crude bounds (3.30), from the maximum principle it follows

$$\left| \frac{d^3 \hat{w}_1}{dx^3} \right| \leq C(\varepsilon_1^{-3/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2} B_{\varepsilon_2}(x)), \quad x \in [0, 1].$$

In some cases, we will need a further decomposition of the singular part and bounds for their derivatives.

**Proposition 3.8.** Suppose that $\varepsilon_1 < \varepsilon_2$. Then, the singular component can be decomposed as

$$\hat{w}_1^{n+1} = \hat{w}_{1, \varepsilon_1}^{n+1} + \hat{w}_{1, \varepsilon_2}^{n+1}, \quad \hat{w}_2^{n+1} = \hat{w}_{2, \varepsilon_1}^{n+1} + \hat{w}_{2, \varepsilon_2}^{n+1},$$
From Proposition 3.6 we know that there are two overlapping boundary layers at 
and also it can be decomposed as

\[ \hat{u}_1^{n+1} = \hat{z}_1^{n+1} + \hat{z}_2^{n+1}, \quad \hat{u}_2^{n+1} = \hat{z}_2^{n+1} + \hat{z}_2^{n+1}, \]

\[ \frac{d^2 \hat{w}_{1,\varepsilon_1}}{d x^2} \leq C \varepsilon_1^{-1} B_{\varepsilon_1}(x), \quad \frac{d^3 \hat{w}_{1,\varepsilon_2}}{d x^3} \leq C \varepsilon_2^{-3/2} B_{\varepsilon_2}(x), \]
\[ \frac{d^2 \hat{w}_{2,\varepsilon_1}}{d x^2} \leq C \varepsilon_2^{-1} B_{\varepsilon_1}(x), \quad \frac{d^3 \hat{w}_{2,\varepsilon_2}}{d x^3} \leq C \varepsilon_2^{-3/2} B_{\varepsilon_2}(x), \]

Proof. It is similar to the proof given in [10, 11]. In this case we consider the functions

\[ \hat{w}_{i,\varepsilon_2}^{n+1} = \begin{cases} 
\sum_{k=0}^{3} \frac{(x-x^*)^k}{k!} \frac{d^k}{d x^k} \hat{w}_i^{n+1}(x^*), & x \in [0, x^*], \\
\hat{w}_i^{n+1}, & x \in [x^*, 1-x^*], \\
\sum_{k=0}^{3} \frac{(x-1+x^*)^k}{k!} \frac{d^k}{d x^k} \hat{w}_i^{n+1}(1-x^*), & x \in [1-x^*, 1], 
\end{cases} \]

\[ \hat{z}_{i,\varepsilon_2}^{n+1} = \begin{cases} 
\sum_{k=0}^{4} \frac{(x-\tilde{x})^k}{k!} \frac{d^k}{d x^k} \hat{w}_i^{n+1}(\tilde{x}), & x \in [0, \tilde{x}], \\
\hat{w}_i^{n+1}, & x \in [\tilde{x}, 1-\tilde{x}], \\
\sum_{k=0}^{4} \frac{(x-1+\tilde{x})^k}{k!} \frac{d^k}{d x^k} \hat{w}_i^{n+1}(1-\tilde{x}), & x \in [1-\tilde{x}, 1], 
\end{cases} \]

where \( x^* = x^*(\varepsilon_1, \varepsilon_2) \in (0, 1/2) \) is such that \( \varepsilon_1^{-3/2} B_{\varepsilon_1}(x^*) = \varepsilon_2^{-3/2} B_{\varepsilon_2}(x^*) \) and \( \tilde{x} = \tilde{x}(\varepsilon_1, \varepsilon_2) \in (0, 1/2) \) is such that \( \varepsilon_1^{-1} B_{\varepsilon_1}(\tilde{x}) = \varepsilon_2^{-1} B_{\varepsilon_2}(\tilde{x}) \) and \( \hat{w}_i^{n+1} = \hat{w}_i^{n+1} - \hat{w}_i^{n+1}, \)
\[ \frac{\tau_{\varepsilon_2}}{1/4, 2\sqrt{\varepsilon_2/\alpha \ln N}}, \quad \tau_{\varepsilon_1} = \min \left\{ \frac{\tau_{\varepsilon_2}}{2, \sqrt{\varepsilon_1/\alpha \ln N}} \right\}, \]

4. The numerical scheme. We discretize (3.1) by the central difference scheme defined on a piecewise uniform mesh \( \Omega^N \) of Shishkin type (see Farrell et al. (2000)). From Proposition 3.6 we know that there are two overlapping boundary layers at \( x = 0 \) and \( x = 1 \). Then, to define the Shishkin mesh we use two transition parameters given by

\[ \tau_{\varepsilon_2} = \min \left\{ 1/4, 2\sqrt{\varepsilon_2/\alpha \ln N} \right\}, \quad \tau_{\varepsilon_1} = \min \left\{ \tau_{\varepsilon_2}/2, 2\sqrt{\varepsilon_1/\alpha \ln N} \right\}, \]
where \( \alpha \) is given in (1.2). In the subintervals \([0, \tau_{\varepsilon_1}], [\tau_{\varepsilon_1}, \tau_{\varepsilon_2}], [\tau_{\varepsilon_2}, 1 - \tau_{\varepsilon_2}], [1 - \tau_{\varepsilon_2}, 1 - \tau_{\varepsilon_1}] \) and \([1 - \tau_{\varepsilon_1}, 1] \) we distribute uniformly \( N/8 + 1, N/8 + 1, N/2 + 1, N/8 + 1 \) and \( N/8 + 1 \) mesh points respectively. So, the mesh points are

\[
x_j = \begin{cases} 
  jh_{\varepsilon_1}, & j = 0, \ldots, N/8, \\
  x_{N/8} + (j - N/8)h_{\varepsilon_2}, & j = N/8 + 1, \ldots, N/4, \\
  x_{N/4} + (j - N/4)H, & j = N/4 + 1, \ldots, 3N/4, \\
  x_{3N/4} + (j - 3N/4)h_{\varepsilon_2}, & j = 3N/4 + 1, \ldots, 7N/8, \\
  x_{7N/8} + (j - 7N/8)h_{\varepsilon_1}, & j = 7N/8 + 1, \ldots, N,
\end{cases}
\]

where

\[
h_{\varepsilon_1} = \frac{8\tau_{\varepsilon_1}}{N}, \quad h_{\varepsilon_2} = \frac{8(\tau_{\varepsilon_2} - \tau_{\varepsilon_1})}{N}, \quad H = \frac{2(1 - 2\tau_{\varepsilon_2})}{N}.
\]

In the case \( \tau_{\varepsilon_1} \neq 1/8 \) and \( \tau_{\varepsilon_2} = 1/4 \), we modify slightly the mesh points that lie in \([\tau_{\varepsilon_2}, 1 - \tau_{\varepsilon_2}] \) in order that the resulting mesh be uniform outside of the boundary layers. Now the mesh points are

\[
x_j = \begin{cases} 
  jh_{\varepsilon_1}, & j = 0, \ldots, N/8, \\
  x_{N/8} + (j - N/8)\hat{H}, & j = N/8 + 1, \ldots, 7N/8, \\
  x_{7N/8} + (j - 7N/8)h_{\varepsilon_1}, & j = 7N/8 + 1, \ldots, N,
\end{cases}
\]

where \( \hat{H} = \frac{4(1 - 2\tau_{\varepsilon_1})}{3N} \). Below we denote the local step sizes by \( h_j = x_j - x_{j-1}, \quad j = 1, \ldots, N \).

On \( \hat{U}^N \) the scheme is given by

\[
\begin{cases} 
  \hat{U}_j^0 = \hat{U}_j, & 0 \leq j \leq N, \\
  \text{For } n = 0, \ldots, M - 1, \\
  \left( I + \frac{\Delta t}{2} L_{x,\varepsilon}^N \right) \hat{U}_{j+1}^n = \left( I - \frac{\Delta t}{2} L_{x,\varepsilon}^N \right) \hat{U}_j^n + \frac{\Delta t}{2} \left( \hat{f}_{j+1}^n + \hat{f}_j^n \right), \\
  \hat{U}_0^{n+1} = \hat{U}_N^{n+1} = 0,
\end{cases}
\]

where

\[
L_{x,\varepsilon}^N \equiv \begin{pmatrix} -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 \end{pmatrix} \delta^2 + AI, \quad \delta^2 Z_j = \frac{2}{h_j + h_{j+1}} \left( Z_j - Z_{j-1} \right),
\]

\[
L_{x,\varepsilon}^N \equiv \begin{pmatrix} -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 \end{pmatrix} \delta^2 + AI, \quad \delta^2 Z_j = \frac{2}{h_j + h_{j+1}} \left( Z_j - Z_{j-1} \right).
\]

**Lemma 4.1.** For each value of \( \Delta t \), the discrete operator \( \left( I + \frac{\Delta t}{2} L_{x,\varepsilon}^N \right) \) is uniformly stable and it satisfies a discrete maximum principle.

**Proof.** It trivially follows using that \( \left( I + \frac{\Delta t}{2} L_{x,\varepsilon}^N \right) \) is an M-matrix. \( \Box \)

To prove the uniform convergence of (4.1), we split the global error at the time \( t_{n+1} \) in the form

\[
\bar{u}(x, t_{n+1}) - \hat{U}_{j+1}^{n+1} = \left( \bar{u}(x, t_{n+1}) - \bar{u}_{j+1}^{n+1}(x_j) \right) + \left( \hat{u}_{j+1}^{n+1}(x_j) - \hat{U}_{j+1}^{n+1} \right) + \left( \hat{U}_{j+1}^{n+1} - \hat{U}_{j+1}^{n+1} \right),
\]

(4.2)

where \( \hat{U}_{j+1}^{n+1} \) is the solution of

\[
\begin{cases} 
  \left( I + \frac{\Delta t}{2} L_{x,\varepsilon}^N \right) \hat{U}_{j+1}^{n+1} = \left( I - \frac{\Delta t}{2} L_{x,\varepsilon}^N \right) \bar{u}(x, t) + \frac{\Delta t}{2} \left( \hat{f}_{j+1}^n + \hat{f}_j^n \right), 0 < j < N, \\
  \hat{U}_0^{n+1} = \hat{U}_N^{n+1} = 0.
\end{cases}
\]

(4.3)
To bound appropriately \( \left( \hat{u}^{n+1}(x_j) - \hat{U}_j^{n+1} \right) \) we consider a further decomposition

\[
\hat{U}^{n+1} = \hat{V}^{n+1} + \hat{W}^{n+1}, \quad n = 0, 1, \ldots, M - 1, \text{ where}
\]

\[
\begin{cases}
(I + \frac{\Delta t}{2} L_{x,\xi}^N) \hat{V}_j^{n+1} = \left( I - \frac{\Delta t}{2} L_{x,\xi}^N \right) \tilde{v}(x_j, t_n) + \frac{\Delta t}{2} (j^1_{j} + j^2_{j}), & 0 < j < N, \\
\hat{V}_0^{n+1} = \tilde{v}(0)^{n+1}, \quad \hat{V}_N^{n+1} = \tilde{v}(1)^{n+1}, \\
\end{cases}
\]

and

\[
\begin{cases}
(I + \frac{\Delta t}{2} L_{x,\xi}^N) \hat{W}_j^{n+1} = \left( I - \frac{\Delta t}{2} L_{x,\xi}^N \right) \tilde{w}(x_j, t_n), & 0 < j < N, \\
\hat{W}_0^{n+1} = \tilde{w}(0)^{n+1}, \quad \hat{W}_N^{n+1} = \tilde{w}(1)^{n+1}. \\
\end{cases}
\]

\[\text{Proposition 4.2. Let } \hat{u}^{n+1}(x) \text{ be the solution of (3.2) and } \{\hat{U}_j^{n+1}\} \text{ the solution of (4.3). Then, it holds}
\]

\[\| \hat{u}^{n+1}(x_j) - \hat{U}_j^{n+1} \|_{\Omega_N} \leq C(N^{-1} \ln N)^2. \]

**Proof.** First, the local error for the central differences scheme, at the mesh point \( x_j \), satisfies

\[
(I + (\Delta t/2) L_{x,\xi}^N) (\hat{u}^{n+1}(x_j) - \hat{U}_j^{n+1}) = \frac{\Delta t}{2} (L_{x,\xi}^N - L_{x,\xi}^\xi) (\hat{u}^{n+1}(x_j) + \tilde{v}(x_j, t_n)).
\]

To find an appropriated bound of \( (L_{x,\xi}^N - L_{x,\xi}^\xi) (\hat{u}^{n+1}(x_j) + \tilde{v}(x_j, t_n)) \), we distinguish several cases.

1.- The spatial mesh is uniform. Then, it is straightforward to prove that

\[
|(L_{x,\xi}^N - L_{x,\xi}^\xi) (\hat{u}^{n+1}(x_j) + \tilde{v}(x_j, t_n))| \leq (N^{-1} \ln N)^2 \tilde{C}.
\]

Then, using the uniform stability of the discrete operator \( (I + (\Delta t/2) L_{x,\xi}^N) \), it follows

\[
|\hat{u}^{n+1}(x_j) - \hat{U}_j^{n+1}| \leq \Delta t (N^{-1} \ln N)^2 \tilde{C}.
\]

2.- The spatial mesh is non uniform and \( \epsilon_1 = \epsilon_2 \) or \( \epsilon_2 = 1 \). Now, using the decomposition into regular and singular components and appropriated Taylor expansions, we can prove that

\[
\begin{align*}
|L_{x,\xi}^N - L_{x,\xi}^\xi| (\hat{u}^{n+1}(x_j) + \tilde{v}(x_j, t_n)) &\leq 2N^{-1}(\epsilon_1^{1/2}, \epsilon_2^{1/2})^T, \\
|L_{x,\xi}^N - L_{x,\xi}^\xi| (\hat{w}^{n+1}(x_j) + \tilde{w}(x_j, t_n)) &\leq (N^{-1} \ln N)^2 \tilde{C},
\end{align*}
\]

which together the uniform stability does not give second order of uniform convergence; the reason is the bound for the local error associate to the regular component. To find a more precise bound, we use a standard barrier function in the context of
singular perturbation problems (see [9, 12]). This barrier function is defined by using the piecewise linear function

\[ \varphi_\gamma(x) = \begin{cases} x\tau_\gamma^{-1}, & x \in [0, \tau_\gamma], \\ 1, & x \in [\tau_\gamma, 1 - \tau_\gamma], \\ (1 - x)\tau_\gamma^{-1}, & x \in [1 - \tau_\gamma, 1]. \end{cases} \]

Defining the barrier functions \( \tilde{\vartheta}(x_i) = CN^{-2}(1 + \tau_\gamma \xi^{1/2}) \varphi_\gamma(x_i) \), if \( \xi = \xi_1 \), and \( \tilde{\vartheta}(x_i) = CN^{-2}(1 + \tau_\gamma \varphi_\gamma(x_i)(\xi^{1/2}, 1)^T) \), if \( \xi = 1 \), we can obtain

\[ |\tilde{w}^{n+1}(x_j) - \tilde{w}_j^{n+1}| \leq (N^{-2} \ln N)^2 \mathcal{C}, \quad |\tilde{\omega}^{n+1}(x_j) - \tilde{W}_j^{n+1}| \leq \Delta t (N^{-1} \ln N)^2 \mathcal{C}. \]

Note that using the barrier function technique we obtain an almost second order of uniform convergence. Nevertheless, the factor \( \Delta t \) disappears from the bound for the error associated to the regular component; this fact is important and its consequence is that we will need to impose a relation between the discretization parameters \( \Delta t \) and \( N \), to deduce the uniform convergence of the totally discrete scheme.

3.- The spatial mesh is non uniform and \( \xi_1, \xi_2 \) are arbitrary. Using similar ideas to these ones developed in [10], we can prove bounds for \((L_{x,\varepsilon}^N - L_{x,\bar{\varepsilon}})(\tilde{w}^{n+1}(x_j) + \tilde{w}(x_j, t_n))\) and \((L_{x,\varepsilon}^N - L_{x,\bar{\varepsilon}})(\tilde{\omega}^{n+1}(x_j) + \tilde{w}(x_j, t_n))\), and from them we obtain a crude bound for \( |(L_{x,\varepsilon}^N - L_{x,\bar{\varepsilon}})(\tilde{w}^{n+1}(x_j) + \tilde{w}(x_j, t_n))| \). Finally, using again a barrier function (see Linß \& Madden (2004b)), which is defined by \( \tilde{\vartheta}(x_i) = N^{-2} \ln N(1 + \varphi_\gamma(x_i) + \varphi_\gamma(x_i))\mathcal{C} \), we obtain \( |\tilde{w}^{n+1}(x_j) - \tilde{U}_j^{n+1}| \leq (N^{-1} \ln N)^2 \mathcal{C} \), which is the required result. Again this bound does not give the factor \( \Delta t \).

**Theorem 4.3.** Let \( \tilde{u}(x, t) \) be the solution of (1.1) and \( \{\tilde{U}_j^{n+1}\} \) the solution of (4.1). Then,

\[ \|\tilde{u}(x_j, t_{n+1}) - \tilde{U}_j^{n+1}\|_{\Omega^N} \leq C(N^{-2+q} \ln N + (\Delta t)^2), \quad 0 < q < 1, \]

where \( N, \Delta t \) and \( q \) are such that \( N^{-q} \leq C \Delta t \).

**Proof.** From (4.2) we must bound three different terms. First, from Lemma 3.1 we have

\[ \|\tilde{u}(x_j, t_{n+1}) - \tilde{w}^{n+1}(x_j)\|_{\Omega^N} \leq C_{n+1}(\Delta t)^3. \]

In second place, from Proposition 4.2 it follows

\[ \|\tilde{w}^{n+1}(x_j) - \tilde{U}_j^{n+1}\|_{\Omega^N} \leq C_{n+1}(N^{-1} \ln N)^2. \]

Finally, to bound \( \|\tilde{U}^{n+1} - \tilde{U}_{n+1}\| \), we take into account that \( \tilde{U}^{n+1} - \tilde{U}_{n+1} \) can be written as the solution of one step of (4.1), taking as source term \( \tilde{f} = \tilde{0} \) together with zero boundary conditions and \( \tilde{u}(x_j, t_n) - \tilde{U}_j^n \) as starting value. Let

\[ R_N = (I + \Delta t \frac{N}{2} L_{x,\varepsilon}^N)^{-1}(I - \Delta t \frac{N}{2} L_{x,\varepsilon}^N), \]

be the transition operator associated to (4.1); then, it holds

\[ \tilde{U}_j^{n+1} - \tilde{U}_{n+1} = R_N \left( \tilde{u}(x_j, t_n) - \tilde{U}_j^n \right). \]
Similarly to previous bounds, we can obtain
\[
\hat{u}(x_j, t_n) - \hat{U}_j^n = \left( \hat{u}(x_j, t_n) - \hat{u}_j^n(x_j) \right) + \left( \hat{u}_j^n(x_j) - \hat{U}_j^n \right),
\]
\[
\|\hat{u}(x_j, t_n) - \hat{u}_j^n(x_j)\|_{\Omega_N} \leq C_n(\Delta t)^3,
\]
\[
\|\hat{u}_j^n(x_j) - \hat{U}_j^n\|_{\Omega_N} \leq C_n(N^{-1}\ln N)^2,
\]
\[
\hat{U}_j^n - \hat{U}_j^{n-1} = R_N \left( \hat{u}(x_j, t_{n-1}) - \hat{U}_j^{n-1} \right).
\]

Then, using a recursive argument we can obtain
\[
\|\hat{u}(x_j, t_{n+1}) - \hat{U}_j^{n+1}\|_{\Omega_N} \leq C \sum_{i=1}^n \|R_N^{-i}\|_{\Omega_N}((\Delta t)^3 + (N^{-1}\ln N)^2) \leq C \Delta t \sum_{i=1}^n \|R_N^{-i}\|_{\Omega_N}((\Delta t)^2 + N^{-2+q}\ln^2 N).
\]

where \( C = \max\{C_1, C_2, \cdots, C_{n+1} \} \) and we have used the hypothesis \( N^{-q} \leq C\Delta t \).

To obtain the desired bound, a sufficient condition is that the powers of the discrete transition operator \( R_N \) be uniformly bounded with respect to the diffusion parameters.

Because the lack of any theoretical result on this subject, at this moment we conjecture that this property is true. We support this assumption by the numerical evidence showed in the experiments which has been performed (see next section). In all cases, the results shown that the spectral radius of \( R_N \) is strictly less than 1, independently of the diffusion parameters \( \varepsilon_1 \) and \( \varepsilon_2 \) and also of the discretization parameters \( N \) and \( \Delta t \). Nevertheless, at the moment, the proof of this property is an open question.

\[ \square \]

Remark 3. Theorem 4.3 proves almost second order of uniform convergence of the method, under the relation \( N^{-q} \leq C\Delta t \). This relation has led to enough large time step sizes are required once the space step size is fixed. Nevertheless, from the numerical point of view (see next section), this condition is an artificial relation that we have never needed. Note that this relation appeared when the barrier function technique was used to prove second order convergence of the regular component.

5. Numerical results. In this section we show the numerical results obtained in some examples by using (4.1). The first problem that we consider is given by

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &- \varepsilon_1 \frac{\partial^2 u_1}{\partial x^2} + 2(1 + x)^2 u_1 - (1 + x^3)u_2 = 2e^x t(1-t), \\
\frac{\partial u_2}{\partial t} &- \varepsilon_2 \frac{\partial^2 u_2}{\partial x^2} - 2 \cos(\pi x/4)u_1 + 4e^{1-x}u_2 = (10x + 1)t(1-t),
\end{align*}
\]

\( u(0, t) = u(1, t) = 0, \quad t \in [0, T], \quad u(x, 0) = 0, \quad x \in (0, 1). \)  

(5.1)

Figure 5.1 shows the solution at \( t = 1 \) obtained for \( \varepsilon_1 = 2^{-20}, \varepsilon_2 = 2^{-15} \) taking \( N = 64 \) and \( \Delta t = 0.1 \); from it we clearly see the boundary layers at both end points. To find an approximation to the pointwise errors \( [\hat{U}_j^n - \hat{u}(x_j, t_n)] \), we use a variant of the double mesh principle. So, we calculate \( \{\hat{Z}_j^n\} \) on the mesh \( \{(\hat{x}_j, \hat{t}_n)\} \) that contains the mesh points of the original mesh and their midpoints, i.e., the mesh points are

\[
\begin{align*}
\hat{x}_j &= x_j, \quad j = 0, \ldots, N, \quad \hat{x}_{j+1} = (x_j + x_{j+1})/2, \quad j = 0, \ldots, N-1, \\
\hat{t}_n &= t_n, \quad n = 0, \ldots, M, \quad \hat{t}_{n+1} = (t_n + t_{n+1})/2, \quad n = 0, \ldots, M-1.
\end{align*}
\]
At the original mesh points \((x_j, t_n)\), the maximum errors and the uniform errors are approximated by

\[
\vec{d}_{\varepsilon, N, \Delta t} = \max_{0 \leq n \leq M} \max_{0 \leq j \leq N} |\tilde{U}_j^n - \tilde{Z}_j^{2n}|, \quad \vec{d}_{N, \Delta t} = \max_S \vec{d}_{\varepsilon, N, \Delta t},
\]

where \(S\) is the set

\[
S = \{(\varepsilon_1, \varepsilon_2) \mid \varepsilon_2 = 2^0, 2^{-2}, \ldots, 2^{-30}, \varepsilon_1 = \varepsilon_2, 2^{-2} \varepsilon_2, \ldots, 2^{-58}, 2^{-60}\},
\]

in order to permit that the maximum errors stabilize. From these values we obtain the corresponding orders of convergence and the uniform orders of convergence in a standard way, by using

\[
\bar{p} = \log_2(\vec{d}_{\varepsilon, N, \Delta t}/\vec{d}_{\varepsilon, 2N, \Delta t/2}), \quad \bar{p}_{uni} = \log_2(\vec{d}_{N, \Delta t}/\vec{d}_{2N, \Delta t/2}).
\]

Table 5.1 displays the results obtained in this case. From it we see that the method gives almost second order of uniform convergence, in agreement with Theorem 4.3.

To corroborate the conjecture about the bounds of powers of operator \(R_N\), we calculate the maximum value of its spectral radius for different values of \(N\) and \(\Delta t\) on set \(S\) defined in (5.2). Table 5.2 displays the maximum spectral radius of this operator; from it, we clearly observe that its value is always less than one. Moreover, we note that the spectral radius stabilizes for any small value of the diffusion parameters, which is necessary in order to deduce that it is uniformly bounded. We also observe that the maximum value is always achieved when both diffusion parameters take the largest values considered, i.e., when \(\varepsilon_1 = \varepsilon_2 = 1\), which corresponds to a classical parabolic reaction-diffusion problem.

In previous example we have obtained the almost second order of uniform convergence, even when we do not have the sufficient compatibility conditions theoretically required. Nevertheless, when the compatibility conditions are weakened, the order reduction is corroborated in practice. To see that, we consider a second problem defined by

\[
\begin{align*}
\frac{\partial u_1}{\partial t} - \varepsilon_1 \frac{\partial^2 u_1}{\partial x^2} + 2u_1 - u_2 &= 1, \\
\frac{\partial u_2}{\partial t} - \varepsilon_2 \frac{\partial^2 u_2}{\partial x^2} - u_1 + 2u_2 &= 1,
\end{align*}
\]

\((x, t) \in (0, 1) \times (0, 1], \quad u(0, t) = u(1, t) = 0, \quad t \in [0, T], \quad u(x, 0) = 0, \quad x \in (0, 1].\)
Table 5.1
Maximum (in $\varepsilon_1$) errors and orders of convergence for the example (5.1)

<table>
<thead>
<tr>
<th>$\varepsilon_2$</th>
<th>$N=16$</th>
<th>$N=32$</th>
<th>$N=64$</th>
<th>$N=128$</th>
<th>$N=256$</th>
<th>$N=512$</th>
<th>$N=1024$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta t = 0.2$</td>
<td>$\Delta t = 0.1$</td>
<td>$\Delta t = 0.05$</td>
<td>$\Delta t = 0.025$</td>
<td>$\Delta t = 0.0025$</td>
<td>$\Delta t = 0.000625$</td>
<td>$\Delta t = 0.0003125$</td>
</tr>
<tr>
<td>$2^{-2}$</td>
<td>8.84E-3</td>
<td>5.99E-3</td>
<td>4.87E-3</td>
<td>3.82E-3</td>
<td>1.72E-3</td>
<td>5.80E-4</td>
<td>1.84E-4</td>
</tr>
<tr>
<td></td>
<td>0.561</td>
<td>0.298</td>
<td>0.350</td>
<td>1.151</td>
<td>1.568</td>
<td>1.657</td>
<td>1.842E-4</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>1.27E-2</td>
<td>5.99E-3</td>
<td>4.87E-3</td>
<td>3.82E-3</td>
<td>1.72E-3</td>
<td>5.80E-4</td>
<td>1.842E-4</td>
</tr>
<tr>
<td></td>
<td>1.089</td>
<td>0.297</td>
<td>0.351</td>
<td>1.151</td>
<td>1.568</td>
<td>1.657</td>
<td>1.842E-4</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>1.56E-2</td>
<td>5.99E-3</td>
<td>4.88E-3</td>
<td>3.82E-3</td>
<td>1.72E-3</td>
<td>5.80E-4</td>
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Table 5.2
Maximum (in $\varepsilon_1$) spectral radius for the example (5.1)

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<th>$N=256$</th>
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<td>$\Delta t = 0.1$</td>
<td>$\Delta t = 0.05$</td>
<td>$\Delta t = 0.025$</td>
<td>$\Delta t = 0.0025$</td>
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<td>0.99029</td>
<td>0.99513</td>
<td>0.99756</td>
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<td>0.99029</td>
<td>0.99513</td>
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Figure 5.2 shows the approximated solution at $t = 1$ obtained for the same values than in example 1. From it, we clearly see again that boundary layers occur at both end points.

Table 5.3 displays the results obtained in this case, showing a lower order of uniform convergence.

Finally, Table 5.4 displays the maximum spectral radius of operator $R_N$ for different values of $N$ and $\Delta t$ (again on the set $S$ defined in (5.2)) associated to test problem (5.3). The results are similar to those ones of previous example, supporting
PARABOLIC REACTION-DIFFUSION SYSTEMS: HIGH ORDER UNIFORM METHODS

Fig. 5.2. Solution of (5.3) for $\varepsilon_1 = 2^{-20}, \varepsilon_2 = 2^{-15}, N = 64, \Delta t = 0.1$.

Table 5.3
Maximum (in $\varepsilon_1$) errors and orders of convergence for the example (5.3)

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<th>$N=1024$</th>
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<td></td>
<td></td>
<td></td>
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<td>1.568E-3</td>
<td>5.286E-4</td>
<td>1.911E-4</td>
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<tr>
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<td>1.140</td>
<td>1.465</td>
<td>1.569</td>
<td>1.468</td>
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<tr>
<td>$2^{-10}$</td>
<td>1.251E-2</td>
<td>1.320E-2</td>
<td>9.541E-3</td>
<td>4.328E-3</td>
<td>1.568E-3</td>
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In practice that our conjecture about the uniform bound of the operator $R_N$ is true.

REFERENCES


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