High order sliding mode observer for linear systems with unbounded unknown inputs

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A global observer is designed for strongly detectable systems with unbounded unknown inputs. The design of the observer is based on three steps. First, the system is extended taking the unknown inputs (and possibly some of their derivatives) as a new state; then, using a global high-order sliding mode differentiator, a new output of the system is generated in order to fulfill, what we will call, the Hautus condition, which finally allows decomposing the system, in new coordinates, into two subsystems; the first one being unaffected directly by the unknown inputs, and the state vector of the second subsystem is obtained directly from the original system output. Such decomposition permits designing of a Luenberger observer for the first subsystem, which satisfies the Hautus condition, i.e. all the outputs have relative degree one w.r.t. the unknown inputs. This procedure enables one to estimate the state and the unknown inputs using the least number of differentiations possible. Simulations are given in order to show the effectiveness of the proposed observer.

Keywords: sliding mode observer; unknown inputs; strong detectability

1. Introduction

The problem of design of exact sliding mode (SM) observers for linear systems with unknown inputs has been extensively studied in the last decade. Particularly in the last years, several sliding mode observers have been proposed (see e.g. Boukhobza, Djemai, and Barbot 2003; Ahmed-Ali, Kenne, and Lamnabi-Lagarrigue 2007; Bejarano, Fridman, and Poznyak 2007; Fridman, Levant, and Davila 2007) to overcome the relative-degree-one restriction w.r.t. the unknown inputs needed for the design of a linear observer (Hautus 1983). The crucial point for the success of SM observers is that they bring with them an implicit or explicit use of a differentiation process, which is the key as to why the relative-degree-one restriction can be surpassed. However, one of the drawbacks of SM observers is that, most of them, require the unknown input vector to be uniformly bounded.

The aim of this article is to design an observer in a more general form, that is, to consider a general class of linear systems (avoiding restrictions over the relative degree of the system and unknown input boundedness), for the systems satisfying structural sufficient and necessary conditions of strong detectability; and also to minimise the error inherent to the differentiation process to be as small as possible.

In Molinari (1976) and Hautus (1983) the strong observability property was studied. This property ensures a one-to-one correspondence between the state and the output (this is analogous to the observability property of linear systems without inputs (wo.i.)). Furthermore, the strong detectability property was studied (analogous to detectability for systems wo.i.). Molinari gave a recursive algorithm dealing with the construction of a series of matrices that allows for the calculation of the weakly unobservable subspace in the last step of the algorithm. Such as algorithm can be interpreted as a differentiation procedure of the output of the system. Thus, if the system is strongly observable (the weakly observable subspace contains only the zero vector), the state can be obtained through a recursive method of differentiation of the output. On the other hand, in Hautus (1983) the same problem is studied in terms of zeros; there conditions are given under which the estimation of the state vector can be done by means of a linear observer whose input is the output of the original system: the system must be strongly detectable (the zeros of the system must be Hurwitz) and some rank condition (it includes the relative-degree-one restriction), relating the output distribution matrix and the input distribution matrices, must be satisfied.

In the sliding mode community there are works where observers have been designed assuming that the Hautus condition is satisfied (Walcott and Zak 1987; Edwards, Spurgeon, and Tan 2002; Hui and Zak 2005).
In other works a differentiation process has been proposed, mainly based on first and second-order sliding mode techniques (Hashimoto, Utkin, Xu, Suzuki, and Harashima 1990; Boukhobza et al. 2003; Ahmed-Ali, Floret, and Lamnabhi-Lagarrigue 2004; Ahmed-Ali et al. 2007; Aurora and Ferrara 2007; Bejarano et al. 2007; Fridman et al. 2007; Pisano and Usai 2007; Floquet and Barbot 2007; Shtessel, Shkolnikov, and Levant 2007). A method for the construction of a new output using differentiation to fulfill the Hautus condition is suggested in Floquet et al. (2007). Nevertheless,

- a condition related to the relative degree of the original system must be satisfied in order to follow such a method, which means that the system is required to meet more than just strong detectability. Furthermore,
- the differentiation procedure is done step-by-step using the super-twisting algorithm (a second-order sliding mode, Levant (1998)), which increases the error due to the sample time of sensors or computer calculations, i.e. the overall observation error is of the order \( r^{1/k} \), where \( r \) represents the sample time and \( k \) is the number of derivatives (steps) needed for the observation process. Also,
- as in the majority of exact sliding mode observers, it is assumed that the unknown input vector is uniformly bounded.

**Main contributions:** In this article, we propose an observer for strongly detectable systems using the least number of derivatives possible: it means that

- a differentiation process is followed only when structurally it is required (i.e. structural conditions of the system decide that). We propose an observation structure that allows to make the needed differentiation process not step-by-step, but in only one step using the high-order differentiator proposed by Levant (2006). Thus, we attempt to
- minimise the observation error due to the numerical differentiator, which in fact is of order \( r w.r.t. \) the sampling time and of order \( \eta^{1/r+1} \) w.r.t. to the noise, where \( \eta \) is the maximal amplitude of the noise and \( r \) is the minimum number of derivatives needed to satisfy the Hautus condition
- estimate the state vector and, at the same time, the estimation of the unknown inputs, which, in general, do not have to be uniformly bounded, but only requires boundedness (not uniformly) of at least one of their high-order derivatives. In the case that none of the high-order derivatives of the unknown input is known to be bounded, but the unknown input is bounded, it is still possible to estimate the state vector.

**Notations:** The following notation is used throughout this article. Let \( X \) be a real matrix of dimension \( n \times m \). The notation \( X^\perp \) means a full row rank orthogonal matrix to \( X \), i.e. \( X^\perp X = 0 \) and \( \text{rank } X^\perp = n - \text{rank } X \). The matrix \( X^{\perp\perp} \) must satisfy the conditions \( \text{rank } X^{\perp\perp} = \text{rank } X \) and \( \det [X^\perp X^{\perp\perp}] \neq 0 \). Meanwhile, \( X_\perp \) is a matrix whose image spans the null space of \( X \), i.e. \( X X_\perp = 0 \) and \( \text{rank } X_\perp = m - \text{rank } X \). Let \( f(t) \) be a vector function, \( f^{(k)} \) represents the \( k \)-th anti-differentiator of \( f(t) \), i.e. \( f^{(k)}(t) = \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} f(\tau_k) d\tau_k \cdots d\tau_2 d\tau_1 \), \( f^{(0)}(t) = f(t) \). For a matrix \( A \), \( \rho_A := \text{rank } A \).

### 2. Problem statement

Let \( \Sigma \) be a linear system whose dynamics is governed by the following equations:

\[
\begin{align*}
\dot{X}(t) &= AX(t) + Dw(t) \\
y(t) &= CX(t) + Fw(t)
\end{align*}
\]

(1)

The state vector is represented by \( x(t) \in \mathbb{R}^n \), \( w(t) \in \mathbb{R}^m \) represents the unknown input vector, and \( y(t) \in \mathbb{R}^p \) is the system output.\(^1\) The fourfold of constant matrices \( (A, C, D, F) \) will be associated to system \( \Sigma \). Without loss of generality, it is assumed that \( \text{rank } [P_F] = m \).

The task is to estimate \( x(t) \) and \( w(t) \) using only the output values \( y(\tau) (\tau \in [0, \ell]) \). Henceforth, the following conditions are assumed to be satisfied.

(A1) System \( \Sigma \) is strongly detectable.\(^2\)

(A2) Vector \( w \) can be partitioned in the following way: \( w^T = [w_0^T \ w_1^T \cdots w_r^T] \), where \( w_0 \in \mathbb{R}^{m_0} \) has an upper-bounded norm, i.e. \( \|w_0(\tau)\| \leq w_0^\tau(t) \); \( w_1 \in \mathbb{R}^{m_1} \) has a derivative bounded as \( \|w_1(t)\| \leq w_1^\tau(t) \), and so forth, until \( w_r \in \mathbb{R}^{m_r} \) which has a bounded norm for its \( r \)-th derivative, i.e. \( \|w_r^{(r)}(\tau)\| \leq w_r^{(r)}(t) \).

Obviously, it must be satisfied that \( \sum_{i=0}^r m_i = m \). Furthermore, it is considered that \( w_i^\tau(t) (i = 0, \ldots, r) \) is a continuous function.

Notice that Assumption A2 is related more with the existence of the bounds of \( w_i \) than with the partition itself since, once the bounds are ensured, \( w \) can always be partitioned in the required form, perhaps with a change of input coordinates.

We can partition matrices \( D \) and \( F \) defining matrices \( d_i \) and \( f_i \) (\( i = 0, \ldots, r \)) by means of the identities \( Dw = \sum_{i=0}^r d_i w_i \) and \( Fw = \sum_{i=0}^r f_i w_i \), respectively. Thus, considering Assumption A2, system \( \Sigma \) can
be rewritten into form (2).
\[
\Sigma_{ex} : \begin{cases} 
\dot{x}_{ex} = \tilde{A}x_{ex} + \tilde{D}w_{ex} \\
y = \tilde{C}x_{ex} + \tilde{F}w_{ex}
\end{cases}
\] (2)
We set, by definition,
\[
x_{ex}^T = \begin{bmatrix} x^T & w_1^T & \cdots & w_{r}^T & \cdots & w_{r-1}^T & \cdots & w_{r}(r-1)^T \end{bmatrix}
\]
\[
w_{ex}^T = \begin{bmatrix} w_1^T & \cdots & w_{r}^T & \cdots & w_{r}(r-1)^T \end{bmatrix},
\]
where \( x_{ex}^T \in \mathbb{R}^{\bar{n}} \) (\( \bar{n} = n + \sum_{i=1}^{r} m_i \)) and \( w_{ex} \in \mathbb{R}^{m} \).

Let us define
\[
\tilde{A} = \text{diag}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_r), \quad \tilde{A}_i = \begin{bmatrix} A & d_1 & \cdots & d_r \\
0 & 0 & \cdots & 0 \end{bmatrix}, \quad \tilde{A}_j := \begin{bmatrix} I \\
0 \end{bmatrix}, \quad A \in \mathbb{R}^{m \times n}, \quad \tilde{D} := \text{diag}(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_r), \quad \tilde{d}_i = \begin{bmatrix} d_0 & 0 \\
0 & I_{m_i} \end{bmatrix},
\]
\[
\tilde{C} := \begin{bmatrix} C & f_1 & f_2 & \cdots & f_r \end{bmatrix}, \quad F = \begin{bmatrix} f_0 & 0 \end{bmatrix}, \quad F \in \mathbb{R}^{m \times m}
\]
It is easy to verify that the detectability property is maintained for system \( \Sigma_{ex} \), i.e. \( \Sigma \) is strongly detectable if, and only if, \( \Sigma_{ex} \) is strongly detectable as well.

Thus, the problem of estimating \( x(t) \) and \( w(t) \) is restated as the problem of estimating extended vector \( x_{ex} \).

3. Construction of a new output

For the triple \((C, D, F)\), identity rank \([CD \mid F] = \text{rank } F + m\) will be referred to as the Hautus condition. The Hautus condition is a necessary condition for the design of an observer for system (1), without the need of any output derivatives.

Let us define \( \bar{Q}_F \in \mathbb{R}^{m \times m} \) as a non-singular matrix \( \bar{Q}_F := [\bar{F}_1 \bar{Q}_1] \) (see notation part in the introduction), which yields the following matrix decomposition:
\[
\begin{bmatrix} \bar{D} \\ \bar{F} \end{bmatrix} \bar{Q}_F = \begin{bmatrix} \bar{\bar{D}}_1 & \bar{D}_2 \\ 0 & \bar{\bar{F}}_2 \end{bmatrix}, \quad \bar{\bar{F}}_2 \in \mathbb{R}^{m \times p_F}, \quad \bar{\bar{D}}_1 = \bar{D}\bar{F}_1 \in \mathbb{R}^{m \times (m-n)}
\] (3)

From (3), rank \( \bar{F}_2 = \text{rank } \bar{\bar{F}}_2 \) (\( \rho_{\bar{F}} = \rho_{\bar{\bar{F}}} \)). Furthermore, from (3), it is easy to verify that, for triple \((C, D, F)\) the Hautus condition is equivalent to satisfying the identity rank \( \bar{\bar{C}}(\bar{\bar{F}}_1 \bar{C}) = \text{rank } \bar{\bar{D}}_1 \). Thus, this section is devoted to the construction of a new output \( y_{m} = M_{n \times n_{xex}} \) so that rank \( M_{n \times \bar{\bar{D}}_1} = \text{rank } \bar{\bar{D}}_1 \). Let us give the following lemma, which will help to remove the influence of the unknown inputs in the output injection from the Luenberger-like observer proposed below.

**Lemma 3.1:** \((A, C, D, F)\) is strongly detectable if, and only if, \((\bar{A} - \bar{\bar{D}}_2 \bar{\bar{C}} \bar{\bar{F}} \bar{\bar{C}} \bar{\bar{F}}_2, \bar{\bar{F}}_2 \bar{\bar{C}} \bar{\bar{F}}_2)\) is strongly detectable as well.

**Proof:** Since fourfold \((A, C, D, F)\) is strongly detectable if, and only if, fourfold \((\bar{A}, \bar{\bar{C}}, \bar{\bar{F}})\) is strongly detectable as well, it is enough to prove the equivalence of the proposition for \((A, C, D, F)\) and \((\bar{A} - \bar{\bar{D}}_2 \bar{\bar{C}} \bar{\bar{F}} \bar{\bar{C}} \bar{\bar{F}}_2, \bar{\bar{F}}_2 \bar{\bar{C}} \bar{\bar{F}}_2)\). Let us define \( R_{\Sigma}(s) \) as the Rosenbrock matrix associated to \( \Sigma \), i.e.
\[
R_{\Sigma}(s) = \begin{bmatrix} sI - \bar{\bar{A}} & -\bar{\bar{D}} \\
\bar{\bar{C}} & \bar{\bar{F}} \end{bmatrix}
\]
Due to the fact that the identity
\[
\begin{bmatrix} sI - \bar{\bar{A}} + \bar{\bar{D}}_2 \bar{\bar{F}}_2 \bar{\bar{C}} - \bar{\bar{D}}_1 - \bar{\bar{D}}_2 \\
\bar{\bar{F}}_2 \bar{\bar{C}} & 0 & 0 \\
0 & 0 & I \end{bmatrix}
\]
\[
= \begin{bmatrix} I & 0 & 0 \\
0 & \bar{\bar{F}}_2 \bar{\bar{C}} & I \\
0 & \bar{\bar{F}}_2 \bar{\bar{C}} & 0 \end{bmatrix} R_{\Sigma}(s) \begin{bmatrix} I & 0 & 0 \\
0 & \bar{\bar{F}}_2 \bar{\bar{C}} & I \\
0 & \bar{\bar{F}}_2 \bar{\bar{C}} & 0 \end{bmatrix}
\] (4)
always holds, it turns out to be that \( s_0 \) is an invariant zero of \((A, C, D, F)\), i.e. rank \( R_{\Sigma}(s_0) < n + m \) if, and only if, rank \( R_{\Sigma_{ex}}(s_0) < n + m - r \) (where the Rosenbrock matrix \( R_{\Sigma_{ex}} \) for \( \Sigma_{ex} \) appears in the extended matrix of the left-hand side of (4)), which in turn is equivalent to asserting that \( s_0 \) is an invariant zero of \((\bar{A} - \bar{\bar{D}}_2 \bar{\bar{C}} \bar{\bar{F}} \bar{\bar{C}} \bar{\bar{F}}_2, \bar{\bar{C}} \bar{\bar{F}}_2)\).

Now, selecting \( \bar{L} \) so that \((\bar{A} - \bar{\bar{D}}_2 \bar{\bar{C}} \bar{\bar{F}} \bar{\bar{C}} \bar{\bar{F}}_2, \bar{\bar{C}} \bar{\bar{F}}_2)\) be Hurwitz, we can design the following Luenberger-like observer whose trajectories \( \tilde{x} \) are governed by the dynamic equations (5).
\[
\dot{\tilde{x}} = (\bar{A} - \bar{\bar{D}}_2 \bar{\bar{C}} \bar{\bar{F}} \bar{\bar{C}} \bar{\bar{F}}_2, \bar{\bar{C}} \bar{\bar{F}}_2) \tilde{x} + \bar{\bar{L}}(\bar{\bar{F}}_2 \dot{y} - \bar{\bar{F}}_2 \bar{\bar{C}} \tilde{x}) + \bar{\bar{D}}_2 \bar{\bar{F}}_2 \dot{y}
\] (5)
Defining \( [\tilde{\bar{y}}_{ex, 1}] := \bar{Q}_F w_{ex} \), \( \tilde{\bar{y}}_{ex, 2} \in \mathbb{R}^{p_{F}, m \times m} \), and since \( \bar{\bar{F}}_2 \dot{y} = \bar{\bar{F}}_2 \bar{\bar{C}} \tilde{x} + \tilde{\bar{y}}_{ex, 2} \), the dynamics of \( e(t) = x_{ex} - \tilde{x} \) takes the following form:
\[
\dot{e}(t) = (\bar{A} - \bar{\bar{D}}_2 \bar{\bar{C}} \bar{\bar{F}} \bar{\bar{C}}) e(t) + \bar{\bar{D}}_1 \tilde{\bar{y}}_{ex, 1}(t)
\]
\[
= \bar{\bar{A}} e(t) + \bar{\bar{D}}_1 \tilde{\bar{y}}_{ex, 1}(t)
\] (6)

3.1 Output extension via sliding mode high-order differentiator

In the sequel, we will construct a new output \( \bar{y} = M_{\bar{\bar{n}}} x_{ex} \) so that Hautus condition can be achieved, i.e. that the identity rank \( M_{\bar{\bar{n}}} \bar{\bar{D}}_1 = \text{rank } \bar{\bar{D}}_1 \) will be true.
For the construction of such an output, we will use the sliding mode high-order differentiator proposed by Levant (2006).

Firstly, let us write a set of matrices $M_k$ so that $M_k x_{ex}$ can be expressed as a differentiation operator depending on $y$. Let the matrices $M_{k+1}$ (which can be taken from algorithm given in Molinari (1976)) be defined by means of the following algorithm:

$$
M_{k+1} = N_{k+1}^\perp N_{k+1}, \quad M_1 = (F^\perp \tilde{C})^\perp F^\perp \tilde{C} \\
N_{k+1} = T_k \left( \begin{array}{l} M_k \tilde{A} \\ \tilde{C} \end{array} \right), \quad T_k = \left( \begin{array}{l} M_k \tilde{D} \\ \tilde{F} \end{array} \right)^\perp
$$

(7)

Notice that $N_{k+1}^\perp$ excludes the linearly dependent rows of $N_{k+1}$, so $M_{k+1}$ has full row rank. Defining $\Phi_1 := J_1 y$, where $J_1 := (F^\perp \tilde{C})^\perp F^\perp$, leads to

$$
\Phi_1 = (F^\perp \tilde{C})^\perp F^\perp x_{ex} = M_1 x_{ex}
$$

Now with $\Phi_2 := N_{k+1}^\perp T_k \left[ \begin{array}{l} J_1 \\ 0 \\ \ldots \\ 0 \end{array} \right]$ and moving the differenciation operator outside of the parenthesis, the following identity is obtained:

$$
\Phi_2 = M_2 x_{ex} = \frac{d}{dt} N_{k+1}^\perp T_k \left[ \begin{array}{l} J_1 \\ 0 \\ \ldots \\ 0 \end{array} \right] y = \frac{d}{dt} J_2 \left[ \begin{array}{l} y \\ y^{(1)} \\ \ldots \\ y^{(k-1)} \end{array} \right]
$$

The matrix $J_2$ is defined by the previous identity. Then, we can generalize the procedure as follows: defining $\Phi_3 := N_{k+1}^\perp T_{k-1} \left[ \begin{array}{l} \frac{d}{dt} M_{k-1} y_{ex} \\ \frac{d}{dt} y_{ex} \end{array} \right] (k = 2, \ldots, n - \rho M)$, the identity

$$
\Phi_3 = M_3 x_{ex} = \frac{d^{k-1}}{dt^{k-1}} N_{k+1}^\perp T_{k-1} \left[ \begin{array}{l} J_{k-1} \\ 0 \\ \ldots \\ 0 \end{array} \right] \left[ \begin{array}{l} y \\ y^{(1)} \\ \ldots \\ y^{(k-1)} \end{array} \right] = \frac{d^{k-1}}{dt^{k-1}} J_k \left[ \begin{array}{l} y \\ y^{(1)} \\ \ldots \\ y^{(k-1)} \end{array} \right]
$$

holds, where $J_k = N_{k+1}^\perp T_{k-1} \left[ \begin{array}{l} J_{k-1} \\ 0 \\ \ldots \\ 0 \end{array} \right]$.

Since the equality rank $M_i := \text{rank } M_{i-1}$ implies rank $M_{i+1} = \text{rank } M_i$, then it means that rank $M_{\hat{n} - \rho M} = \text{rank } M_{\hat{n} - \rho M} + 1$, i.e. at most $\hat{n} - \rho M$, differentiations are needed, which means that, for a strongly observable $\Sigma$ (i.e. det $M_{\hat{n} - \rho M} \neq 0$), the state vector $x_{ex}$ can be expressed by the identity

$$
x_{ex} = \frac{d^{\hat{n} - \rho M}}{dt^{\hat{n} - \rho M}} M_{\hat{n} - \rho M}^{-1} J_{\hat{n} - \rho M} + \frac{d^{\hat{n} - \rho M}}{dt^{\hat{n} - \rho M}} J_{\hat{n} - \rho M} y_{ex}
$$

Nevertheless, two drawbacks appear in order to reconstruct $x_{ex}$ in such a manner. The first drawback has to do with the assumption that the system is strongly detectable but not strongly observable; therefore, a transformation must first be done in order to decompose the system into the strongly observable part and the detectable part (see e.g. Bejarano, Fridman, and Poznyak 2009). The second drawback has to do with the fact that, during the implementation of the differentiator, some errors appear due to the computation sample time and noises in the sensors.

From Assumption A1 and Lemma 3.1, the identity rank $M_{\hat{n}} \tilde{D}_1 = \text{rank } \tilde{D}_1$ holds (Bejarano et al. 2009). Thus, we can define $\hat{n}_H (2 \leq \hat{n}_H \leq \hat{n} - \rho M + 1)$ as the least natural number such that the matrix $M_{\hat{n}_H}$, calculated by (7), satisfies the rank condition $M_{\hat{n}_H} \tilde{D}_1 = \text{rank } \tilde{D}_1$, and define the extended output $\tilde{y} = M_{\hat{n}_H} x_{ex}$ as the new output of the system. The new output $\tilde{y}$ will be calculated using (9), then a differentiator will be needed.

**Remark 1:** Notice that, from (9), the construction of $\tilde{y} = M_{\hat{n}_H} x_{ex}$ can be guaranteed by means of a derivative of order $\hat{n}_H - 1$; however, some terms of the vector $\tilde{y}$ could be estimated with a derivative of order less than $\hat{n}_H - 1$. Let us exemplify this point, suppose that $\tilde{y}_j$ (the $j$-th term of $\tilde{y}$) is calculated using (9) and the $j$-th row of $J_{\hat{n}_H}$ has the form $J_{\hat{n}_H,j} = \left[ \begin{array}{l} 0 \\ J_e \end{array} \right]$, i.e. the first term of $J_{\hat{n}_H,j}$ is zero. Then $\tilde{y}_j$ becomes

$$
\tilde{y}_j = \frac{d^{\hat{n}_H - 1}}{dt^{\hat{n}_H - 1}} J_{\hat{n}_H,j} \left[ \begin{array}{l} y^T \\ (y^{(1)})^T \\ \ldots \\ (y^{(\hat{n}_H - 1)})^T \end{array} \right]^T = \frac{d^{\hat{n}_H - 2}}{dt^{\hat{n}_H - 2}} J_e \left[ \begin{array}{l} y^T \\ (y^{(1)})^T \\ \ldots \\ (y^{(\hat{n}_H - 2)})^T \end{array} \right]^T
$$

whence, we conclude that $\tilde{y}_{\hat{n}_H,j}$ can be calculated using a differentiator of order $\hat{n}_H - 2$ instead of one of order $\hat{n}_H - 1$.

The reduction of the order in the differentiator is important since as it is known that higher the order of the differentiator, higher is the error due to noises.

In the sequel, we will give a method for calculating a derivative of order $\hat{n}_H - 1$; however, it is advised to take into account Remark 1. From (9), the vector $M_{\hat{n}_H} e$ can be expressed as

$$
M_{\hat{n}_H} e = \frac{d^{\hat{n}_H - 1}}{dt^{\hat{n}_H - 1}} J_{\hat{n}_H} \left[ \begin{array}{l} y^T \\ (y^{(1)})^T \\ \ldots \\ (y^{(\hat{n}_H - 1)})^T \end{array} \right]^T - (M_{\hat{n}_H} \tilde{x})
$$
Defining the vector
\[ H(t) = J_{\tilde{\theta}_i} \left[ y^T \ (y^{(1)})^T \ldots \ (y^{(\tilde{\theta}_{i-1})})^T \right] - M_{\tilde{\theta}_i} \tilde{x}^{(\tilde{\theta}_{i-1})} \]
(11)
where the \( j \)-th term of \( M_{\tilde{\theta}_i} \) is calculated as \( \frac{d^{\vartheta_j-1}}{dt^{\vartheta_j}} H(t) \) (\( j = 1, \ldots, \tilde{\theta}_i \)), where \( \tilde{\theta}_i \) is the dimension of \( \tilde{y} \).

**Remark 2:** In the case that \( \text{dim} \ \tilde{y} = n \), then the vector state can be reconstructed by a differentiation process only, then in this case \( H \) must be defined as
\[
H(t) = M_{\tilde{\theta}_i}^{-1} J_{\tilde{\theta}_i} \left[ y^T \ (y^{(1)})^T \ldots \ (y^{(\tilde{\theta}_{i-1})})^T \right]^T - \tilde{x}^{(\tilde{\theta}_{i-1})}.
\]
Thus, the \( j \)-th term of \( M_{\tilde{\theta}_i} \) can be estimated by means of a sliding mode differentiator of high-order which has the form (for more details, see Levant (2003)):
\[
\begin{align*}
\dot{z}_{j,0} &= -\lambda_0 \left| z_{j,0} - H_j \right|^{\vartheta_j} \text{sign}(z_{j,0} - H_j) + z_{j,1} \\
\dot{z}_{j,1} &= -\lambda_1 \left| z_{j,1} - z_{j,0} \right|^{\vartheta_j} \text{sign}(z_{j,1} - z_{j,0}) + z_{j,2} \\
&\vdots
\dot{z}_{j,\tilde{\theta}_i-2} &= -\lambda_{\tilde{\theta}_i-2} \left| z_{j,\tilde{\theta}_i-2} - \dot{z}_{j,\tilde{\theta}_i-3} \right|^{\vartheta_j} \text{sign}(z_{j,\tilde{\theta}_i-2} - \dot{z}_{j,\tilde{\theta}_i-3}) + z_{j,\tilde{\theta}_i-1} \\
\dot{z}_{j,\tilde{\theta}_i-1} &= -\lambda_{\tilde{\theta}_i-1} \text{sign}(z_{j,\tilde{\theta}_i-1} - \dot{z}_{j,\tilde{\theta}_i-2})
\end{align*}
\]
(12)
where \( \tilde{\theta}_i - 1 \) is the order of the differentiator.

It was shown in Levant (2003) that, with the proper choice of constants \( \lambda_i \) \( (i = 0, \ldots, \tilde{\theta}_i - 1) \), there exists a finite time \( t_j \) such that the identity \( z_{j,\tilde{\theta}_i-1}(t) = d^{\vartheta_j-1} H_j(t) \) is achieved for all \( t \geq t_j \). Thus, every term of \( \dot{y} \) can be calculated using a sliding mode differentiator of high-order. \( \lambda_i \) is a continuous function and, at time \( t \), \( K(t) \) is a known local Lipschitz constant for \( \frac{d^{\vartheta_j-1}}{dt^{\vartheta_j}} H_j(t) = M_{\tilde{\theta}_i} \tilde{x}^{(\tilde{\theta}_{i-1})} \) and \( \lambda_{\tilde{\theta}_i} \) is calculated for the case when \( K(t) = 1 \) \((\lambda_{\tilde{\theta}_i} \) can be calculated using simulations). A value of \( \lambda_{\tilde{\theta}_i} \) \((i = 0, \ldots, \tilde{\theta}_i - 1) \) for a fifth order differentiator was given in Levant (2006), with \( \lambda_{00} = 12, \lambda_{01} = 8, \lambda_{02} = 5, \lambda_{03} = 3, \lambda_{04} = 1.5, \) and \( \lambda_{05} = 1.1 \).

Thus, defining the vector \( \tilde{z}_{\tilde{\theta}_i-1} = \left[ z_{1,\tilde{\theta}_i-1} \ldots z_{\tilde{\theta}_i,\tilde{\theta}_i-1} \right]^T \), we achieve the identity \( \tilde{z}_{\tilde{\theta}_i-1} = M_{\tilde{\theta}_i} \tilde{x}^{(\tilde{\theta}_{i-1})} \), and consequently, the identity
\[
\hat{\tilde{y}} := z_{\tilde{\theta}_i-1} + M_{\tilde{\theta}_i} \tilde{x}^{(\tilde{\theta}_{i-1})} = \tilde{y}
\]
(13)
holds for all \( t \geq \max_{i=1}^{\tilde{\theta}_i} t_j \).

A function \( K(t) \) required by the differentiator might be calculated in the following manner.

**Lemma 3.2:** Under Assumptions A1 and A2, there exist a time \( \tilde{t} \) and known positive constants \( \gamma, \varphi, \xi, \) and \( \delta \) such that
\[
\left\| \frac{d}{dt} M_{\tilde{\theta}_i} \epsilon(t) \right\| \leq K(t)
\]
where the \( t \)-th term of \( M_{\tilde{\theta}_i} \) is calculated as \( \frac{d^{\vartheta_j-1}}{dt^{\vartheta_j}} H_j(t) \) (\( j = 1, \ldots, \tilde{\theta}_i \)), where \( \tilde{\theta}_i \) is the dimension of \( \tilde{y} \).

**Proof:** Since \( \tilde{A} \) is Hurwitz, the exponential matrix \( e^{\tilde{A}(t-t_0)} \) has a bounded norm, i.e. \( \| e^{\tilde{A}(t-t_0)} \| \leq \| \tilde{e}^{\tilde{A}(t-t_0)} \| \) for known positive constants \( \tilde{\varphi}, \delta \). Thus, from the solution of (6), we have that
\[
\| \epsilon(t) \| \leq \| e^{\tilde{A}(t-t_0)} \| \| \epsilon(0) \| + \| \tilde{D}_1 \| \| Q^T_F \| F \times \int_0^t \| e^{\tilde{A}(t-t_0)} \| w^+(t) \| \tilde{w}_M(t) \| dt
\]
Thus, for an arbitrary constant \( \delta \), there exists a time \( \tilde{t} \) such that the term \( \| e^{\tilde{A}(t-t_0)} \| \| \epsilon(0) \| \) is less than \( \tilde{y} = \tilde{y}^*(t) \), for all \( t \geq \tilde{t} \). Hence, we obtain
\[
\| \epsilon(t) \| < \tilde{y} + \tilde{\varphi} \int_0^t e^{\tilde{A}(t-t_0)} \| \tilde{w}_M(t) \| dt
\]
with \( \tilde{\varphi} = \tilde{\varphi} \| \tilde{D}_1 \| \| Q^T_F \| F \) and in turn yields the inequality
\[
\| \frac{d}{dt} M_{\tilde{\theta}_i} \epsilon \| \leq \| M_{\tilde{\theta}_i} \| \| \epsilon(t) \| + \| \tilde{D}_1 \| \| Q^T_F \| F \times \int_0^t e^{\tilde{A}(t-t_0)} \| \tilde{w}_M(t) \| dt
\]
Thus, the lemma is proven with \( \tilde{y} = \tilde{y} \| M_{\tilde{\theta}_i} \| \tilde{A} \), \( \tilde{\varphi} = \tilde{\varphi} \| M_{\tilde{\theta}_i} \| \tilde{A} \), and \( \xi = \| M_{\tilde{\theta}_i} \| \| \tilde{D}_1 \| \| Q^T_F \| F \). \qed

4. Asymptotic observer

With \( \tilde{y} \) estimated exactly by means of \( \tilde{\tilde{y}} \). The system with its new output takes the form
\[
\begin{align*}
\dot{x}_c &= (\tilde{A} - \tilde{D}_2 \tilde{F}_2 \tilde{C}) x_c + \tilde{D}_1 \tilde{w}_{ex,1} + \tilde{D}_2 \tilde{F}_2 \tilde{y} \\
\tilde{y} &= M_{\tilde{\theta}_i} x_c
\end{align*}
\]
Since \( M_1 = (\tilde{C} \tilde{C})^{-1} \tilde{F}_1 \tilde{C} \), it means that \( M_{\tilde{\theta}_i} \) is strongly detectable (the strong detectability is not lost by output injection), the triple \((\tilde{A} - \tilde{D}_2 \tilde{F}_2 \tilde{C}, M_{\tilde{\theta}_i}, \tilde{D}_2) \) is strongly detectable as well. In fact, it can be proven that the converse is true as well.

Now let us make a change of coordinates to design an asymptotic observer. Let the state and output
transformations be given by means of the matrices $T$ and $U$, respectively, defined as follows:

$$T = \begin{bmatrix} \tilde{D}_1^+ \\ (M_{\tilde{H}} \tilde{D}_1)^+ + M_{\tilde{H}} \end{bmatrix}, \quad U = \begin{bmatrix} (M_{\tilde{H}} \tilde{D}_1)^+ \\ (M_{\tilde{H}} \tilde{D}_1) \end{bmatrix}$$ (14)

where the inverse matrices are $T^{-1} = \begin{bmatrix} I - \tilde{D}_1 (M_{\tilde{H}} \tilde{D}_1)^+ + M_{\tilde{H}} \tilde{D}_1^+ \\ (I - M_{\tilde{H}} \tilde{D}_1 (M_{\tilde{H}} \tilde{D}_1)^+ + M_{\tilde{H}} \tilde{D}_1) \end{bmatrix}$ and $U^{-1} = \begin{bmatrix} I - \tilde{D}_1 (M_{\tilde{H}} \tilde{D}_1)^+ + M_{\tilde{H}} \tilde{D}_1^+ \\ (I - M_{\tilde{H}} \tilde{D}_1 (M_{\tilde{H}} \tilde{D}_1)^+ + M_{\tilde{H}} \tilde{D}_1) \end{bmatrix}$. Thus, the equations governing the dynamics of the system, in the new coordinates $z = Tx$ and $\tilde{y} = UM_{\tilde{H}}x$, take the form

$$\begin{aligned}
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{w}_{ex,1} \end{bmatrix} \\
&+ \begin{bmatrix} (M_{\tilde{H}} \tilde{D}_1)^+ + M_{\tilde{H}} \\ (M_{\tilde{H}} \tilde{D}_1) \end{bmatrix} \tilde{D}_2 \tilde{F}_2^+ y \\
\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} &= \begin{bmatrix} C_1 z_1 \\ z_2 \end{bmatrix}
\end{aligned}$$ (15a)

which corresponds to triple $(\tilde{A} - \tilde{D}_2 \tilde{F}_2^+ \tilde{C}, M_{\tilde{H}} C, \tilde{D}_1)$, construct an asymptotic observer based on the design described in this subsection, that is,

(a) with the change of coordinates $z = Tx$ and $\tilde{y} = UM_{\tilde{H}}x$, system (15) is obtained;
(b) since $y_2 \equiv z_2$, state $z_1$ can be observed from output $\tilde{y}_1$ using a usual Luenberger observer:

$$\dot{z}_1 = A_1 \tilde{z}_1 + L_1 (\tilde{y}_1 - C_1 \tilde{z}_1) + A_2 \tilde{y}_2 + \tilde{D}_2^+ \tilde{D}_2 \tilde{F}_2^+ y$$ (16)

then the observer for the original system is designed as

$$\dot{x}_{ex} = T^{-1} \begin{bmatrix} \dot{z}_1 \\ \dot{y}_2 \end{bmatrix}$$ (17)

Thus, the main result can be summarised through the following theorem.

**Theorem 4.2:** Assuming that $(A, C, D, F)$ is strongly detectable and $L$ is designed so that $(A_1 - L_1 C_1)$ is Hurwitz, then observer $\dot{x}_{ex}$ designed using the two-step procedure given above, converges asymptotically to the state vector $x_{ex}$.

**Proof:** It comes directly from Lemma 4.1 and comparing (15a) with (16). □

**Remark 3:** For the case when $K(t) = K$, we can compare the precision of the observer proposed in this article with respect to already published observers that are based on the Molinari’s algorithm for the design of the observer and that use high-order sliding modes too. For instance, in the observer designed in Bejarano et al. (2007) the precision of the observer with respect to a Lebesgue measurable and bounded noise appearing in the system output $||y(t)|| \leq \varepsilon$ is of order $O(\varepsilon^2)$ where $k$ is the number of times to derive in order to reconstruct the extended state vector $(x, w)$. Meanwhile, the precision with respect to a sample time $\tau$ is of order $O(\tau^2)$. These errors are due to the proper design of the observer which is based on the consecutive use of a second-order differentiator (super-twisting algorithm).

The observer designed in Fridman et al. (2007) has a precision of order $O((\varepsilon)^{\frac{1}{r}})$ w.r.t. noise and of order $O(\varepsilon^{\frac{1}{r}})$ w.r.t. the sample time $\tau$, where $r$ is the maximum of the terms of the vector of relative degrees of the system output with respect to the unknown inputs.

At difference with the previous two observers, the observer proposed here needs to use only once an exact sliding mode differentiator of order $(\tilde{n}_H - 1)$, which in
general is less than the constant \( k \) defined above. Thus according to Levant (2003), the precision of the observer w.r.t. noise is of order \( O(e^{m/2}) \) and w.r.t. sample time is \( O(\tau) \).

5. Simulations

Example 5.1 (Bounded unknown inputs): Let the matrices of system (1) take the following values:

\[
A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

For simulation purposes, the unknown inputs are \( w^T = [w_1 \ w_2 \ w_3] \), with \( w_1 = -2\sqrt{\sin(3\tau)} + 0.4\sqrt{\cos(9.5\tau)} + 2.2 \), \( w_2 = 1 + 0.5|\sin(8\tau)| - 1.5\cos(2\tau) \) and \( w_3 \) being a sum of the signal \( 0.2 \sin(2.5\tau) \) and a continuous sawtooth wave of amplitude 1 and frequency 17.5.

In this case the unknown inputs are not differentiable that is why \( x_{\text{ex}} = x \). We have that rank \( M_2 = 3 \) and rank \( M_3 = 4 \). That is why to estimate \( x \) it is necessary to differentiate twice. By (10), the states can be represented in the following form:

\[
\dot{x}_1(t) = \frac{d^2}{dt^2} \left( \int_0^t y_1(\tau)d\tau \right) = \ddot{y}_1(t)
\]

\[
\dot{x}_2(t) = \frac{d^2}{dt^2} \left( \int_0^t y_2(\tau)d\tau - y_1(t) \right)
\]

\[
\dot{x}_3(t) = y_1(t)
\]

\[
\dot{x}_4(t) = y_2(t)
\]

Thus, for the design of the observer \( \hat{x} \), we design \( H(t) \) according to (11) and \( \hat{x} \) according to (5),

\[
H_1(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 & (y_1^T) & y_1^T & y_2^T \end{bmatrix}^T - \dddot{x}_1^2
\]

\[
H_2(t) = \begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 & (y_1^T) & y_1^T & y_2^T \end{bmatrix}^T - \dddot{x}_2^2
\]

The extended state vector is \( x_{\text{ex}} = (x^T \ w^T) \), and \( \tilde{A} = \begin{bmatrix} A \ 0 \end{bmatrix}, \tilde{C} = \begin{bmatrix} C & F \end{bmatrix}, \tilde{D} = \begin{bmatrix} 0 & I \end{bmatrix} \) and \( \tilde{F} = 0 \). It can be verified that rank \( M_1 = 11 \), and that the matrices \( M_4 \) to \( M_5 \) have rank equal to 4, 7, 9, 10 and 12.

Example 5.2 (Unbounded unknown inputs): Consider that the matrices of the system are the following:

\[
A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ D_{11} = D_{33} = D_{93} = D_{82} = 1, \text{ otherwise } D_{ij} = 0 \\ C_{11} = C_{25} = C_{39} = C_{48} = 1, \text{ otherwise } C_{ij} = 0 \\ F_{43} = 1, \text{ otherwise } F_{ij} = 0 
\]

The unknown inputs used for the simulations are \( w = [w_1 \ w_2 \ w_3] \), where \( w_1 = -0.2t + \sin(2t + 3) + 0.3\cos(2t) + 0.1t^2 - 6, w_2 = 0.6t \sin(0.6t) - 0.4 \sin(0.4t^2) - 0.5t + 4, w_3 = 0.1 + 0.5t - 0.5\cos(2t) - 0.2t^2 \).

The next step is to use the differentiator according to (12). Since for the first state it is enough to differentiate once (this exemplifies Remark 1), we redefine \( H_1 = y_1 - \int_0^t x_1(\tau)d\tau \), and use a first-order differentiator instead of the one of second-order; thus we have that \( \dot{x}_1 = z_{1,1} \). To estimate the second state necessarily we have to use a second-order differentiator; therefore, \( \dot{x}_2 = z_{2,2} \).

The states are shown in Figure 1, where the trajectories of the observer designed using a super-twisting observer (STO) (Bejarano et al. 2007) are compared with the trajectories of the observer designed in this article (HODO), with a sample time of \( 10^{-3} \). The observation error is depicted in Figure 2.

Figure 1. States \( x_1 \) and \( x_2 \) and its estimates: STO (dashed), HODO (dotted).
11, respectively, which means that a derivative of fourth order should be estimated online if a differentiation process is followed to estimate the state vector $x_{\text{ex}}(t)$. On the other hand, in order to satisfy the Hautus condition, only a second-order derivative must be carried out (rank $M_3D = \text{rank } D$), and so the observer proposed in (17) can be designed.

Thus, the first step is to reconstruct a new output $\tilde{y} = M_3x_{\text{ex}}$. According to (9), we have that

$$\tilde{y}_1 = \frac{d^2}{dt^2} \left(0.36 y_1 - 0.74 y_2 + 0.37 y_3 - 0.42 y_4^{[1]}\right)$$

$$\tilde{y}_2 = \frac{d^2}{dt^2} y_1^{[1]} = \frac{d}{dt} y_1$$

$$\tilde{y}_3 = \frac{d^2}{dt^2} y_2^{[1]} = \frac{d}{dt} y_2$$

$$\tilde{y}_4 = \frac{d^2}{dt^2} y_3^{[1]} = \frac{d}{dt} y_3$$

$$\tilde{y}_5 = \frac{d^2}{dt^2} \left(-0.50 y_1 - 0.18 y_2 + 0.68 y_3 + 0.48 y_4^{[4]}\right)$$

$$\tilde{y}_6 = \frac{d^2}{dt^2} y_1^{[2]} = y_1$$

Thus, we have that only for $\tilde{y}_1$ and $\tilde{y}_5$ we need to use a second-order differentiator, meanwhile for $\tilde{y}_6$ we do not require any differentiation. Thus, $H(t)$ is defined as follows:

$$H_1 = 0.36 y_1 - 0.74 y_2 + 0.37 y_3 - 0.42 y_4^{[1]} - M_3(1)\tilde{x}^{[2]}$$

$$H_2 = y_1 - M_3(2)\tilde{x}^{[1]}$$

$$H_3 = y_2 - M_3(3)\tilde{x}^{[1]}$$

$$H_4 = y_3 - M_3(4)\tilde{x}^{[1]}$$

$$H_5 = \frac{d^2}{dt^2} \left(-0.50 y_1 - 0.18 y_2 + 0.68 y_3 + 0.48 y_4^{[4]}\right)$$

$$- M_3(5)\tilde{x}^{[2]}$$

and so $H_1$ and $H_5$ are to be differentiated using a second-order differentiator of the form (12), we use a first-order differentiator to obtain the derivative of $H_2$ to $H_4$. Thus, each term of $\mathbf{\tilde{y}}$ takes the form

$$\mathbf{\tilde{y}}_1 = z_{1,2} + M_3(1)\tilde{x},$$

$$\mathbf{\tilde{y}}_2 = z_{2,1} + M_3(2)\tilde{x},$$

$$\mathbf{\tilde{y}}_3 = z_{3,1} + M_3(3)\tilde{x},$$

$$\mathbf{\tilde{y}}_4 = z_{4,1} + M_3(4)\tilde{x},$$

$$\mathbf{\tilde{y}}_5 = z_{5,2} + M_3(5)\tilde{x},$$

where $M_3(i)$ is the $i$-th row of $M_3$. Once the procedure to obtain the new output has been designed, then the observer of the state $x_{\text{ex}}$ is designed using (17).

The sample time used in the simulation is $10^{-4}$. Figures 3–5 show the observation error for the vector...
6. Conclusions

We have proposed an observer for strongly detectable systems using the least number of derivatives possible, i.e. using the proposed methodology, differentiation is done only when structurally it is required. We have proposed an observation structure that allows to use the high-order differentiator proposed by Levant (2006), with the advantages that it offers. Thus, in this sense, we minimise the observation error due to the use of the numerical differentiation. Moreover, the proposed observation scheme, in general, does not require the unknown input to be uniformly bounded, but only requires boundedness (not uniformly) of at least one of their high-order derivatives.

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Notes

1. Clearly all results obtained here can be applied to the more general sort of systems of the form \( \dot{x} = Ax + Bu + Dw, \quad y = Cx + Eu + Fw \). In that case it is enough to construct the system \( \dot{z} = A\tilde{z} + Bu, \quad y = C\tilde{z} + Eu \), and define \( \tilde{x} = x - z \). Thus, we get a new system \( \dot{\tilde{x}} = A\tilde{x} + Dw \) and \( \tilde{y} = C\tilde{x} + Fw \), which belongs to the sort of systems considered in (1).

2. Recall that \( \Sigma \) is strongly detectable if \( \dot{y} = 0 \) implies \( x(t) \to 0 \). Some times it is said that the fourfold \((A, C, D, F)\) is strongly detectable meaning that this property is fulfilled for system \( \Sigma \), associated to the fourfold. It was proven in Hautus (1983) that \( \Sigma \) is strongly detectable if, and only if, the set of zeros of \((A, C, D, F)\) lies within the interior of the left half side of the complex plane.

References


