On the linear relaxation of the $p$-median problem

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**A R T I C L E   I N F O**

Article history:
Received 22 April 2009
Received in revised form 20 October 2010
Accepted 16 December 2010
Available online 15 January 2011

Keywords:
$p$-median problem
Uncapacitated facility location problem
Odd cycle

**A B S T R A C T**

We study a well-known linear programming relaxation of the $p$-median problem. We give a characterization of the directed graphs for which this system of inequalities defines an integral polytope. As a consequence, we obtain that the $p$-median problem is polynomial in that class of graphs. We also give an algorithm to recognize these graphs.

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1. Introduction

One of the most relevant problems in location theory is the $p$-median problem ($p$MP), in which we want to select exactly $p$ locations (nodes) in a graph and assign the remaining nodes to one of those selected at a minimum total selection plus assignment cost. When the number of locations is not specified, this is called the uncapacitated facility location problem (UFLP). These problems have numerous applications in computer science as well as in operations research, including location of bank accounts [1], placement of web proxies in a network [2], and semi-structured data bases [3,4].

The study of the $p$-median problem goes back to the sixties, where the first heuristics and linear programming relaxations were proposed. Indeed, the first heuristic methods for $p$MP were developed by Kuehn and Hamburger [5] and Maranzana [6], while Balinski [7] presented the natural linear programming relaxations for UFLP, which was adapted by ReVelle and Swain [8] to $p$MP. Since then, a vast number of solution methods combining heuristics with the natural linear programming relaxation have been proposed. We refer the reader to [9] for a detailed account of references.

Both UFLP and $p$MP are NP-hard in general [10]. However, there are polynomially solvable cases such as when the underlying graph is a tree [10]. Furthermore, there has also been a significant body of work in the approximation algorithms community. In particular, algorithms with increasingly better performance guarantee have been presented. Currently the best such algorithms are due to Charikar et al. [11], Charikar and Guha [12], and Jain and Vazirani [13]. Interestingly, many of the known approximation algorithms are based on rounding the optimal fractional solution of the natural linear programming relaxation for $p$MP.

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The latter considerations motivate the study of the natural linear programming relaxation of $p$MP which we now present. A better understanding of this relaxation may lead to better algorithms and heuristics for the problem. Formally, in the classic $p$-median problem we are given a simple directed graph $G = (V, A)$, and costs $c(u, v)$ and $w(v)$, for each arc $(u, v) \in A$ and node $v \in V$. The problem consists in selecting $p$ nodes, usually called centers, and then assign, through a single arc, each non-selected node to a selected one. The goal is thus to select exactly $p$ nodes minimizing the total cost of the selected nodes

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**10.1016/j.disopt.2010.12.002**
plus the assignment cost. Thus, if we associate a variable \(x(u, v)\) with each arc \((u, v)\in A\) and a variable \(y(v)\) with each \(v\in V\), the linear programming relaxation of \(pMP\) is

\[
\min \sum_{(u, v)\in A} c(u, v)x(u, v) + \sum_{u\in V} w(u)y(u),
\]

\[
\sum_{v\in V} y(v) = p,
\]

\[
\sum_{v:(u, v)\in A} x(u, v) + y(u) = 1 \quad \forall u \in V,
\]

\[
x(u, v) \leq y(v) \quad \forall (u, v) \in A,
\]

\[
y(v) \geq 0 \quad \forall v \in V,
\]

\[
x(u, v) \geq 0 \quad \forall (u, v) \in A.
\]

Analogously (1) and (3)−(6) give the natural linear programming relaxation of the UFLP proposed by Balinski [7].

Our main result in this paper is a complete characterization of the graphs for which such a linear program defines an integral polytope. Therefore the \(p\)-median problem may be solved in polynomial time is such graphs. In this direction there are few known results. Brimberg and ReVelle [14] prove, for a very special type of objective function, that when possible locations for centers are specified in advance there is an optimal integral solution. On the other hand, Baiou and Barahona [15] give a complete description of the polytope when the graph does not contain a certain structure, called a \(Y\). An alternative proof of this result was recently obtained by Stauffer [16], by establishing an interesting connection to the matching polytope. Furthermore, Avella and Sassano [17] investigated the facets of the \(p\)-median problem using the previous linear program as starting point, and established close relations with the stable set polytope. The facets of the \(p\)-median polytope have also been studied by de Farias [18], whereas there are many papers dealing with facets for the uncapacitated facility location polytope including [19−23]. Related polyhedral results but in a slightly different model, sometimes called \(K\)-median, have been obtained, and can be found in [24].

Our main contributions. Denote by \(P_p(G)\) the polytope defined by (2)−(6), and let \(pMP(G)\) be the convex hull of \(P_p(G) \cap \{0, 1\}^{A+V}\). Also let \(P(G)\) be the polytope defined by (3)−(6). In this paper we characterize all directed graphs such that \(P_p(G) = pMP(G)\) for any integer \(p\). Clearly, \(P_p(G) \neq pMP(G)\) in general. Indeed, Fig. 1 depicts four graphs for which there is a fractional extreme point of \(P_p(G)\) with \(p = |V| − 2\). The numbers close to the nodes correspond to the \(y\) variables, all the arc variables are equal to \(\frac{1}{2}\).

Furthermore, there is another configuration where \(P_p(G)\) is not integral. To illustrate it we need some definitions and notations. A simple cycle \(C\) is an ordered sequence \(v_0, a_0, v_1, a_1, \ldots, a_{t−1}, v_t = v_0\), where all \(v_i\)’s, for \(i = 0, \ldots, t − 1\), are distinct and such that for \(0 \leq i < t\) − 1: either \(v_i\) is the tail of \(a_i\) and \(v_{i+1}\) is the head of \(a_i\), or \(v_i\) is the head of \(a_i\) and \(v_{i+1}\) is the tail of \(a_i\). Let \(V(C)\) and \(A(C)\) denote the nodes and the arcs of \(C\), respectively. By setting \(a_0 = a_0\), we associate with \(C\) three more sets as below.

- We denote by \(\hat{C}\) the set of nodes \(v_i\) such that \(v_i\) is the head of \(a_{i−1}\) and also the head of \(a_i\), \(1 \leq i < t\).
- We denote by \(\hat{C}\) the set of nodes \(v_i\) such that \(v_i\) is the tail of \(a_{i−1}\) and also the tail of \(a_i\), \(1 \leq i < t\).
- We denote by \(\hat{C}\) the set of nodes \(v_i\) such that either \(v_i\) is the head of \(a_{i−1}\) and also the tail of \(a_i\), or \(v_i\) is the tail of \(a_{i−1}\) and also the head of \(a_i\), \(1 \leq i < t\).

Notice that \(|\hat{C}| = |\hat{C}|\). A cycle will be called \(g\)-odd if \(|\hat{C}| + |\hat{C}|\) is odd, otherwise it will be called \(g\)-even. A cycle \(C\) with \(V(C) = \hat{C}\) is a directed cycle. The notion of \(g\)-odd (\(g\)-even) cycle generalizes the notion of odd (even) directed cycle.

**Definition 1.** A simple cycle is called a \(Y\)-cycle if for every \(v \in \hat{C}\) there is an arc \((v, \bar{v})\) in \(A\), where \(\bar{v}\) is in \(V \setminus \hat{C}\).
Note that when \( \hat{C} = \emptyset \), then \( C \) is a directed cycle and also a \( Y \)-cycle. A \emph{chain} is defined in a similar way to a cycle, but without asking the condition \( v_0 = v_t \). For a chain from \( v_0 \) to \( v_t \), the nodes \( v_1, \ldots, v_{t-1} \) are called \emph{internal}.

In Fig. 2 we show a fractional extreme point of \( P_p(G) \) with \( p = 4 \), different from those given in Fig. 1. It consists of a \( g \)-odd \( Y \)-cycle with an arc having both its endnodes not in the cycle. The values of the arc variables are all \( \frac{1}{2} \) and those corresponding to the nodes are shown near each node in the figure.

Interestingly, it turns out that the configurations that should be forbidden in order to have an integral polytope are exactly those in Figs. 1 and 2. Now, we are ready to present the main result of this paper.

**Theorem 2.** Let \( G = (V, A) \) be a directed graph, then \( P_p(G) \) is integral for any integer \( p \) if and only if

(C1) it does not contain any of the graphs \( H_1, H_2, H_3 \) nor \( H_4 \) of Fig. 1, as a subgraph, and

(C2) it does not contain a \( g \)-odd \( Y \)-cycle \( C \) and an arc \((u, v)\) with neither \( u \) nor \( v \) in \( V(C) \).

A variation of the \( p \)-MP that is common in the literature is when \( V \) is partitioned into \( V_1 \) and \( V_2 \). The set \( V_1 \) corresponds to the customers, and the set \( V_2 \) corresponds to the potential facilities. Each customer in \( V_1 \) should be assigned to an opened facility in \( V_2 \). This is obtained by considering \( A \subseteq V_1 \times V_2 \), and using the following linear programming relaxation

\[
\sum_{v \in V_2} y(v) = p, \tag{7}
\]

\[
\sum_{(u, v) \in A} x(u, v) = 1 \quad \forall u \in V_1, \tag{8}
\]

\[
x(u, v) \leq y(v) \quad \forall (u, v) \in A, \tag{9}
\]

\[
y(v) \leq 1 \quad \forall v \in V_2, \tag{10}
\]

\[
x(u, v) \geq 0 \quad \forall (u, v) \in A. \tag{11}
\]

We call this the \emph{bipartite case}. Here we also characterize the bipartite graphs for which (7)–(11) define an integral polytope.

**Sketch of the Proof.** The proof of Theorem 2 is divided into two parts. First, we prove the result for graphs with no \( g \)-odd \( Y \)-cycle. This proof uses the key Lemma 11. In this lemma we show that we cannot have a fractional extreme point \( \bar{z} \) of \( P_p(G) \) where \( \bar{z}(u, v) = \bar{z}(v) \) for each arc \((u, v)\) such that \( \delta^+(v) \neq \emptyset \). The proof of the lemma proceeds by induction on the number of pairs of nodes \{u, v\} having one arc from u to v and another from v to u. The long part in the proof is the base case of the induction. In particular, it involves the proof of Theorem 2 when restricted to directed graphs (i.e., a graph in which if \((u, v)\) belongs to the arc-set then \((v, u)\) does not; see [25]). The proof for oriented graphs, also goes by induction, but on the number of \( Y \) configurations, where \( Y \) is some basic configuration in the graph. It consists, roughly, of two arcs entering a given node, and one arc leaving this same node. When this number is zero, the graph is \( Y \)-free and the result in [15] applies readily, as it states that \( P_p(G) \) is integral on \( Y \)-free graphs with no odd directed cycle, though significant further work is needed to complete the proof. In the second part, we prove Theorem 2 when \( G \) contains a \( g \)-odd \( Y \)-cycle. To this end, we need to show that the node-set of all \( g \)-odd \( Y \)-cycles must coincide. \( \square \)

The paper is organized as follows. Section 2 contains preliminary definitions and notations. The graphs that satisfy conditions (C1) and (C2) of Theorem 2 with no \( g \)-odd \( Y \)-cycle are considered in Section 3 and those containing a \( g \)-odd \( Y \)-cycle are studied in Section 4. Section 5 gives the proof of Theorem 2. The bipartite case is studied in Section 6. The bipartite graphs for which the system defined by (7)–(11) is integral are characterized. In Section 7 we show how to test in polynomial time conditions (C1) and (C2) of Theorem 2. Finally Section 8 concludes this paper with an application of Theorem 2 on undirected graphs.
2. Preliminaries

Let \( G = (V, A) \) be a simple directed graph. For \( W \subseteq V \), we denote by \( \delta^+(W) \) the set of arcs \((u, v)\) in \( A \), with \( u \in W \) and \( v \in V \setminus W \). Also we denote by \( \delta^-(W) \) the set of arcs \((u, v)\), with \( v \in W \) and \( u \in V \setminus W \). We write \( \delta^+(v) \) and \( \delta^-(v) \) instead of \( \delta^+(\{v\}) \) and \( \delta^-(\{v\}) \). If there is a risk of confusion relative to which graph is considered, we use \( \delta_G^+ \) and \( \delta_G^- \). Let \( u \) be a

Let \( I : V \cup A \rightarrow \mathbb{Z} \) be a labeling function that associates integer values to each node and arc of \( G \). A vector \((x, y) \in P_p(G)\) will be denoted by \( z \), i.e., \( z(u) = y(u) \) for all \( u \in V \) and \( z(u, v) = x(u, v) \) for all \((u, v) \in A\). Given a vector \( z \) and a labeling function \( I \), we define a new vector \( z_I \) from \( z \) as follows.

\[
z_I(u) = z(u) + I(u) \epsilon, \quad \text{for all} \; u \in V,
\]

\[
z_I(u, v) = z(u, v) + I(u, v) \epsilon, \quad \text{for all} \; (u, v) \in A,
\]

where \( \epsilon \) is a sufficiently small positive scalar. We say that an arc \((u, v)\) is tight for \( z \in P_p(G) \) if \( z(u, v) = z(v) \).

**Observation 3.** When we assign labels to some nodes and arcs without specifying the labels of the remaining nodes and arcs, it means that they are assigned the label 0.

Let \( C = \nu_0, a_0, v_1, a_1, \ldots, a_{t-1}, v_t \) be a \( g \)-even cycle, not necessarily a \( Y \)-cycle. The following labeling procedure will assign labels to the nodes and arcs in \( C \).

The labeling procedure.

- If \( C \) is a directed cycle then, set \( l(v_0) := 1; l(a_0) := -1 \). Otherwise, assume \( v_0 \in \mathcal{C} \) and set \( l(v_0) := 0; l(a_0) := 1 \).
- For \( i = 1 \) to \( t-1 \) do the following:
  - If \( v_i \) is the head of \( a_{i-1} \) and is the tail of \( a_i \), then \( l(v_i) := l(a_{i-1}) \), \( l(a_i) := -l(v_i) \).
  - If \( v_i \) is the head of \( a_{i-1} \) and is the head of \( a_i \), then \( l(v_i) := l(a_{i-1}) \), \( l(a_i) := l(v_i) \).
  - If \( v_i \) is the tail of \( a_{i-1} \) and is the head of \( a_i \), then \( l(v_i) := -l(a_{i-1}) \), \( l(a_i) := l(v_i) \).
  - If \( v_i \) is the tail of \( a_{i-1} \) and is the tail of \( a_i \), then \( l(v_i) := 0 \), \( l(a_i) := -l(a_{i-1}) \).

Next we give two useful properties of the labeling procedure. The first is given in the following observation and is easy to see.

**Observation 4.** If \( C \) is a directed even cycle then \( l(a_{t-1}) = l(v_0) = 1 \) and \( \sum l(v_i) = 0 \).

The second property is given in the following lemma and it concerns cycles that are not necessarily directed.

**Lemma 5.** If \( C \) is a \( g \)-even cycle of size \( t \), then \( l(a_{t-1}) = -l(a_0) = -1 \) and \( \sum l(v_i) = 0 \).

**Proof.** Let \( v_{j(0)}, v_{j(1)}, \ldots, v_{j(k)} \) be the ordered sequence of nodes in \( \mathcal{C} \), with \( v_{j(0)} = v_{j(k)} \). A chain in \( C \)

\[
v_{j(0)}, v_{j(1)}, \ldots, a_{j(i+1)-1}, v_{j(i+1)}
\]

from \( v_{j(i)} \) to \( v_{j(i+1)} \) will be called a segment and denoted by \( S_i \). A segment is odd (resp. even) if it contains an odd (resp. even) number of arcs. Let

\[
l(S_i) = \sum_{v \in S_i \cap V} l(v).
\]

Let \( r \) be the number of even segments and \( r' \) the number of odd segments. We have that \( r + r' = |\mathcal{C}| \), and since the parity of \( |\mathcal{C}| \) is equal to the parity of \( r' \), we have that \( r' = |\mathcal{C}| \) is even. Therefore \( r = |\mathcal{C}| - r' \) is also even. The labeling has the following properties.

(a) If \( S_i \) is odd then \( l(a_{j(i)}) = -l(a_{j(i+1)-1}) \).
(b) If \( S_i \) is even then \( l(a_{j(i)}) = l(a_{j(i+1)-1}) \).
(c) If \( S_i \) is odd then \( l(S_i) = 0 \).
(d) If \( S_i \) is even then \( l(S_i) = l(a_{j(i)}) \).
(e) Let \( S_1, \ldots, S_r \) be the ordered sequence of even segments in \( C \). Then \( l(S_i) = -l(S_{i+1}) \), for \( i = 1, \ldots, r-1 \).

Since there is an even number of even segments, properties (a) and (b) imply \( l(a_0) = l(a_{p-1}) \). Properties (c) and (e) imply \( \sum l(v_i) = 0 \). \( \square \)

Given a fractional extreme point \( z \in P_p(G) \), the labeling procedure is used to produce a new vector \( z_I \) from \( z \). The observation and the lemma above are useful to show that \( z_I \) is in fact in \( P_p(G) \) and satisfies with equality the constraints of \( P_p(G) \) that are satisfied with equality by \( z \). This contradicts the fact that \( z \) is an extreme point. This is the contradiction we use in several lemmas.

From now on the graphs we consider satisfy condition (C1) of Theorem 2. In this context, a \( Y \)-cycle may be redefined as follows.
Assumethat (a1) and (a2) are true. Let

Definition 7. Let \( G = (V, A) \) be a directed graph that does not contain any of the graphs of Fig. 1 as a subgraph. In this case, a simple cycle \( C \) is a Y-cycle if for every \( v \in \hat{C} \) at least one of the following holds:

(i) there exists an arc \((v, \bar{v}) \notin A(C), \bar{v} \notin \bar{V}(C)\), or

(ii) there exists an arc \((v, \bar{v}) \notin A(C), \bar{v} \in \hat{C} \) and \( v \) is one of the two neighbors of \( v \) in \( C \).

For a simple cycle \( C \), denote by \( \hat{C}_0 \) the set of nodes in \( \hat{C} \) that satisfy condition (i) of the above definition. Notice that we may have nodes in \( \hat{C} \) that satisfy both (i) and (ii).

Definition 7. Let \( C \) be a Y-cycle in a directed graph \( G = (V, A) \). A node \( v \in V(C) \) is called a blocking node, (see Fig. 3), if one of the following holds:

(i) \( v \in \hat{C}_0, (v, u) \in A(C), (u, v) \in A \setminus A(C) \) and \( u \in \hat{C} \), or

(ii) \( v \in \hat{C}_0, (u, v) \in A(C), (v, u) \in A \setminus A(C), (v, w) \in A \setminus A(C) \) and both \( u \) and \( w \) are in \( \hat{C} \).

Lemma 8. Let \( G = (V, A) \) be a simple directed graph satisfying condition (C1) of Theorem 2. If the following assumptions hold:

(a1) \( G \) admits a g-even Y-cycle \( C \) of size greater than or equal to three with no blocking node, and

(a2) \( P_p(G) \) contains a vector \( \bar{z} \) with

\[
\begin{align*}
0 &< \bar{z}(v) < 1 \text{ for each node } v \in \hat{C} \cup \hat{C}_0; \\
0 &< \bar{z}(u, v) < 1 \text{ for each arc } (u, v) \in A(C); \\
\text{and } 0 &< \bar{z}(u, v) < 1 \text{ for each arc } (u, v) \text{ with } u \in \hat{C},
\end{align*}
\]

then \( \bar{z} \) is not an extreme point of \( P_p(G) \).

Proof. Assume that (a1) and (a2) are true. Let

\[ C = v_0, a_0, v_1, a_1, \ldots, a_{l-1}, v_l \]

be a g-even Y-cycle with no blocking node.

Assign labels to the arcs and nodes of \( C \) following the labeling procedure above. Extend this labeling as follows: for each node \( v_i \in \hat{C} \) if there is an arc \((v_i, u) \in A \setminus A(C) \) with \( u \in \hat{C}, \) then \( l(v_i, u) := -l(v_i) \). Notice that \( u = v_{i-1} \) or \( u = v_{i+1} \) and since \( v_i \) is not a blocking node, such an arc is unique if it exists. If there is no such arc, by the definition of a Y-cycle we must have an arc \((v_i, u) \in A \setminus A(C) \) with \( u \notin V(C) \), in this case also set \( l(v_i, u) := -l(v_i) \). Now assign the label 0 to each node and arc with no label. Call this labeling function \( l \).

Next we will show that \( \bar{z}_i \) satisfies with equality each constraint among (2)–(6) that is satisfied with equality by \( \bar{z} \). Since \( \bar{z} \neq \bar{z}_i \), this contradicts the fact that \( \bar{z} \) is an extreme point of \( P_p(G) \).

Assumption (a2) shows that for the nodes and arcs that received a non-zero label, their corresponding variables take a fractional value. This implies that each inequality among (5) and (6) that is satisfied with equality by \( \bar{z} \), is also satisfied with equality by \( \bar{z}_i \). Observation 4 and Lemma 5 imply \( \sum l(v_i) = 0 \), in both cases, whether \( C \) is directed or not. Hence equality (2) is satisfied by \( \bar{z}_i \). When \( C \) is directed, equalities (3) are satisfied by \( \bar{z}_i \) by definition. When it is not directed, by definition these equalities are satisfied for every node \( v \neq v_0 \). By Lemma 5 we have \( l(a_{l-1}) = -l(a_0) \). This shows that equality (3) with respect to \( v_0 \) is also satisfied by \( \bar{z}_i \).

Now we will show that every arc that is tight for \( \bar{z} \) is also tight for \( \bar{z}_i \). Let \((u, v) \in A(C)\), the labeling procedure gives \( l(v) = l(u, v) \), hence \( \bar{z}_i(u, v) = \bar{z}_i(v) \). Also, for every arc \((u, v) \in A \setminus A(C) \) with \( u, v \notin V(C) \), we have \( l(u, v) = 0 \) and \( l(u) = l(v) = 0 \). Let us examine the three other cases.

(i) \((u, v) \in A \setminus A(C) \), with \( u \) and \( v \) in \( V(C) \). We have three sub-cases.

- If \( v \in \hat{C} \), then \( l(v) = 0 \) and \( l(u, v) = 0 \).
Let $v \in \check{C}$, since $G$ does not contain any of the graphs $H_1$, $H_3$ and $H_4$ as a subgraph, the nodes $u$ and $v$ must be consecutive in $C$. So $(u, v) \in A(C)$. By assumption $(a1)$, $v$ is not a blocking node, so $u$ must be in $\check{C}$. Let $u'$ be the other node of the cycle adjacent to $u$. The node $u$ is not a blocking node. Thus if $(u, u') \in A$, then $u' \in \check{C}$. Hence when extending the labeling of $C$, we get $l(u, v) = -l(u)$ which is equal to $l(v)$ by the labeling procedure of $C$.

- The case $v \in \check{C}$ cannot exist since $G$ is simple and does not contain neither $H_2$ nor $H_4$ as a subgraph.

(ii) $(u, v) \in A \setminus A(C)$, with $u \in V(C)$ and $v \notin V(C)$. By definition $l(v) = 0$. If $u \in (\check{C} \setminus \check{C})$, then $l(u, v) = 0$. And if $u \in \check{C}$, since $G$ does not contain $H_1$, $H_2$ or $H_4$ as a subgraph, $v$ must be a sink node, so $\tilde{z}(u, v) < \tilde{z}(v) = 1$.

(iii) $(u, v) \in A \setminus A(C)$, with $u \notin V(C)$ and $v \in V(C)$. The node $v$ must be in $\check{C}$, otherwise one of the graphs $H_1$, $H_2$ or $H_4$ exists in $G$. Thus by the labeling procedure, $l(v) = 0$; and when extending this labeling $(u, v)$ takes the label 0 since $u \notin V(C)$. □

We will follow this section with the following basic polyhedral fact.

**Observation 9.** Let $P$ and $P'$ be two polytopes and $P' = \{x \in P | cx = d\}$. Let $\hat{x}$ be an extreme point of $P'$. If $\hat{x}$ is not an extreme point of $P$, then $\hat{x}$ is a convex combination of two different extreme points of $P$. Moreover, if the extreme points of $P$ are all 0–1 vectors, then all the components of $\hat{x}$ are in $\{0, 1, \alpha, 1-\alpha\}$, for some number $\alpha \in [0, 1]$.

### 3. Graphs with no $g$-odd $Y$-cycle

In this section we prove the following simplified version of our main result.

**Theorem 10.** If $G = (V, A)$ is a simple directed graph that satisfies (C1) and (C2) and has no $g$-odd $Y$-cycle, then $P_p(G)$ is integral for any integer $p$.

In this section we assume that the graph $G = (V, A)$ is simple, directed, with no $g$-odd $Y$-cycle and satisfies condition (C1) of Theorem 2, that is, it does not contain any of the graphs $H_1$, $H_2$, $H_3$ or $H_4$ of Fig. 1 as a subgraph.

The proof of Theorem 10 uses the following lemma.

**Lemma 11.** $P_p(G)$ does not contain a fractional extreme point $\tilde{z}$ where $\tilde{z}(u, v) = \tilde{z}(v)$, for all $(u, v)$ with $v$ not a sink node.

This section is organized as follows. In Section 3.1, we will prove Lemma 11. Section 3.2 gives the proof of Theorem 10.

But first, let us give some useful implicit properties of the graph $G = (V, A)$ defined above and its associated polytope $P_p(G)$.

A bidirected chain $P$ of $G = (V, A)$ is an ordered sequence of nodes $P = v_1, \ldots, v_t$, where $(v_i, v_{i+1})$ and $(v_{i+1}, v_i)$ belong to $A$, for $i = 1, \ldots, t - 1$. The size of $P$ is $t$. As for a chain, a node $v_i$ of $P$ is called internal if $i \notin \{1, t\}$.

**Observation 12.** If $P = v_1, \ldots, v_t$ is a bidirected chain of $G$, then for each internal node $v_i$ we have $\delta^-(v_i) = \{(v_{i-1}, v_i), (v_{i+1}, v_i)\}$.

Let $\hat{z}$ be a fractional extreme point of $P_p(G)$.

**Lemma 13.** We may assume that $\tilde{z}(u, v) > 0$ for all $(u, v) \in A$.

**Proof.** Let $G'$ be the graph obtained after removing all arcs $(u, v)$ with $\tilde{z}(u, v) = 0$. The graph $G'$ has the same properties as $G$, i.e., it does not contain any of the graphs in Fig. 1 nor a $g$-odd $Y$-cycle. Let $\tilde{z}'$ be the restriction of $\tilde{z}$ on $G'$. Then $\tilde{z}'$ is a fractional extreme point of $P_p(G')$. □

**Lemma 14.** We may assume that $\tilde{z}(v) > 0$ for all $v \in V$ with $|\delta^-(v)| > 1$.

**Proof.** It is straightforward from Lemma 13 and constraints (4). □

**Lemma 15.** Let $(u, v)$ and $(v, w)$ be two arcs in $G$. Then $\tilde{z}(v, w)$, $\tilde{z}(u, v)$ and $\tilde{z}(v, u)$ are fractional.

**Proof.** Lemma 14 implies $\tilde{z}(v) > 0$, and Lemma 13 implies $\tilde{z}(v, w) > 0$ and $\tilde{z}(u, v) > 0$. Using Eq. (3) with respect to $v$ we get $\tilde{z}(w) < 1$ and $\tilde{z}(v, w) < 1$. And using inequalities (4) we obtain $\tilde{z}(u, v) < 1$. □

**Lemma 16.** We may assume that every sink node $v$ in $G$ is a pendent node.

**Proof.** If $v$ is a sink node in $G$ and $\delta^-(v) = \{(u_1, v), \ldots, (u_k, v)\}$, we can split $v$ into $k$ pendent nodes $\{v_1, \ldots, v_k\}$ and replace every arc $(u_i, v)$ with $(u_i, v_i)$. Then we define $z'$ such that $z'(u_i, v_i) = z(u_i, v)$, $z'(v_i) = 1$, for all $i$, and $z'(u) = z(u)$, $z'(u, w) = z(u, w)$ for every other node and arc. Let $G'$ be this new graph. This graph transformation does not create cycles nor any of the graphs $H_1, \ldots, H_4$. So $G'$ has the same properties as $G$. Moreover, it is easy to check that $z'$ is a fractional extreme point of $P_{p+k-1}(G')$. □

From now on, in this section, all the sink nodes are pendent nodes.
Observation 17. Let $v \in V$, with $\delta^-(v) = \{(u_1, v), (u_2, v)\}$. If $(v, t) \in A$, then $t$ is a pendant node or it coincides with $u_1$ or $u_2$.

Lemma 18. We may assume that $G$ does not contain a bidirected chain $P = v_1, v_2, v_3$, where $\delta^-(v_1) = \{(v_2, v_1)\}$, $\delta^-(v_3) = \{(v_2, v_3)\}$, the inner node $v_2$ is only adjacent to $v_1$ and $v_3$ and where all the arcs of $P$ are tight for $\bar{z}$ except for $(v_2, v_3)$ that may or may not be tight.

Proof. Let $P$ be the chain defined in the lemma. Define $G'$ as the graph obtained from $G$ by identifying the nodes $v_1$ and $v_3$, call $v^*$ the resulting node, and by removing the node $v_2$ with its incident arcs. Add a new node $t$ and the arc $(v^*, t)$, (see Fig. 4).

Let $\delta = \bar{z}(v_3) - \bar{z}(v_2, v_3)$. Define $z'$ from $\bar{z}$ as follows.

$$
\bar{z}'(v) = \begin{cases} 
\delta & \text{if } v = v^*, \\
1 & \text{if } v = t, \\
\bar{z}(v) & \text{otherwise,}
\end{cases}
\quad
\bar{z}'(u, v) = \begin{cases} 
\bar{z}(v_1, v) & \text{if } u = v^* \text{ and } (v_1, v) \in A, \\
\bar{z}(v_3, v) & \text{if } u = v^* \text{ and } (v_3, v) \in A, \\
\bar{z}(v_2) & \text{if } u = v^* \text{ and } v = t, \\
\bar{z}(u, v) & \text{otherwise.}
\end{cases}
$$

The graph $G'$ is simple. In fact, let $a_1$ and $a_2$ be two multiple arcs in $G'$. The node $v^*$ must be their tail and let $u$ be their head. Since $|\delta^-(u)| \geq 2$, by Lemma 16, $u$ is not a sink node. Let $(u, t') \in A$, by the definition of $P$, $t'$ is different from $v_1, v_2$ and $v_3$. The cycle $C' = v_1, (v_1, v_2), v_2, (v_2, v_3), v_3, (v_3, u), (u, v_1), v_1$ is a $g$-odd $Y$-cycle $(u \in \hat{C}'$, which is not possible. Now, notice that $G'$ does not contain a $g$-odd $Y$-cycle. Otherwise, let $C'$ be such a cycle. We should assume that $v^* \in \hat{C}'$. Assume also that $(u^*, u)$ and $(v^*, v)$ are the two arcs in $C'$ incident to $v^*$, where $(u^*, u)$ was obtained from $(v_1, u)$ and $(v^*, v)$ was obtained from $(v_3, v)$. Then by removing $(u^*, u)$, $(v^*, v)$ from $C'$ and adding $(v_1, u)$, $(v_1, (v_2, v_1), v_2, (v_2, v_3), v_3, (v_3, v)$, we obtain a $g$-odd $Y$-cycle in $G$, which is impossible. Now to see that $G'$ has the same properties as $G$, it suffices to check that $G'$ does not contain as a subgraph none of the graphs of Fig. 1. But this is easy to state.

To complete the proof of our lemma, we need to show that $z'$ is a fractional extreme point of $P_p(G')$. Lemma 15 implies that $\bar{z}(v_2)$ is fractional. So at least $z'(v^*, t)$ is fractional.

Let us examine the validity of $z'$. By the definition of $z'$, we only need to show that $\sum z'(v) = p$ and that Eq. (3) with respect to $v^*$ is satisfied.

Notice that the validity of $\bar{z}$ implies that

$$
\bar{z}(v_2) + \bar{z}(v_2, v_1) + \bar{z}(v_2, v_3) = 1.
$$

Since $\bar{z}(v_2, v_1) = \bar{z}(v_1)$ and that $\bar{z}(v_2, v_3) = \bar{z}(v_3) - \delta$, then when replacing in (12) we obtain that

$$
\bar{z}(v_2) + \bar{z}(v_1) + \bar{z}(v_3) = 1 + \delta,
$$

so that

$$
\sum_{v \in V} \bar{z}(v) - \bar{z}(v_1) - \bar{z}(v_2) - \bar{z}(v_3) + z'(v^*) + z'(t) = p - (1 + \delta) + \delta + 1 = p.
$$

Now let us show that Eq. (3) with respect to $v^*$ is satisfied as well. The validity of $\bar{z}$ implies that

$$
\bar{z}(\delta^+(v_1) \backslash \{(v_1, v_2)\}) + \bar{z}(v_1, v_2) + \bar{z}(v_1) = 1,
$$

(14)

$$
\bar{z}(\delta^+(v_2) \backslash \{(v_2, v_3)\}) + \bar{z}(v_2, v_3) + \bar{z}(v_2) = 1.
$$

(15)

Adding Eqs. (14) and (15) and replacing $\bar{z}(v_1, v_2)$ and $\bar{z}(v_3, v_2)$ by $\bar{z}(v_2)$, we obtain

$$
\bar{z}(\delta^+(v_1) \backslash \{(v_1, v_2)\}) + \bar{z}(\delta^+(v_2) \backslash \{(v_2, v_3)\}) + 2\bar{z}(v_2) + \bar{z}(v_1) + \bar{z}(v_3) = 2.
$$

By combining this last equation with (13), we obtain

$$
\bar{z}(\delta^+(v_1) \backslash \{(v_1, v_2)\}) + \bar{z}(\delta^+(v_2) \backslash \{(v_2, v_3)\}) + \bar{z}(v_2) + \delta = 1.
$$

By definition this last equation corresponds to Eq. (3) with respect to $v^*$.

Now, let us show that $z'$ is an extreme point of $P_p(G')$. Suppose the contrary, then there must exist $z'' \in P_p(G')$ where every constraint tight for $z'$ is also tight for $z''$. Let

$$
\alpha = \sum_{u : (v_1, u) \in A} z''(v^*, u); \quad \beta = \sum_{u : (v_3, u) \in A} z''(v^*, u).
$$
Notice that \( z''(v^*) + z''(v^*, t) + \alpha + \beta = 1 \). Let \( z^* \) be the extension of \( z'' \) to \( P_p(G) \) defined as follows.

\[
\begin{align*}
    z^*(v) &= \begin{cases} 
        \beta + z''(v^*) & \text{if } v = v_1, \\
        z''(v^*, t) & \text{if } v = v_2, \\
        \alpha + z''(v^*) & \text{if } v = v_3, \\
        z''(v) & \text{otherwise,}
    \end{cases} \\
    z^*(u, v) &= \begin{cases} 
        z''(v, v) & \text{if } u = v_1 \text{ and } v \neq v_2, \\
        z''(v^*, v) & \text{if } u = v_3 \text{ and } v \neq v_2, \\
        z''(v^*, t) & \text{if } v_2 \text{ and } u = v_1 \text{ or } v_3, \\
        \alpha & \text{if } u = v_2 \text{ and } v = v_3, \\
        \beta + z''(v^*) & \text{if } u = v_2 \text{ and } v = v_1, \\
        z''(u, v) & \text{otherwise.}
    \end{cases}
\end{align*}
\]

It is easy to check that \( z^* \in P_p(G) \) and that every constraint tight for \( \tilde{z} \) is also tight for \( z^* \). To see that \( \tilde{z} \neq z^* \), notice that if \( z'(u, v) \neq z''(u, v) \) for an arc \( (u, v) \in G' \), \( v \neq t \), or \( z'(u) \neq z''(u) \) for some node \( u \in G' \), \( u \neq v^*, t \), then \( \tilde{z} \neq z^* \).

Suppose now that \( z'(u, v) = z''(u, v) \) for all \( (u, v) \in G' \), \( v \neq t \), and \( z'(u) = z''(u) \) for all \( u \in G' \), \( u \neq v^*, t \), then \( \tilde{z}(v_2) = z''(v^*, t) \neq z''(v^*, t) = z^*(v_2) \). This contradicts the fact that \( \tilde{z} \) is an extreme point of \( P_p(G) \). \( \square \)

### 3.1. Proof of Lemma 11

In this sub-section we assume that \( \tilde{z} \) is a fractional extreme point of \( P_p(G) \), such that

\[
\tilde{z}(u, v) = z(v) \quad \text{for every arc } (u, v) \in A, \quad \text{when } v \text{ is not a sink node.}
\]

(16)

The proof of Lemma 11 will be given in Sections 3.1.1 and 3.1.2. Next, we give several lemmas useful for that proof.

**Lemma 19.** Let \((v, w), (w, v)\) and \((w, t)\) be three arcs in \(A\). Then \(|\delta^+(v)| \geq 2\).

**Proof.** Suppose the contrary, that is \(\delta^+(v) = \{(v, w)\}\). Since \(v\) and \(w\) are not sink nodes, assumption (16) implies \(\tilde{z}(w, v) = \tilde{z}(v)\) and \(\tilde{z}(v, w) = \tilde{z}(w)\). Constraint (3) with respect to \(v\) implies \(\tilde{z}(v, w) = 1 - \tilde{z}(v)\). Thus \(\tilde{z}(w) = 1 - \tilde{z}(v) = 1 - \tilde{z}(w, v)\). Hence constraint (3) with respect to \(w\) implies that \(\tilde{z}(w, t) = 0\), which contradicts Lemma 13. \(\square\)

**Lemma 20.** We may assume that \(G\) does not contain a bidirected chain \(P\) of size four, where its internal nodes are adjacent to only their neighbors in \(P\).

**Proof.** Assume the contrary. Let \(P = v_1, v_2, v_3, v_4\) be a bidirected chain of size four, where \(\delta^+(v_2) = \{(v_2, v_1), (v_2, v_3)\}, \delta^+(v_3) = \{(v_3, v_2), (v_3, v_4)\}\). Define \(z'\) from \(\tilde{z}\) as follows.

\[
\begin{align*}
    z'(v) &= \begin{cases} 
        \tilde{z}(v_2, v_1) & \text{if } v = v^* \text{ and } (v_2, v_1) \in A, \\
        \tilde{z}(v) & \text{otherwise,}
    \end{cases} \\
    z'(u, v) &= \begin{cases} 
        \tilde{z}(v_1, v) & \text{if } u = v^* \text{ and } (v_1, v) \in A, \\
        \tilde{z}(u, v_1) & \text{if } v = v^* \text{ and } (u, v_1) \in A, \\
        \tilde{z}(v_4, v) & \text{if } u = v^* \text{ and } (v_4, v) \in A, \\
        \tilde{z}(u, v_4) & \text{if } v = v^* \text{ and } (u, v_4) \in A, \\
        \tilde{z}(u, v) & \text{if } u \neq v^* \text{ and } v \neq v^*.
    \end{cases}
\end{align*}
\]

We will prove that (i) \(G'\) has the same properties as \(G\) and that (ii) \(z'\) is a fractional extreme point of \(P_p(G')\), for some positive integer \(p\).

(i) We need to show that \(G'\) is simple, with no \(g\)-odd \(Y\)-cycle and does not contain any of the graphs \(H_i\), \(1 \leq i \leq 4\), as a subgraph.

To see that \(G'\) is simple, it will be shown that \(v_1\) and \(v_4\) have no neighbor in common. This is more, but this fact helps to see that we cannot create the subgraph \(H_4\) in \(G'\). Let \(u\) be a common neighbor of \(v_1\) and \(v_4\). If \((v_1, u)\) and \((u, v_4)\) are in \(A\), then the ordered sequence \(v_1, u, v_4, v_3, v_2, v_1\) defines an odd directed cycle, which is not possible. The
same contradiction holds when \((u, v_1)\) and \((v_4, u)\) are in \(A\). Now let \((u, v_1)\) and \((u, v_4)\) in \(A\). By Lemma 19, \(|\delta^+(v_1)| \geq 2\). Thus there must exist an arc \((v_1, v')\), with \(v' \not\in \{(v_1, v_2), (v_3, v_4)\}\). Suppose \(v' = u\). Then the ordered sequence \(u, v_4, v_2, v_1, v, u\) defines a directed odd cycle in \(G\), which is impossible. And if \(v' \neq u\), then the cycle \(C' = u, (u, v_1), v_1, (v_2, v_1), v_2, (v_2, v_3), v_3, (v_3, v_4), v_4, (u, v_4)\) is a g-odd Y-cycle, \((v_1, v_4)\) are in \(\tilde{C}'\) and \(v_2, v_3 \in \tilde{C}'\). This contradicts the fact that \(G\) does not contain a g-odd Y-cycle. Finally, if \((v_1, u)\) and \((v_4, u)\) are in \(A\). Lemma 16 implies that \(u\) is not a sink node. Thus we must have an arc \((u, v)\) \(\in A\). The node \(v\) is different from \(v_2, v_3\). Suppose that \(v\) is different from \(v_1, v_4\). Then \(C' = u, (v_1, u), v_1, (v_2, v_1), v_2, (v_2, v_3), v_3, (v_3, v_4), v_4, (v_4, u)\) is a g-odd Y-cycle \((u, v_2)\) and \((u, v_3)\) are in \(\tilde{C}'\) and \(v_3 \in \tilde{C}'\). If \(v = v_4\), then the ordered sequence \(u, v_3, v_2, v_1, u\) define an odd directed cycle. Also if \(v = v_1\) one can construct by symmetry an odd directed cycle. In all cases, \(G\) contains a g-odd Y-cycle, which is not possible.

Now let us see that \(G'\) does not contain a g-odd Y-cycle. Assume the contrary and let \(C'\) be a g-odd Y-cycle in \(G'\). The cycle \(C'\) must contain the node \(v^*\), otherwise \(C'\) is a g-odd Y-cycle in \(G\) too, which is impossible. We distinguish four cases as shown in Fig. 6.

(a) \(v^* \in \tilde{C}'\). Let \((v_1, v) \in A\) and \((u, v_4) \in A\). Let \(C\) be the Y-cycle in \(G\) obtained from \(C'\) by removing the node \(v^*\) and the arcs \((v^*, u)\) and \((u, v^*)\), and by adding the nodes \(v_1, v_2, v_3, v_4\) and the arcs \((v_1, v), (v_1, v_2), (v_2, v_3), (v_3, v_4), (u, v_4)\). We have \(|V(C)| = |V(C')| + 3\) and \(|\tilde{C}'| = |\tilde{C}'| + 1\). These imply that \(|V(C)| + |\tilde{C}'| = |V(C')| + |\tilde{C}'| + 4\). Thus \(C\) is g-odd, which is impossible.

(b) \(v^* \in \tilde{C}'\). Let \((v_1, v) \in A\) and \((u, v_4) \in A\). Suppose that the arc \((v^*, t) \in A', t \not\in V(C')\) exist. Let \((v^*, t)\) be obtained from \((v_1, t) \in A\). Let \(C\) be the Y-cycle in \(G\) obtained from \(C'\) by removing the node \(v^*\) and the arcs \((u, v^*)\) and \((v^*, u)\), and by adding the nodes \(v_1, v_2, v_3, v_4\) and the arcs \((v_1, v), (v_2, v_1), (v_2, v_3), (v_3, v_4), (v_4, v)\) and \((u, v_4)\). We have that \(|V(C)| + |\tilde{C}'| = |V(C')| + |\tilde{C}'| + 4\). So \(C\) is a g-odd Y-cycle of \(G\). Now if there is no arc \((v^*, t) \in A', t \not\in V(C')\), we must have at least one of the nodes \(u\) or \(v\) in \(\tilde{C}'\). Let say \(u \in \tilde{C}'\). We also have \((u, v^*), u \in A'\). Let \(C\) be the Y-cycle in \(G\) obtained from \(C'\) by removing the node \(v^*\) and the arcs \((u, v^*)\) and \((v^*, u)\), and by adding the nodes \(v_1, v_2, v_3, v_4\) and the arcs \((v_1, v), (v_2, v_1), (v_2, v_3), (v_3, v_4), (u, v_4)\). We have that \(|V(C)| + |\tilde{C}'| = |V(C')| + |\tilde{C}'| + 4\). Thus \(C\) is g-odd, which is impossible.

(c) \(v^* \in \tilde{C}'\). Let \((v_1, v) \in A\) and \((u, v_4) \in A\). Let \(C\) be the Y-cycle in \(G\) obtained from \(C'\) by removing the node \(v^*\) and the arcs \((u, v^*)\) and \((v^*, u)\), and by adding the nodes \(v_1, v_2, v_3, v_4\) and the arcs \((v_1, v), (v_2, v_1), (v_2, v_3), (v_3, v_4), (u, v_4)\). We have that \(|V(C)| + |\tilde{C}'| = |V(C')| + |\tilde{C}'| + 4\). Thus \(C\) is g-odd, which is impossible.

(d) This case is similar to Case (c).

The fact that \(G'\) has none of the graphs \(H_1, H_2, H_3\) or \(H_4\) as a subgraph is easy and it is left to the reader.

(ii) First, let us see that \(z' \in F_{p-1}(C')\). The definition of the chain \(P\), assumption (16) and equalities (3) with respect to \(v_1, v_2, v_3\) and \(v_4\) imply the following:

\[
\tilde{z}(v_2) + \tilde{z}(v_1, v_3) + \tilde{z}(v_2, v_3) = 1,
\]
\[
\tilde{z}(v_1) = \tilde{z}(v_2, v_1),
\]
\[
\tilde{z}(\delta^+(v_1) \setminus \{(v_1, v_2)\}) = \tilde{z}(v_2, v_3),
\]
\[
\tilde{z}(v_3) = \tilde{z}(v_2, v_3),
\]
\[
\tilde{z}(v_4) = \tilde{z}(v_2, v_1),
\]
\[
\tilde{z}(\delta^+(v_4) \setminus \{(v_4, v_3)\}) = \tilde{z}(v_2).
\]

(17) (18) (19) (20) (21) (22)

Any constraint that does not contain \(z'(v^*)\) is satisfied by definition. Let us examine those constraints that contain \(z'(v^*)\).

- Let us show that \(z'\) satisfies equality (2).

\[
\sum_{v \in V'} z'(v) = \sum_{v \in V \setminus \{(v_1, v_2, v_3, v_4)\}} \tilde{z}(u) + z'(v^*)
\]
\[
= p - \tilde{z}(v_1) - \tilde{z}(v_2) - \tilde{z}(v_3) - \tilde{z}(v_4) + z'(v^*).
\]

By (18) \(\tilde{z}(v_1) = \tilde{z}(v_2, v_1)\) and by (20) \(\tilde{z}(v_3) = \tilde{z}(v_2, v_3)\). Replacing this in (17), we obtain \(\tilde{z}(v_1) + \tilde{z}(v_2) + \tilde{z}(v_3) = 1\). Also from (21) and the definition of \(z'(v^*)\) we have that \(\tilde{z}(v_4) = z'(v^*)\). Thus \(\sum_{v \in V'} z'(v) = p - 1\).

- Let us show that \(z'\) satisfies equality (3) with respect to \(v^*\). We have

\[
z'(\delta^+(v^*)) + z'(v^*) = \tilde{z}(\delta^+(v_1) \setminus \{(v_1, v_2)\}) + \tilde{z}(\delta^+(v_4) \setminus \{(v_4, v_3)\}) + z'(v^*).
\]
If we combine the above equality with (19) and (22), we obtain

$$z'(\delta^+(v^*)) + z'(v^*) = \tilde{z}(v_2, v_3) + \tilde{z}(v_2) + z'(v^*).$$

Since \(z'(v^*) = \tilde{z}(v_2, v_1), (17)\) implies \(z'(\delta^+(v^*)) + z'(v^*) = 1\).

• Finally, let us show that \(z'\) satisfies (4) with respect to \(v^*\). Let \((u, v^*)\) be an arc in \(G'\) and let us show that \(z'(u, v^*) \leq z'(v^*)\).

By definition \(z'(u, v^*) = \tilde{z}(u, v_1)\) or \(z'(u, v^*) = \tilde{z}(u, v_4)\). The definition of \(z'(v^*)\), (18) and (21) imply \(z'(v^*) = \tilde{z}(v_1) = \tilde{z}(v_4)\). Hence the fact that \(\tilde{z}(u, v_1) \leq \tilde{z}(v_1)\) or \(\tilde{z}(u, v_4) \leq \tilde{z}(v_4)\) implies immediately \(z'(u, v^*) \leq z'(v^*)\). Also notice that \(z'(v^*, u) \leq z'(u)\) for all \((v^*, u) \in A\).

To finish the proof of this lemma, let us see that \(z'\) is a fractional extreme point of \(P_{p-1}(G')\). We yet proved that \(z' \in P_{p-1}(G')\). Lemma 15 and the definition of \(z'\) imply that \(z'\) is fractional. Suppose that \(z'\) is not an extreme point of \(P_{p-1}(G')\). Thus there must exist \(z'' \in P_{p-1}(G'), z'' \neq z'\), where each constraint that is tight for \(z'\) is also tight for \(z''\). Let

$$\alpha = \sum_{u \in (v_3, u) \in A} z''(v^*, u); \quad \beta = \sum_{u \in (v_4, u) \in A} z''(v^*, u).$$

Notice that \(z''(v^*) + \alpha + \beta = 1\). Let \(z^*\) be the extension of \(z''\) to \(P_p(G)\) defined as follows.

$$z^*(v) = \begin{cases} 
z''(v^*) & \text{if } v \neq v_1, v_4, \\
\beta & \text{if } v = v_2, \\
\alpha & \text{if } v = v_3, \\
z'(v) & \text{otherwise}, 
\end{cases}$$

$$z^*(u, v) = \begin{cases} 
z''(v^*, u) & \text{if } u = v_1 \text{ and } v \neq v_2, \\
z'(u, v^*) & \text{if } u \neq v_2 \text{ and } v = v_1, \\
z'(v^*, u) & \text{if } u = v_4 \text{ and } v \neq v_3, \\
z'(u, v^*) & \text{if } u \neq v_3 \text{ and } v = v_4, \\
\beta & \text{if } v = v_2 \text{ and } u \neq v_1 \text{ or } v_3, \\
\alpha & \text{if } v = v_3 \text{ and } u \neq v_2 \text{ or } v_4, \\
z''(v^*) & \text{if } (u, v) = (v_2, v_1) \text{ or } (v_3, v_4), \\
z'(u, v) & \text{otherwise}. 
\end{cases}$$

It is easy to check that \(z^* \in P_p(G)\) and that every constraint tight for \(\tilde{z}\) is also tight for \(z^*\). To see that \(z^* \neq \tilde{z}\), notice that if \(z''(u, v) \neq z'(u, v)\) for some arc \((u, v)\) of \(G'\) or \(z''(v) \neq z'(v)\) for \(v \neq v^*\), then \(z^* \neq \tilde{z}\). And if \(z''(v^*) \neq z'(v^*)\), then by definition \(z^*(v_2, v_1) = z''(v^*) \neq z'(v^*) = \tilde{z}(v_2, v_1)\). This contradicts the fact that \(\tilde{z}\) is an extreme point of \(P_p(G)\).

**Lemma 21.** \(G\) does not contain a bidirected chain \(P = v_1, v_2, v_3, v_4\), satisfying the following:

(i) \((v_3, t) \in A \) with \(t\) a pendant node, and

(ii) \(\delta^- (v_1) = \{ (v_2, v_1) \}\).

**Proof.** Suppose the contrary and let \(P = v_1, v_2, v_3\) be a bidirected chain satisfying (i) and (ii). Let \(l\) be a labeling function, where the node \(v_2\) with the arcs \((v_1, v_2)\) and \((v_3, v_2)\) receive the label 1; the node \(v_1\) with the arcs \((v_2, v_1)\) and \((v_3, t)\) receive the label \(-1\); and all other nodes and arcs receive the label 0.

The vector \(\tilde{z}\) satisfies with equality each constraint among (2)–(6) that was satisfied with equality by \(\tilde{z}\). In fact, Lemma 15 implies that the value of \(\tilde{z}\), corresponding to the nodes and arcs that received a label different from 0, is fractional. This implies that any inequality (5) or (6) that is satisfied with equality by \(\tilde{z}\) remain satisfied with equality by \(\tilde{z}\). Let us see that equations Eq. (3) are satisfied. The arcs that receive a non-zero label are incident to the nodes \(v_1, v_2, v_3, v_4\). Eq. (3) with respect to \(v_1\) is satisfied since \(v_1\) and \((v_1, v_2)\) receive opposite labels, the same holds for \(v_2\). Also, the unique arcs incident to \(v_1\) that receive a non-zero label are \((v_3, v_2)\) and \((v_3, t)\), and they receive opposite labels. Since \(v_3\) receives a zero label, then Eq. (3) with respect to \(v_3\) is satisfied. Equality (2) is satisfied since \(v_1\) and \(v_2\) received opposite labels and the other nodes received the label 0.

Now we consider inequalities (4). The unique nodes with labels different from 0 are \(v_1\) and \(v_2\). Notice that \((v_2, v_1)\) received the same label as \(v_1\) and by hypothesis (ii) is the unique arc directed into \(v_1\). Also by Observation 12 the only arcs directed into \(v_2\) are \((v_1, v_2)\) and \((v_3, v_2)\) and they received the same label as \(v_2\). Hence any inequality (4) that is satisfied with equality by \(\tilde{z}\) remains satisfied with equality by \(\tilde{z}\). This is in contradiction with the fact that \(\tilde{z}\) is an extreme point of \(P_p(G)\).

**Lemma 22.** \(G\) does not contain a bidirected chain \(P = v_1, v_2, v_3, v_4\), such that \(v_1\) and \(v_4\) are adjacent to a pendant node.

**Proof.** Suppose the contrary and let \(P = v_1, v_2, v_3, v_4\) be a bidirected chain such that \((v_1, t)\) and \((v_4, t')\) are in \(A\), where \(t\) and \(t'\) are pendant nodes.

Assign the label 1 to the node \(v_3\) and the arcs \((v_1, t)\), \((v_2, v_3)\) and \((v_4, v_3)\); assign the label \(-1\) to the node \(v_2\) and the arcs \((v_1, v_2)\), \((v_3, v_2)\) and \((v_4, t')\); assign the other nodes and arcs the label 0. Call this labeling \(l\).

As in the proof of Lemma 21, one can easily check that \(\tilde{z}\) satisfies with equality any constraint among (2)–(6) that is satisfied with equality by \(\tilde{z}\). This contradicts the fact that \(\tilde{z}\) is an extreme point of \(P_p(G)\).
Lemma 23. If \( G \) contains a cycle of size at least three, then it contains a \( Y \)-cycle of the same size.

**Proof.** Let \( C' = v_0, a_0, v_1, a_1, \ldots, a_{l-1}, v_l \) be a simple cycle with \( p \geq 3 \). Suppose that \( C' \) is a not a \( Y \)-cycle. There must exist a node \( v_i \in \hat{C}' \) where conditions (i) and (ii) of Definition 6 are not satisfied. Let \((v_{i-1}, v_i)\) and \((v_{i+1}, v_i)\) be the two arcs of \( C' \) directed into \( v_i \). By Lemma 14, \( \hat{Z}(v_i) > 0 \). Since \( v_i \) is not a pendent node, there must exist an arc \((v_i, u)\) in \( G \). The fact that (i) is not satisfied implies that \( u \in V(C') \). If \( u \) is different from \( v_{i-1} \) and \( v_{i+1} \), then \( C' \) is of size at least four. In this case, \( G \) must contain one of the graphs \( H_1 \) or \( H_2 \) as a subgraph, which is impossible. Thus \( \delta^+(v_i) \) consists of one of the arcs \((v_i, v_{i-1})\) or \((v_i, v_{i+1})\), or both. Assume \((v_i, v_{i-1}) \in A\), since Definition 6(ii) is not satisfied \( v_{i-1} \) must be in \( C' \), so \((v_{i-1}, v_{i-2}) \in A(C') \) with \( v_{i-2} \in V(C') \). Then Lemma 19 implies that \((v_i, v_{i+1}) \in A\). Also \((v_i, v_{i+1}) \in A \) implies \( v_{i+1} \in \hat{C}' \), so \((v_{i+1}, v_{i+2}) \in A(C') \), (see Fig. 7).

Thus we may suppose that for any node \( v_i \in \hat{C}' \) that does not satisfy Definition 6(i) and (ii), \( \delta^+(v_i) = \{(v_i, v_{i-1}), (v_i, v_{i+1})\} \) and both nodes \( v_{i-1} \) and \( v_{i+1} \) are in \( \hat{C}' \). Define \( C \) from \( C' \), recursively, by the following procedure.

Step 1. \( A(C) := A(C'), V(C) := V(C'), C := C' \).

Step 2. If there exist \( v_i \in \hat{C} \), a node not satisfying Definition 6(i) and (ii), go to Step 3. Otherwise stop, \( C \) is a \( Y \)-cycle.

Step 3. \( A(C) := (A(C) \setminus \{(v_{i-1}, v_i), (v_{i+1}, v_i)\}) \cup \{(v_i, v_{i-1}), (v_i, v_{i+1})\} \). \( C \) is the new cycle defined by \( A(C) \). Go to Step 2.

Each Step 3 decreases by one the number of nodes in \( \hat{C} \). Thus the procedure must end. \( \square \)

Lemma 24. Let \( C = v_0, a_0, v_1, a_1, \ldots, a_{r-1}, v_r, r \geq 3 \), be a \( g \)-even \( Y \)-cycle with \( |\hat{C}(0)| \) maximum. Then \( C \) does not contain a blocking node.

**Proof.** Suppose that \( C \) contains a blocking node \( v_i \).

Case 1. \( v_i \) is a blocking node satisfying Definition 7(i). Thus \( v_i \in \hat{C}, (v_i, v_{i-1}), (v_i, v_{i+1}) \in A(C), (v_{i+1}, v_i) \in A \setminus A(C) \) and \( v_{i+1} \in \hat{C} \). Thus \((v_{i+1}, v_{i+2}) \in A(C) \) (see Fig. 8). Notice that \( v_{i+2} \neq v_{i-1} \), otherwise \( C \) is a directed odd cycle.

**Claim 1.** If \((v_i, u) \in A\), then \( v \in V(C) \).

**Proof.** Suppose the contrary, let \((v_i, u) \in A \) with \( u \not\in V(C) \). The node \( v_{i+2} \) is not in \( \hat{C} \), otherwise the cycle \( C' \), where \( V(C') = V(C) \) and \( A(C') = (A(C) \setminus \{(v_i, v_{i+1})\}) \cup \{(v_{i+1}, v_i)\} \), is a \( g \)-odd \( Y \)-cycle. Thus \( v_{i+2} \) must be in \( \hat{C} \). If the cycle \( C' \) as defined previously is a \( Y \)-cycle, then it is \( g \)-odd. Thus \( C' \) is not a \( Y \)-cycle, which implies that \((v_{i+2}, v_{i+1}) \in A \setminus A(C) \) and \( v_{i+2} \not\in \hat{C}(0) \). Replace the arcs \((v_i, v_{i+1})\) and \((v_{i+1}, v_{i})\) by \((v_{i+1}, v_i)\) and \((v_{i+2}, v_{i+1})\). Call \( C'' \) the resulting cycle. It is easy to check that \( C'' \) is a \( Y \)-cycle with \( |\hat{C}(0)| = |\hat{C}(0)| + 1 \), this contradicts the fact that \( C \) is a \( Y \)-cycle with \( |\hat{C}(0)| \) maximum. \( \square \)

**Claim 2.** \( \delta^+(v_i) = \{(v_i, v_{i-1}), (v_i, v_{i+1})\} \) and \( \delta^-(v_i) = \{(v_{i-1}, v_i), (v_{i+1}, v_i)\} \).

**Proof.** Lemma 19 implies that \( |\delta^+(v_i)| \geq 2 \). Let \((v_i, u) \in \delta^+(v_i)\), where \( u \neq v_{i+1} \). Claim 1 implies that \( u \in V(C) \). If \( u \neq v_{i-1} \), then \( G \) contains one of the graphs of Fig. 1 as a subgraph. Thus \( u = v_{i-1} \) and \( \delta^{-}(v_i) = \{(v_i, v_{i-1}), (v_i, v_{i+1})\} \). Now notice that since \( G \) does not contain \( H_4 \) as a subgraph the only arcs in \( A \) directed into \( v_i \) are \((v_{i-1}, v_i)\) and \((v_{i+1}, v_i)\). \( \square \)
The node $v_{i-1}$ must be in $\hat{C}$. Assume the contrary. It follows that $v_{i-1} \in \hat{C}$, thus $(v_{i-2}, v_{i-1}) \in A(C)$. Notice that $v_{i-1}$ is a blocking node satisfying Definition 7(i). Thus Claim 2 may be applied to $v_{i-1}$, so $\delta^+(v_{i-1}) = \{(v_{i-1}, v_{i-2}), (v_{i-1}, v_i)\}$ and $\delta^-(v_{i-1}) = \{(v_{i-2}, v_{i-1}), (v_i, v_{i-1})\}$. Thus the sequence $P = v_{i-2}, v_{i-1}, v_i, v_{i+1}$ is a bidirected chain of size four, where its internal nodes $v_i$ and $v_{i-1}$ are adjacent to only their neighbors in $P$. This contradicts Lemma 20. Thus $v_{i-1} \in \hat{C}$ and $(v_{i-1}, v_{i-2}) \in A(C)$, as shown by Fig. 9. Notice that $v_{i-2} \neq v_{i+2}$, otherwise the $Y$-cycle $C$ would be g-odd.

$P = v_{i-1}, v_i, v_{i+1}$ is a bidirected chain of size four. Lemma 18 implies that at least one of the arcs $(u, v_{i-1})$ or $(u, v_{i+1})$ exists, with $u \neq v_i$.

Suppose $(u, v_{i-1}) \in A$. The case when $(u, v_{i+1}) \in A$ is symmetric. Since $v_{i-2}$ is not a pendant node, Observation 17 implies that $u = v_{i-2}$, so $\delta^-(v_{i-1}) = \{(v_i, v_{i-1}), (v_{i-2}, v_{i-1})\}$. If $\delta^+(v_{i-1}) = \{(v_{i-1}, v), (v_{i-1}, v_1)\}$, then $P = (v_{i-2}, v_{i-1}, v, v_{i+1})$, which contradicts Lemma 20. Hence we may assume that $(v_{i-1}, t) \in A$ and $t$ is a pendant node.

If $\delta^-(v_{i-1}) = \{(v_i, v_{i+1})\}$, then $P = (v_{i-1}, v_i, v_{i+1})$ is a bidirected chain satisfying conditions (i) and (ii) of Lemma 20, which is impossible. Thus we must have an arc $(u, v_{i+1}) \in A$ with $u \neq v_i$. Since $v_{i+2}$ is not a pendant node, Observation 17 implies that $u = v_{i+2}$. There must exist also an arc $(v_{i+1}, t') \in A$, where $t'$ is a pendant node, otherwise the bidirected chain $P = v_{i-1}, v_i, v_{i+1}, v_{i+2}$ contradicts Lemma 20. The situation is summarized in Fig. 10.

If $v_{i+2} \in \hat{C}$, then $v_{i+1}$ is a blocking node satisfying Definition 7(i). But since $(v_{i+1}, t') \in A$ and $t' \notin V(C)$, this contradicts Claim 1. Thus $v_{i+2} \notin \hat{C}$. Indeed, suppose the contrary let $(v_{i+2}, u) \in A$ and $u \notin V(C)$. The node $u$ must be a pendant node, otherwise $G$ contains one of the graphs $H_1, H_2$ or $H_4$ as a subgraph. Thus, the sequence $P = v_{i-1}, v_i, v_{i+1}, v_{i+2}$ is a bidirected chain of size four, where $v_{i-1}$ and $v_{i+2}$ are adjacent to pendant nodes, which is impossible by Lemma 22.

Now it is easy to check that the cycle $C'$ obtained from $C$ by removing $(v_{i+1}, v_{i+2})$ and adding $(v_{i+2}, v_{i+1})$ is a Y-cycle with $|\hat{C}(0)| = |\hat{C}(0)| + 1$. This contradicts the fact that $C$ is chosen so that $|\hat{C}(0)|$ is maximum.

Case 2. $v_i$ is a blocking node satisfying Definition 7(ii). Thus $v_i \notin \hat{C}$; $(v_i, v_{i-1}), (v_{i+1}, v_i)$ belong to $A(C)$; $(v_i, v_{i+1}), (v_i, v_{i-1})$ belong to $A \setminus A(C)$; and $v_{i-1}, v_{i+1} \in \hat{C}$. It follows that $(v_{i+2}, v_{i+1})$ and $(v_{i-2}, v_{i-1})$ are in $A(C)$ (see Fig. 11). Notice that $v_{i+2} \neq v_{i-2}$, otherwise $C$ is a g-odd Y-cycle.

Lemma 19 implies that $(v_i, u) \in A$ and $(v_{i+1}, u') \in A$, with $u \neq v_i, u' \neq v_i$. By Observation 17, $u$ is a pendant node or $u = v_{i-2}$, and also $u'$ is a pendant node or $u' = v_{i+2}$. Also both nodes $v_{i-1}$ and $v_{i+1}$ cannot be adjacent to a pendant node. Otherwise, the cycle obtained from $C$ by removing $(v_{i-1}, v_i)$ and $(v_{i+1}, v_{i+2})$, and by adding $(v_i, v_{i+1})$ and $(v_i, v_{i+2})$ is a g-odd Y-cycle, which is not possible. Thus we have two sub-cases:

(a) $u = v_{i-2}$ and $v_{i-1}$ is not adjacent to a pendant node, or
(b) $u' = v_{i+2}$ and $v_{i+1}$ is not adjacent to a pendant node.

Below we treat sub-case (a); sub-case (b) is symmetric. Let $u = v_{i-2}$; $v_{i-1}$ is not adjacent to a pendant node and $(v_{i-1}, v_{i-2}) \in A \setminus A(C)$. The node $v_i$ must be adjacent to a pendant node $t$, otherwise the bidirected chain $P = v_{i-2}, v_{i-1}, v_i, v_{i+1}$ contradicts Lemma 20. The situation is described in Fig. 12.

The node $v_{i-2}$ must be in $\hat{C}$. Otherwise, $v_{i-2}$ is a blocking node by Definition 7(i), which is impossible as shown in Case 1. Thus, $(v_{i-2}, v_{i-3}) \in A(C)$. By Lemma 21, we must have an arc $(u', v_{i-2})$, $u' \neq v_{i-1}$. Since $v_{i-3}$ is not a pendant node, Observation 17 implies $u' = v_{i-3}$. Also, Lemma 20 implies that $v_{i-2}$ is adjacent to a pendant node $t'$. 

![Fig. 9. Dashed lines represent arcs in C.](image1)

![Fig. 10. Dashed lines represent arcs in C.](image2)
Lemma 25. \( G \) does not contain a cycle of size at least three.

Proof. Assume the contrary. Suppose that \( G \) admits such a cycle. From Lemma 23, we may assume that \( G \) contains a \( g \)-even \( Y \)-cycle. Among all these \( Y \)-cycles, let \( C = v_0, a_0, v_1, a_1, \ldots, a_{r-1}, v_r \) be a \( g \)-even \( Y \)-cycle such that \( |\hat{C}(0)| \) is maximum. Lemma 24 implies that \( C \) does not contain a blocking node. Hence assumption (a1) of Lemma 8 is satisfied. Also \( \bar{z} \in P_p(G) \) and Lemma 15 implies that assumption (a2) of Lemma 8 is satisfied. Also the graph \( G \) is a simple directed graph and satisfies (C1) of Theorem 2. It follows from Lemma 8 that \( \bar{z} \) is not an extreme point of \( P_p(G) \), a contradiction. 

Now we can prove the main result of this sub-section. Denote by Pair(\( G \)) the set of pair of nodes \( \{u, v\} \) such that both arcs \( (u, v) \) and \( (v, u) \) belong to \( A \). We use an induction on \( |\text{Pair}(G)| \). We first prove Lemma 11 in case where \( |\text{Pair}(G)| = 0 \), and then we use the induction to prove it when \( |\text{Pair}(G)| \geq 1 \).
3.1.1. Proof of Lemma 11 when |Pair(G)| = 0

In this case G is an oriented graph. Recall that in an oriented graph, if (u, v) belongs to the arc-set then (v, u) does not belong to it. Here the subgraph H_4 cannot appear.

Let us first discuss a special class of graphs. A node t is called a Y-node, if there exist three different nodes u_1, u_2, w in V such that (u_1, t), (u_2, t) and (t, w) belong to A. We denote by Y_G the set of Y-nodes in G. The graph G is called Y-free if it does not contain a Y-node. Observe that if C is a Y-cycle in an oriented graph, then each node in C is a Y-node. When G is Y-free, we have C = ∅ and C is a directed cycle.

The consequences of Theorems 14 and 20 in [15] are summarized in the following theorem. The same result is given in [16] where the proof is based on the matching polytope.

**Theorem 26.** If G is a Y-free graph with no odd directed cycle, then P_p(G) for any integer p, and P(G) are integral.

We prove Lemma 11 when |Pair(G)| = 0 we will show, in fact, that P_p(G) is integral. This proof is done by induction on the number of Y-nodes. If |Y_G| = 0, then the graph is Y-free with no odd directed cycle; it follows from Theorem 26 that P_p(G) is integral. Assume that P_p(G) is integral for any positive integer p and for any oriented graph G', with |Y_{G'}| < |Y_G|, that satisfies condition (C1) and does not contain a g-odd Y-cycle.

**Lemma 27.** The graph G must contain at least one Y-node.

**Proof.** Suppose the contrary. Then G is a Y-free graph with no odd directed cycle. Theorem 26 implies that P_p(G) is an integral polytope. This contradicts the fact that z̃ is a fractional extreme point of P_p(G). □

**Lemma 28.** There is a Y-node t in G, and arcs (u_1, t), (u_2, t), (t, w), such that

- V can be partitioned into W_1 and W_2 so that \{u_1, t, w\} ⊆ W_1 and u_2 ∈ W_2.
- The only arc in G between W_1 and W_2 is (u_2, t). See Fig. 14.

**Proof.** Let t be a Y-node in G, Lemma 27 shows that such a node exists. Let G_1 = (S_1, A_1) be the connected component of G that contains t. It follows from Lemma 25 that (u_2, t) does not belong to any cycle in G. Hence if we remove (u_2, t) from G then we disconnect G_1 into two connected components. Let S'_1 and S'_2 be the node-sets of these two components, containing u_1 and u_2, respectively. Define W_1 to be S'_1 and W_2 = V \ S'_1. □

Recall that P(G) is the polytope defined by (3)–(6).

**Lemma 29.** P(G) is an integral polytope.

**Proof.** P(G) is a face of the polytope studied in [26]. Here G has no g-odd Y-cycle. It follows from Lemma 16 that any g-odd cycle is a Y-cycle, hence G has no g-odd cycle. It was shown in [26] that if G has no g-odd cycle then P(G) is integral. □

**Lemma 30.** The values of z̃ are in \{0, 1, α, 1 − α\}, for some number α ∈ [0, 1].

**Proof.** Since Lemma 29 shows that P(G) is an integral polytope and P_p(G) is obtained from P(G) by adding exactly one equation, the result follows from Observation 9. □

**Lemma 31.** Each component of z̃ is in \{0, 1, 1/2\}.

**Proof.** Based on Lemma 28, we define the graphs G^1 and G^2 as follows. Let A(W_1) and A(W_2) be the set of arcs in G having both endnodes in W_1 and W_2, respectively. Let G^1 = (W_1, A(W_1)) and G^2 = (W_2 ∪ \{t', v', w'\}, A(W_2) ∪ \{(u_2, t'), (t', v'), (v', w')\}); see Fig. 15. Let G' = G^1 ∪ G^2.

Notice that from (16) we have \vec{z}(u_1, t) = \vec{z}(u_2, t) = \vec{z}(t). Define \vec{z}' to be \vec{z}'(u_2, t') = \vec{z}(u_2, t), \vec{z}'(t') = \vec{z}(u_2, t), \vec{z}'(t', v') = 1 − \vec{z}(u_2, t), \vec{z}'(t', w') = \vec{z}(u_2, t), \vec{z}'(w') = 1 and \vec{z}'(u) = \vec{z}(u), \vec{z}'(u, v) = \vec{z}(u, v) for all other nodes and arcs. We have that \vec{z}' ∈ P_{p+2}(G') and G' is a graph satisfying (C1) and does not contain a g-odd Y-cycle,
Thus from Lemma 31 consider the graphs $G^1$ and $G^2$ as defined in the proof of Lemma 33.

Noticing that $z$ are not tight for $z$' for vectors $z^t$ are also tight for $z$'. Thus $z' = \sum_{i=1}^r \lambda_i z_i'$, $\sum_{i=1}^r \lambda_i = 1$ and $\lambda_i \geq 0$ for $i = 1, \ldots, r$. If there exists a vector $z_i'$ with $z_i'(t) = z_k'(t')$, then we can define from $z_k'$ a 0-1 vector $z'' \in \mathcal{P}_p(G)$ such that the same constraints tight for $z$ are also tight for $z''$. The vector $z''$ is obtained setting $z''(t) = z_i'(t) = z_k'(t')$, and $z''(u) = z_k'(u)$, $z''(u, v) = z_k'(u, v)$, for all other nodes and arcs in $G$.

Thus we may suppose that for all $z_i', i = 1, \ldots, r$, we have $z_i'(t) \neq z_i'(t')$. Let $z_i'(t) = 1, z_i'(t') = 0$, for $i = 1, \ldots, r_1$, and $z_i'(t) = 0, z_i'(t') = 1$, for $i = r_1 + 1, \ldots, r$. Then $z'(t) = \sum_{i=1}^{r_1} \lambda_i z_i'$ and $z'(t') = \sum_{i=r_1+1}^r \lambda_i z_i$. Since by definition $z'(t) = z'(t')$ and $z'(t) + z'(t') = \sum_{i=1}^r \lambda_i = 1$. So $z(t) = \frac{1}{2}$ and the result is obtained from Lemma 30.

Define $p_1 = \sum_{v \in W_1} z(v)$ and $p_2 = \sum_{v \in W_2} z(v)$, so $p = p_1 + p_2$. We distinguish two cases: $p_1$ and $p_2$ are integer; and they are not.

**Lemma 32.** If the numbers $p_1$ and $p_2$ are integer then $z$ cannot be an extreme point.

**Proof.** Consider the graphs $G^1$ and $G^2$ of Fig. 15, as defined above. Let $z_1$ be the restriction of $z$ to $G^1$. Clearly $z_1 \in \mathcal{P}_{p_1}(G^1)$.

Define $z_2$ as follows, $z_2(t_1, t) = \tilde{z}(u, t) = \frac{1}{2}, z_2(t', t) = z_2(t', t') = \frac{1}{2}$, $z_2(v', v') = \frac{1}{2}$, $z_2(u', w') = \frac{1}{2}$, $z_2(u, v) = 1$ and $z_2(u) = \tilde{z}(u)$, $z_2(u, v) = \tilde{z}(u, v)$ for all other nodes and arcs of $G^2$. We have that $z_2 \in \mathcal{P}_{p_2+2}(G^2)$.

Both graphs $G^1$ and $G^2$ satisfy (C1) and do not contain a g-odd Y-cycle. Moreover, $|Y_{G^1}| < |Y_{G^2}|$ and $|Y_{G^2}| < |Y_{G^1}|$. Since $z_1$ and $z_2$ are both fractional, the induction hypothesis implies that they are not extreme points of $\mathcal{P}_{p_1}(G^1)$ and $\mathcal{P}_{p_2+2}(G^2)$, respectively. Thus there must exist a 0-1 vector $z_1' \in \mathcal{P}_{p_1}(G^1)$ with $z_1'(t) = 0$ so that the same constraints that are tight for $z_1$ are also tight for $z_1'$. Also there must exist a 0-1 vector $z_2' \in \mathcal{P}_{p_2+2}(G^2)$ with $z_2'(t') = 0$ such that the same constraints that are tight for $z_2$ are also tight for $z_2'$. Combine $z_1'$ and $z_2'$ to define a solution $z' \in \mathcal{P}_p(G)$ as follows.

\[
\begin{align*}
  z'(u) &= z_1'(u), & \text{for every node } u \text{ of } G^1, \\
  z'(u, v) &= z_2'(u, v), & \text{for every arc } (u, v) \text{ of } G^1, \\
  z'(u_1, t) &= 0, \\
  z'(u, v) &= z_2'(u, v), & \text{for every node } v \in W_2, \\
  z'(u, v) &= z_2'(u, v), & \text{for every arc } (u, v) \in A(W_2).
\end{align*}
\]

Notice that $\sum_{v \in V} z'(v) = p$. Also any constraint among (3)–(6), that is tight for $z$ is also tight for $z'$. Then the same constraints of $\mathcal{P}_p(G)$ that are tight for $z$ are also tight for $z'$. This contradicts the fact that $z$ is an extreme point of $\mathcal{P}_p(G)$.

**Lemma 33.** If the numbers $p_1$ and $p_2$ are not integer then $z$ cannot be an extreme point.

**Proof.** Thus from Lemma 31, $\sum_{v \in W_1} z(v) = p_1 = \alpha + \frac{1}{2}$ and $\sum_{v \in W_2} z(v) = p_2 = \beta - \frac{1}{2}$, where $\alpha$ and $\beta$ are integers and $\alpha + \beta = p$. Define $G^1$ and $G^2$ from $G$ as follows. $G^1 = (W_1 \cup \{u_1\}, (A(W_1) \setminus \{(u_1, t)\}) \cup \{(u_1, u_1'), (u_1', t)\})$ and $G^2 = (W_2 \cup \{t', w'\}, A(W_2) \cup \{(u_2, t'), (t', w')\})$; see Fig. 16.
Define $z^1$ to be
\[ z^1(u_1, u_1') = z^1(u_1') = z^1(u_1, t) = \frac{1}{2}, \]
\[ z^1(u) = \bar{z}(u) \text{ for all other nodes of } G^1, \]
\[ z^1(u, v) = \bar{z}(u, v) \text{ for all other arcs of } G^1. \]

Let $z^2$ be defined by
\[ z^2(u_2, t') = z^2(t') = z^2(t', w') = \frac{1}{2}, \]
\[ z^2(w') = 1, \]
\[ z^2(u) = \bar{z}(u) \text{ for all other nodes of } G^2, \]
\[ z^2(u, v) = \bar{z}(u, v) \text{ for all other arcs of } G^2. \]

Notice that $z^1 \in P_{\alpha+1}(G^1)$ and $z^2 \in P_{\beta+1}(G^2)$. Notice also that the graphs $G^1$ and $G^2$ satisfy (C1) and do not contain a g-odd $Y$-cycle. The induction hypothesis may be applied to $G^1$ and $G^2$ since $|Y_{G^1}| < |Y_G|$ and $|Y_{G^2}| < |Y_G|$. Thus there must exist a $0$–$1$ vector $\tilde{z}^1 \in P_{\alpha+1}(G^1)$ such that the same constraints that are tight for $z^1$ are also tight for $\tilde{z}^1$, and such that $\tilde{z}^1(u_1, u_1') = 0$. Also there must exist a $0$–$1$ vector $\tilde{z}^2 \in P_{\beta+1}(G^2)$ such that the same constraints that are tight for $z^2$ are also tight for $\tilde{z}^2$ and such that $\tilde{z}^2(t') = 0$. Now from $\tilde{z}^1$ and $\tilde{z}^2$ define $\tilde{z}^* \in P_p(G)$ as follows.

\[ z^*(u_2, t) = 0, \]
\[ z^*(u) = \tilde{z}^1(u), \quad \text{for all } u \in W_1 \setminus \{t\}, \]
\[ z^*(u, v) = \tilde{z}^2(u, v), \quad \text{for all } (u, v) \in A(W_1) \setminus \{(u_1, t), (t, w)\}, \]
\[ z^*(t) = 0, \]
\[ z^*(u_1, t) = 0, \]
\[ z^*(t, w) = 1, \]
\[ z^*(u) = \tilde{z}^2(u), \quad \text{for all } u \in W_2, \]
\[ z^*(u, v) = \tilde{z}^2(u, v), \quad \text{for all } (u, v) \in A(W_2). \]

It is easy to see that $z^* \in P_p(G)$ and that the same constraints that are tight for $\tilde{z}$ are also tight for $z^*$. Thus $\tilde{z}$ cannot be an extreme point. \hfill \Box

These last two lemmas give the desired contradiction, so $P_p(G)$ is integral and Lemma 11 is proved when $|\text{Pair}(G)| = 0$.

3.1.2. Proof of Lemma 11 when $|\text{Pair}(G)| \geq 1$

Suppose that the lemma is true for every simple directed graph $H$ with no g-odd $Y$-cycle, satisfying (C1) and $|\text{Pair}(H)| \leq m, m \geq 0$. Let $G = (V, A)$ with same properties as $H$ having $|\text{Pair}(G)| = m + 1$.

Let $\bar{z}$ be a fractional extreme point of $P_p(G)$ where $\bar{z}(u, v) = \bar{z}(v)$ for each arc $(u, v)$ with $v$ not a sink node. Notice that Lemma 25 applies, so $G$ does not contain a cycle.

Let $(u, v)$ and $(v, u)$ be two arcs in $G$. Denote by $G(u, v)$ the graph obtained from $G$ by removing the arc $(u, v)$ and adding a new arc $(u, t)$, where $t$ is a new pendent node. Define $\tilde{z} \in P_p(G(u, v))$, $\bar{z}(u, t) = \bar{z}(u, v) = \tilde{z}(u, t) = 1$ and $\tilde{z}(r) = \bar{z}(r), \tilde{z}(s) = \bar{z}(s)$ for every other node and arc.

The graph $G(u, v)$ is simple and satisfies condition (C1) of Theorem 2. Since $G$ does not contain a cycle, we have that $G(u, v)$ has no g-odd $Y$-cycle. Moreover, $|\text{Pair}(G(u, v))| \leq m$, hence the induction hypothesis applies for $G(u, v)$. We have that $\tilde{z}$ is a fractional vector in $P_p(G(u, v))$ with $\bar{z}(u, v) = \bar{z}(v)$ for each arc $(u, v)$ such that $v$ is not a sink node. By the induction hypothesis $\tilde{z}$ is not an extreme point. Thus, there must exist a set of extreme points of $P_p(G(u, v))$, $z^1, \ldots, z^k$, where each constraint that is tight for $\tilde{z}$ is also tight for each of $z^1, \ldots, z^k$, and $\tilde{z}$ is a convex combination of $z^1, \ldots, z^k$. Let us see that all these extreme points are integral. In fact, suppose that $z^1$ is a fractional extreme point of $P_p(G(u, v))$. By the induction hypothesis, we must have an arc $(u', v')$ in $G(u, v)$ where $v'$ is not a sink node and $z^1(u', v') < z^1(v')$. Then by construction the arc $(u', v')$ is in $G$ too. Thus we must have $\tilde{z}(u', v') < \tilde{z}(v')$. But this implies that $v'$ must be a sink node, a contradiction.

Since all the extreme points $z^1, \ldots, z^k$ are integral and $\bar{z}(v, u) > 0$, there must exist one vector among $z^1, \ldots, z^k$, say $z^1$, with $\bar{z}(v, u) = 1$. From $z^1$ define $z'' \in P_p(G)$ as follows: $z''(u, v) = z^1(u, t)$ and $z''(r, s) = z^1(r, s), z''(r) = z^1(r)$, for all other nodes and arcs. All constraints that are tight for $\tilde{z}$ are also tight for $z''$. To see this, it suffices to observe that $z''(v) = z^1(v) = 0$ and $z''(u, v) = z^1(u, t) = 0$. This contradicts the fact that $\tilde{z}$ is an extreme point of $P_p(G)$. Thus the proof of Lemma 11 is complete.

3.2. The proof of Theorem 10

Assume that $\tilde{z}$ is a fractional extreme point of $P_p(G)$. In this sub-section, we will not further suppose that $\tilde{z}(u, v) = \tilde{z}(v)$ when $v$ is not a sink node. The proof of Theorem 10 will be given in Sections 3.2.1 and 3.2.2. For that proof we need the following preliminaries.
**Lemma 34.** Let $S = \{(u_1, v_1), \ldots, (u_k, v_k)\}$ be a subset of arcs in $A$ where for $i = 1, \ldots, k$, $v_i$ is not a sink node. Let $G'$ be the graph obtained from $G$ by removing the arc $(u_i, v_i)$ and adding a new pendant node $t_i$ and the arc $(u_i, t_i)$ for each $i = 1, \ldots, k$. If $G'$ does not contain a g-odd Y-cycle, then we may assume that $\tilde{z}(u, v) = \tilde{z}(v)$ for at least one arc in $S$.

**Proof.** Suppose that $\tilde{z}(u_i, v_i) < \tilde{z}(v_i)$ for all $i = 1, \ldots, k$. Define $z'(u_i, t_i) = \tilde{z}(u_i, v_i), z'(t_i) = 1$, for all $i = 1, \ldots, k$, and $z'(s, t) = \tilde{z}(s, t), z'(r) = \tilde{z}(r)$ for all other arcs and nodes. It is easy to check that $G'$ share the same properties as $G$ and that $\zeta'$ is a fractional extreme point of $P_{p+1}(G')$. Thus one may consider $G'$ and $\zeta'$ instead of $G$ and $\zeta$. □

**Definition 35.** Let $v$ be a node in $G$. We call $v$ a knot if $\delta^-(v) = \{(u, v), (w, v)\}, u \neq w$ and both $(v, u)$ and $(v, w)$ belong to $\delta^+(v)$.

**Lemma 36.** Let $(u, v)$ and $(v, w)$ be two arcs in $A$. If $u = w$ or $v$ is a knot, then the graph $G'$ obtained from $G$ by removing $(u, v)$ and $(v, w)$ and adding two new pendant nodes $v'$ and $v''$ and two arcs $(u, v')$ and $(v, v'')$ does not contain a g-odd Y-cycle.

**Proof.** Recall that $G$ does not contain any of the graphs $H_1, H_2, H_3$ and $H_4$ of Fig. 1 as a subgraph. From this, it is easy to check that any g-odd Y-cycle in $G'$ is also a g-odd Y-cycle in $G$. But this contradicts the fact that $G$ does not contain a g-odd Y-cycle, which is assumed in this section. □

**Lemma 37.** Let $(u, v)$ and $(v, u)$ be two arcs in $G$. If $\delta^+(u) = \{(u, v), (v, u)\}$ and $\tilde{z}(v, u) = \tilde{z}(u, v)$, then $\tilde{z}(u, v) = \tilde{z}(v)$ for $\tilde{z} \in P_p(G)$.

**Proof.** Immediate from the validity of $\tilde{z}$. □

**Lemma 38.** We cannot have two arcs $(u, v)$ and $(v, w)$ where both are not tight for $\tilde{z}$.

**Proof.** In fact, Assign the label 1 to $(v, u)$ and $-1$ to $(v, w)$. Call this labeling $l$. Every constraint of $P_p(G)$ that is tight for $\tilde{z}$ remains tight for $\tilde{z}$, which contradicts the fact that $\tilde{z}$ is an extreme point of $P_p(G)$. □

**Lemma 39.** Let $v_1$ be a knot with $\delta^+(v_1) = \{(v_1, v_2), (v_1, v_0)\}$, and $\delta^-(v_1) = \{(v_2, v_1), (v_0, v_1)\}$. Then we cannot have $\tilde{z}(v_0, v_1) < \tilde{z}(v_1)$ and $\tilde{z}(v_1, v_0) < \tilde{z}(v_1)$. First notice that neither $v_0$ nor $v_2$ can be adjacent to a pendant node. Otherwise, this would contradict Lemma 38.

Let $G'$ be the graph obtained by removing the arc $(v_0, v_1)$, adding a pendant node $t$ and adding the arc $(v_0, t)$. It follows from Lemma 34 that $G'$ has a g-odd Y-cycle $\tilde{C}'$. Since $C'$ is a g-odd cycle of $G$ but not a Y-cycle, the arcs $(v_0', v_0), (v_1, v_0)$ and $(v_1, v_2)$ are in $C'$. This means that $v_0 \in C'$ and $v_1 \in \tilde{C}'$.

If there is an arc $(v_0, w)$ in $G$, we have seen that $w$ cannot be pendant, but if $w \neq v_0'$ then either $H_1$ or $H_3$ would be present, so $w = v_0'$. Now let us see that such an arc must exist. Suppose $\delta^+(v_0) = \{(v_0, v_1)\}$. If we take $S = \{(v_0, v_1), (v_1, v_0)\}$ in Lemma 34 and we combine this with Lemma 36, we have $\tilde{z}(v_0, v_1) = \tilde{z}(v_0)$. Now Lemma 37 implies that $\tilde{z}(v_0, v_1) = \tilde{z}(v_1)$, which is not possible. Thus

$$\delta^+(v_0) = \{(v_0, v_0'), (v_0, v_1)\}.$$

Because of the existence of $(v_0, v_0')$ and since $C'$ is a g-odd cycle of $G$ but not a Y-cycle we must have $v_0' \in \tilde{C}'$. By symmetry we have that there are two arcs $(v_2, v_3)$ and $(v_3, v_2)$, and

$$\delta^+(v_2) = \{(v_2, v_3), (v_3, v_2)\}.$$

Now we have two cases to study.

- Assume that $(v_3, v_2) \in C'$. Since $C'$ is a Y-cycle of $G'$ and $v_1 \in \tilde{C}'$, then $v_3 \in \tilde{C}'$. The cycle obtained from $C'$ by removing the arcs $(v_1, v_2)$ and $(v_0', v_0)$, and adding $(v_2, v_1)$ and $(v_0, v_0')$ is a g-odd Y-cycle in $G$ which is impossible.

- Assume that $(v_2, v_3) \in C'$. If $v_3 \in \tilde{C}'$, then the cycle obtained from $C'$ by replacing $(v_1, v_0)$ and $(v_1, v_2)$ by $(v_0, v_1)$ and $(v_2, v_1)$, is a g-odd Y-cycle which is impossible.

- If $v_2 \in \tilde{C}'$, then the cycle obtained from $C'$ by replacing the arcs $(v_2, v_3)$, $(v_1, v_2)$ and $(v_0, v_0')$ by the arcs $(v_3, v_2)$, $(v_2, v_1)$ and $(v_0, v_0')$, is a g-odd Y-cycle in $G$, which is impossible. □

**Definition 40.** Let $\tilde{z}$ be a fractional extreme point of $P_p(G)$. Let $v$ be a knot in $G$ with $\delta^-(v) = \{(u, v), (w, v)\}, u \neq w$ and both $(v, u)$ and $(v, w)$ belong to $\delta^+(v)$. Recall that $\tilde{z}(r, s) > 0$ for all $(r, s)$ and that from Lemma 15, $\tilde{z}(v, u) = \tilde{z}(v, w)$ and $\tilde{z}(w, v)$ are fractional. The node $v$ is called a fragile knot and we say that the pair $(G, \tilde{z})$ contains a fragile knot, if $\tilde{z}(u, v) < \tilde{z}(v)$ or $\tilde{z}(v, w) < \tilde{z}(v)$ and $(v, t) \in A$ where $t$ is a pendant node.

Notice that $(v, t)$ is unique from Lemma 38. Moreover, if $G$ satisfies (C1) then $\delta^+(v) = \{(v, t), (v, u), (v, w)\}$.

Let $(G, \tilde{z})$ be a pair containing a fragile knot $v$. The arcs incident to $v$ are $(u, v), (v, u), (w, v), (v, w)$ and $(v, t)$ with $t$ a pendant node. Assume that $\tilde{z}(u, v) < \tilde{z}(v)$. Define the graph $G(v)$ form $G$ as follows. Remove $v$ and its incident arcs. Add four nodes $v', v'', s'$ and a pendant node $t'$. Add the arcs $(u, v'), (v', u), (v', s'), (s', t'), (v'', w), (w, v'')$ and the arc $(v'', t)$. 

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Lemma 41. If $G$ is a simple directed graph, with no $g$-odd $Y$-cycle and satisfies condition (C1) of Theorem 2, then $G(v)$, as defined above, has the same properties.

Proof. Any $g$-odd $Y$-cycle in $G(v)$ is also a $g$-odd $Y$-cycle in $G$. By definition $G(v)$ is simple. Now suppose that $G(v)$ contains one of the graphs $H_1, H_2$ or $H_3$ as a subgraph, call it $H$. Notice that $H$ cannot contain $(s', t')$. If it contains $(v', s')$, then by replacing it by $(v, w)$ one obtains the same subgraph in $G$. If $H$ does not contain $(v', s')$ and contains the node $v'$, then the set of nodes in $H$ where $v'$ is replaced by $v$ induces the same subgraph in $G$, which is not possible. Similar arguments can be used with $v''$. Finally, if $H$ does not contain $v'$ nor $v''$, then $H$ is also a subgraph in $G$.

Lemma 42. Let $G = (V, A)$ be a directed graph. If $P_p(G)$ admits a fractional extreme point $\tilde{z}$, where $(G, \tilde{z})$ contains a fragile knot $v$, $(\tilde{z}(u, v) < \tilde{z}(v))$, then $P_p(G(v)) \neq \tilde{z}MP(G(v))$, with $\tilde{p} = p + 2$.

Proof. Let $v$ be a fragile knot. The arcs incident to $v$ are $(u, v), (w, v), (v, u), (v, t)$ and $(v, t)$, where $t$ is a pendant node, and we have $\tilde{z}(u, v) < \tilde{z}(v)$. Suppose that $P_p(G(v)) = \tilde{z}MP(G(v))$. Define $\bar{z} \in P_p(G(v))$ to be

$$\bar{z}(l) = \begin{cases} \tilde{z}(v) & \text{if } l = v' \text{ or } l = v'' \\
1 - \tilde{z}(v) & \text{if } l = t' \\
1 & \text{if } l = t \\
\tilde{z}(l) & \text{otherwise} \end{cases}$$

The vector $\tilde{z}$ is fractional, so $\bar{z}$ is not an extreme point of $P_p(G(v))$. In the following we will construct a solution $z''$ so that the same constraints that are tight for $\bar{z}$ are also tight for $z''$, then we do not need that $z'' \in P_p(G)$ to contradict the fact that $\tilde{z}$ is an extreme point of $P_p(G)$.

Since $P_p(G(v))$ is integral, there is a 0–1 vector $z^* \in P_p(G(v))$ with $z^*(v', s') = 1$ so that the same constraints that are tight for $\bar{z}$ are also tight for $z^*$. From $z^*$ define $z''$ as follows.

$$z''(l) = \begin{cases} z^*(v'') & \text{if } l = v \\
z^*(l) & \text{otherwise} \end{cases}$$

All constraints that are tight for $\bar{z}$ are also tight for $z''$. To see this, it suffices to notice that $\sum_{v \in V} z''(v) = p$ since $z^*(v') = z^*(v'') = 1$. Also notice that $z^*(u, v') = z^*(v'') = 0$ and that $z^*(v''')$ may be equal to 0 or 1, so we may have $z''(u, v) < z''(v') = 1$ but this inequality was not tight for $\bar{z}$.

Let $P = v_1, v_2, v_3, v_4$ be a bidirected chain of size four where its internal nodes are adjacent to only their neighbors in $P$. Define $G(P)$ the graph obtained from $G$ by identifying the nodes $v_1$ and $v_4$, call $v^*$ the resulting node, and by removing the nodes $v_2$ and $v_3$ with their incident arcs.

Lemma 43. $G(P)$ is simple, satisfies condition (C1) of Theorem 2 and does not contain a $g$-odd $Y$-cycle.

Proof. The graph $G(P)$ is exactly the graph $G'$ as defined in the proof of Lemma 20. Therefore, the proof is exactly part (i) of the proof of Lemma 20.

Lemma 44. Let $G = (V, A)$ be a directed graph and $P = v_1, v_2, v_3, v_4$ be a bidirected chain of size four where its internal nodes are adjacent to only their neighbors in $P$. If $P_p(G)$ admits a fractional extreme point $\tilde{z}$ where all the arcs of $P$ are tight for $\tilde{z}$ except for $(v_2, v_3)$ or (exclusive) for $(v_3, v_4)$, then $P_{p-1}(G(P))$ is not integral.

Proof. Define $z'$ from $z''$ as follows.

$$z'(v) = \begin{cases} z(v_1) & \text{if } v = v^* \text{ and } z(v_2, v_3) < z(v_3), \\
z(v_4) & \text{if } v = v^* \text{ and } z(v_3, v_4) < z(v_4), \\
z(v) & \text{otherwise}, \end{cases}$$

$$z'(u, v) = \begin{cases} z(v_1, v) & \text{if } u = v^* \text{ and } (v_1, v) \in A, \\
z(u, v_1) & \text{if } v = v^* \text{ and } (u, v_1) \in A, \\
z(v_4, v) & \text{if } u = v^* \text{ and } (v_4, v) \in A, \\
z(u, v_4) & \text{if } v = v^* \text{ and } (u, v_4) \in A, \\
z(u, v) & \text{if } u \neq v^* \text{ and } v \neq v^*. \end{cases}$$
We claim that $z'$ is a fractional vector of $P_{p-1}(G(P))$. Obviously $z'$ is fractional. Let us examine its validity. By the definition any constraint where $z'(v^*)$ does not appear is satisfied. We have two cases to consider, when $\bar{z}(v_2, v_3) < \bar{z}(v_3)$ and when $\bar{z}(v_3, v_4) < \bar{z}(v_4)$. Let us examine the first one. Symmetrical arguments hold for the second case.

Thus $\bar{z}(v_2, v_3) < \bar{z}(v_3)$. Let us show that $\sum z'(v) = p - 1$. We have that $\sum z'(v) = \sum_{v \in V} \bar{z}(v) - \bar{z}(v_2) - \bar{z}(v_3) - \bar{z}(v_4)$. Notice that the validity of $\bar{z}$ implies that

$$\bar{z}(v_2) + \bar{z}(v_3) + \bar{z}(v_4) = 1.$$  \hspace*{1cm} (23)

Since all the arcs of $P$ are tight for $\bar{z}$ except $(v_2, v_3)$, Eq. (23) is equivalent to

$$\bar{z}(v_2) + \bar{z}(v_3) + \bar{z}(v_4) = 1.$$  \hspace*{1cm} (24)

Then we have that $\sum z'(v) = \sum_{v \in V} \bar{z}(v) - \bar{z}(v_2) - \bar{z}(v_3) - \bar{z}(v_4) = p - 1$.

Let us see that Eq. (3) with respect to $v^*$ is satisfied, that is $z'(v^*) + z'(\delta^+(v^*)) = 1$. By definition we have

$$z'(v^*) + z'(\delta^+(v^*)) = \bar{z}(v_1) + \bar{z}(\delta^+(v_1)) = \bar{z}(v_4) = 1.$$  \hspace*{1cm} (25)

Eq. (3) with respect to $v_1$ and $v_4$ implies

$$\bar{z}(v_1) + \bar{z}(\delta^+(v_1) \setminus \{(v_1, v_2)\}) + \bar{z}(v_2) = 1,$$  \hspace*{1cm} (26)

which implies that $\bar{z}(v_1) + \bar{z}(\delta^+(v_1) \setminus \{(v_1, v_2)\}) + \bar{z}(v_4, v_3) = 1$. Hence $z'(v^*) + z'(\delta^+(v^*)) = 1$.

Let us show that $z'(u, v^*) \leq z'(v^*)$ for any arc $(u, v^*)$ in $G(P)$. Let $(u, v^*)$ be an arc of $G(P)$. Then $(u, v_1)$ or $(u, v_4)$ exists in $G$. If $(u, v_1)$ is in $G$, then from the definition of $z'$ and the validity of $\bar{z}$ it follows that $z'(u, v^*) = \bar{z}(u, v_1) \leq \bar{z}(v_1) = z'(v^*)$.

Now assume that $(u, v_4)$ is in $G$.

The validity of $\bar{z}$ implies that

$$\bar{z}(v_2) + \bar{z}(v_3) + \bar{z}(v_4) = 1.$$  \hspace*{1cm} (27)

Since $\bar{z}(v_2, v_1) = \bar{z}(v_1)$, then Eq. (27) is equivalent to

$$\bar{z}(v_2) + \bar{z}(v_3) + \bar{z}(v_4) = 1.$$  \hspace*{1cm} (28)

Combining (24) with (28) we obtain

$$\bar{z}(v_1) + \bar{z}(v_2, v_3) = \bar{z}(v_3) + \bar{z}(v_4).$$  \hspace*{1cm} (29)

Since $\bar{z}(v_2, v_3) < \bar{z}(v_3)$, it follows from (29) that $\bar{z}(v_1) > \bar{z}(v_4)$. Therefore,

$$z'(u, v^*) = \bar{z}(u, v_4) \leq \bar{z}(v_1) = z'(v^*).$$

Now let us begin the proof of the lemma. Assume that is false, $P_{p-1}(G(P))$ is integral. It follows, from the above discussion, that $z'$ is not an extreme point of $P_{p-1}(G(P))$. So $z'$ can be written as a convex combination of 0–1 vectors that satisfy with equality each constraint that is satisfied with equality by $z'$. Among these 0–1 solutions we will choose a solution $z^*$ following the above two cases.

(i) The case $\bar{z}(v_2, v_3) < \bar{z}(v_3)$. If there is an arc $(u, v^*)$ in $G(P)$ that corresponds to $(u, v_4)$, then one can choose $z^*$ with $z^*(u, v^*) = 1$, since $z'(u, v^*) > 0$. Otherwise, choose $z^*$ so that $z^*(v^*) = 1$. Notice that $(u, v_4)$ is unique, otherwise the graph $H_4$ is present.

(ii) The case $\bar{z}(v_3, v_4) < \bar{z}(v_4)$. If there is an arc $(u, v^*)$ in $G(P)$ that corresponds to $(u, v_1)$, then one can choose $z^*$ with $z^*(u, v^*) = 1$, since $z'(u, v^*) > 0$. Otherwise, choose $z^*$ so that $z^*(v^*) = 1$. Notice that $(u, v_1)$ is unique, otherwise the graph $H_4$ is present.

In both cases, we have $z^*(v^*) = 1$. From $z^*$ define $z'' \in P_p(G)$ as follows.

$$z''(v) = \begin{cases} 0 & \text{if } v \in \{v_2, v_3\}, \\ 1 & \text{if } v \in \{v_1, v_4\}, \\ z^*(v) & \text{otherwise}, \end{cases}$$

$$z''(u, v) = \begin{cases} 1 & \text{if } (u, v) \in \{(v_2, v_1), (v_3, v_4)\}, \\ 0 & \text{if } (u, v) \in \{(v_1, v_2), (v_3, v_2), (v_2, v_3), (v_4, v_3)\}, \\ z^*(u, v) & \text{if } u \in \{v_1, v_4\} \text{ and } v \in V \setminus \{v_1, v_2, v_3, v_4\}, \\ z^*(u, v) & \text{otherwise}. \end{cases}$$

It is easy to see that $z'' \in P_p(G)$. Now we have to see that $z''$ satisfies with equality any constraint that is satisfied with equality by $\bar{z}$. For this, it suffices to see the following.
• Case (i). If the arc \((u, v)\) exists, \(u \neq v_3\), then this arc is unique and by definition \(z'(u, v_4) = z''(v_4) = 1\). If there is an arc \((u, v_1)\) with \(z(u, v_1) = z(v_1)\), then by definition \(z'(u, v') = z(u, v_1) = z(v_1) = z'(v')\). Since each constraint satisfied as equation by \(z'\) is also satisfied as equation by \(z''\) and that \(z''(v') = 1\), we have \(z'(u, v') = z''(v') = 1\) and \(z''(v_1) = z'(v_1) = 1\) follows by definition.

• Case (ii) is similar to Case (i) by exchanging \((u, v_4)\) with \((u, v_1)\) and vice versa. □

All the material defined above permits us to begin the proof Theorem 10. Denote by knot(G) the set of knots in G. The proof is by induction on \(|\text{knot}(G)|\). In Section 3.2.1 we give the proof when \(|\text{knot}(G)| = 0\), and in Section 3.2.2 we complete it for \(|\text{knot}(G)| \geq 1\).

3.2.1. Proof of Theorem 10 when \(|\text{knot}(G)| = 0\)

Suppose that the theorem is false. Let \(\tilde{z}\) be a fractional extreme point of \(P_p(G)\). By Lemma 11, there must exist an arc \((u, v)\) with \(\tilde{z}(u, v) < \tilde{z}(v)\) and \(v\) is not a sink node. Lemma 34 implies that the graph \(G'\) obtained from \(G\) by removing \((u, v)\) and adding a pendant node \(v'\) with the arc \((u, v')\) contains a g-oed Y-cycle \(C\). Notice that in this case \(u\) must be in \(C\) and is not adjacent to a pendant node in \(G\). Also, since \(G\) contains no knot and none of the graphs \(H_1, \ldots, H_4\) as a subgraph, this implies that \(\delta_{C}(u) = \{(u, v)\}\) and \(\delta_{C}(u) = \{(s, u), (v, u)\}\), where \(s\) and \(v\) are the two neighbors of \(u\) in \(C\).

If we take \(S = \{(u, v), (v, u)\}\) in Lemma 36, then with Lemma 34 we have that \(\tilde{z}(u, v) = \tilde{z}(u)\) since \(\tilde{z}(u, v) < \tilde{z}(v)\). But then, Lemma 37 implies that \(\tilde{z}(u, v) = \tilde{z}(v)\), a contradiction.

3.2.2. Proof of Theorem 10 when \(|\text{knot}(G)| \geq 1\)

Suppose that the theorem is true for every simple directed graph, with no g-oed Y-cycle, satisfying condition (C1) of Theorem 2 and having at most \(m\) knots, with \(m \geq 0\). Let \(G = (V, A)\) be a graph with these properties and \(|\text{knot}(G)| = m + 1\). Assume that \(\tilde{z}\) is a fractional extreme point of \(P_p(G)\).

Claim 1. \((G, \tilde{z})\) does not contain a fragile knot.

Proof. Suppose the contrary and let \(v\) be a fragile knot. We have that \(|\text{knot}(G(v))| \leq m\) and by Lemma 41 the graph \(G(v)\) is simple, with no g-oed Y-cycle and satisfies condition (C1) of Theorem 2. Thus the induction hypothesis applies, so \(P_{p+2}(G(v))\) is integral. This contradicts Lemma 42. □

Claim 2. The graph \(G\) does not contain a bidirected chain of size four satisfying the hypothesis of Lemma 44.

Proof. Suppose the contrary and let \(P\) be a bidirected chain satisfying the hypothesis of Lemma 44. We have that \(|\text{knot}(G(P))| \leq m\) and by Lemma 43 the graph \(G(P)\) is simple, with no g-oed Y-cycle and satisfies condition (C1) of Theorem 2. Thus the induction hypothesis applies, so \(P_{p-1}(G(P))\) is integral. This contradicts Lemma 44. □

Claim 3. If there is an arc \((v, w)\) with \(\tilde{z}(v, w) < \tilde{z}(w)\) and \(w\) is not a sink node, then \(v\) is a knot where \(\delta^+(v) = \{(v, w), (v, u)\}\) and \(\delta^-(v) = \{(u, v), (w, v)\}\) and \((v, u), (u, v)\) and \((w, v)\) are tight for \(\tilde{z}\). Moreover, there are two arcs \((u, u')\) and \((w, w')\) where \(u', w', v\) are three different nodes and both \(u'\) and \(w'\) are not sink nodes.

Proof. Let \((v, w)\) an arc of \(G\) with \(\tilde{z}(v, w) < \tilde{z}(w)\) and \(w\) is not a sink node. Lemma 34 implies that the graph \(G'\) obtained from \(G\) by removing \((v, w)\) and adding a new pendant node \(v'\) and the arc \((v, v')\) contains a g-oed Y-cycle \(C\). The fact that \(G\) does not contain a g-oed Y-cycle implies that \(C\) is a g-oed cycle in \(G\) where \(v \in C\) and does not satisfy neither Definition 6(i) nor (ii) with respect to \(G\). Hence \(\delta^-(v) = \{(w, v), (u, v)\}\) (both arcs are in \(C\)), otherwise the graph \(H_4\) is present. Notice that \(w\) is in \(C\).

Lemma 34 with \(S = \{(v, w), (w, v)\}\) together with Lemma 36 imply that \(\tilde{z}(w, v) = \tilde{z}(v)\). Since \(\tilde{z}(v, w) < \tilde{z}(w)\), Lemma 37 implies that we must have an arc \((v, t)\) different from \((v, w)\). Since \(v\) is in \(C\) and it does not satisfy neither Definition 6(i) nor (ii) and \(G\) satisfies condition (C1) of Theorem 2, we must have \(t = u\) and \(u\) must be in \(C\). Thus \(\delta^+(v) = \{(v, u), (v, w)\}\), so \(v\) is a knot. Again Lemma 34 with \(S = \{(u, v), (v, w)\}\) together with Lemma 36 (\(v\) is a knot) imply that \(\tilde{z}(u, v) = \tilde{z}(v)\) and by Lemma 38 we cannot have \(\tilde{z}(v, u) < \tilde{u}(u)\).

We noticed that \(u\) and \(w\) are in \(C\). Thus we must have two arcs \((u, u')\) and \((w, w')\) in \(C\) with \(u' \neq v\) and \(v' \neq v\) and both are not sink nodes. Also \(u' \neq w'\) follows from the fact that \(C\) is g-oed. □

Now let us finish the proof of Theorem 10. By Lemma 11, \(G\) must contain an arc \((v_2, v_3)\) with \(\tilde{z}(v_2, v_3) < \tilde{z}(v_3)\) and \(v_3\) is not a sink node. By Claim 3 we must have a node \(v_1\) with the following:

• \(\delta^-(v_3) = \{(v_1, v_2), (v_3, v_2)\}\); \(\delta^+(v_2) = \{(v_2, v_1), (v_2, v_3)\}\); \(\delta^-(v_2) = \{(v_2, v_1), (v_2, v_3)\}\); \(\delta^+(v_1) = \{(v_1, v_2), (v_1, v_3)\}\).

We cannot have \(\delta^-(v_1) = \{(v_2, v_1)\}\) and \(\delta^-(v_3) = \{(v_2, v_3)\}\). Otherwise Lemma 18 is contradicted. We distinguish two cases.

Case 1. \(\delta^-(v_3) \neq \{(v_2, v_3)\}\). Let \((v_4, v_3) \in A\) with \(v_4 \neq v_2\). Since \(G\) satisfies Condition (C1) of Theorem 2, it follows that \((v_4, v_3)\) as defined is unique. Notice that \(v_4 \neq v_1\), otherwise the arcs \((u_1, v_3), (v_3, v_2)\) and \((v_2, v_1)\) form a g-oed Y-cycle. Claim 3 implies that \((v_3, u)\) exists with \(u \neq v_2\) and is not a sink node. We have \(u = v_4\), otherwise \(H_1\) or \(H_3\) is present. Thus
$v_3$ is a knot and from Claim 1 it is not a fragile knot, that is, $\delta^+(v_3) = \{(v_3, v_2), (v_3, v_4)\}$ and using Lemma 39 we have $\tilde{z}(v_4, v_3) = \tilde{z}(v_3)$. If we take $S = \{(v_2, v_3), (v_3, v_4)\}$ in Lemma 34, then with Lemma 36 we have that $\tilde{z}(v_3, v_4) = \tilde{z}(v_4)$ since $\tilde{z}(v_2, v_3) < \tilde{z}(v_3)$. But then the bidirected chain $P = v_1, v_2, v_3, v_4$ contradicts Claim 2.

Case 2. $\delta^-(v_1) \neq \{(v_2, v_1)\}$. Let $(u, v_1)$ be an arc in $G$ with $u \neq v_2$. Claim 3 implies that $(v_1, v_0)$ is an arc of $G$ with $v_0 \neq v_2$ and $v_0$ is not a sink node. Condition (C1) of Theorem 2, implies that $u = v_0$. Suppose that $\tilde{z}(v_1, v_0) < \tilde{z}(v_0)$ (resp. there exist an arc $(v_1, t)$ and a pendant node). Define the following labeling function $l$. Assign the label 1 to the arcs $(v_1, v_0)$ (resp. $(v_1, t)$) and $(v_2, v_3)$ and to the node $v_3$; assign the label $-1$ to the arcs $(v_1, v_2)$ and $(v_3, v_2)$ and to the node $v_2$; for all other arcs and nodes assign the label 0. Then any constraint that is tight for $z$ is also tight for $\tilde{z}$. This is true because from Case 1 there is no arc different from $(v_2, v_3)$ directed into $v_3$. We have a contradiction with the fact that $\tilde{z}$ is an extreme point. Hence we must have $\tilde{z}(v_1, v_0) = \tilde{z}(v_0)$ and $\delta^+(v_1) = \{(v_1, v_0), (v_1, v_2)\}$. We cannot have $\tilde{z}(v_0, v_1) = \tilde{z}(v_1)$, otherwise the bidirected chain $P = v_0, v_1, v_2, v_3$ contradicts Claim 2. Thus $\tilde{z}(v_0, v_1) < \tilde{z}(v_1)$. Claim 3 implies that we must have a node $u$ with the following:

- $\delta^-(v_0) = \{(v_1, v_0), (u, v_0)\}; \delta^+(v_0) = \{(v_0, v_1), (v_0, u)\};$  
- $\tilde{z}(u, v_0) = \tilde{z}(v_1, v_0) = \tilde{z}(v_0); \tilde{z}(v_0, u) = \tilde{z}(u).$

Notice that $u \neq v_2$, otherwise we have a directed cycle of size three. But now the bidirected chain $P = u, v_0, v_1, v_2$ contradicts Claim 2. This completes the proof of Theorem 10.

4. Graphs with $g$-odd $Y$-cycles

In this section we prove another simplified version of Theorem 2, that will be combined with Theorem 10 to complete the proof of the main result.

**Theorem 45.** If $G = (V, A)$ is a simple directed graph satisfying conditions (C1) and (C2) of Theorem 2 and containing a $g$-odd $Y$-cycle, then $P_G(G)$ is integral.

The graph $G = (V, A)$ we consider here is simple, directed; it satisfies conditions (C1) and (C2) of Theorem 2 and contains a $g$-odd $Y$-cycle $C = v_0, a_0, v_1, a_1, \ldots, a_{k-1}, v_l$. The proof of Theorem 45 is done by induction on $|\text{Pair}(G)|$ (the number of pair of nodes $(u, v)$ such that $(u, v)$ and $(v, u)$ are in $G$). In Section 4.1 we prove it for oriented graphs that is when $|\text{Pair}(G)| = 0$. The proof is completed, by induction, in Section 4.2. Next we give several lemmas to show that the node-set of any cycle in $G$ must coincide with $V(C)$. This define the structure of $G$ and it is useful to prove Theorem 45.

Let $\tilde{z}$ be a fractional extreme point of $P_G(G)$.

**Lemma 46.** We may assume that $\tilde{z}(u, v) > 0$ for all $(u, v) \in A$.

**Proof.** Let $G'$ be the graph obtained after removing all arcs $(u, v)$ with $\tilde{z}(u, v) = 0$. Let $z'$ be the restriction of $\tilde{z}$ to $G'$. Since $\tilde{z}$ is a fractional extreme point of $P_G(G)$, this implies that $z'$ is a fractional extreme point of $P_{G'}(G')$. Notice that $G'$ satisfies conditions (C1) and (C2) of Theorem 2. Also $G'$ contains a $g$-odd $Y$-cycle, otherwise from Theorem 10 we have that $P_{G'}(G')$ is integral, this would contradict the fact that $z'$ is an extreme point of $P_{G'}(G')$. □

We also notice that from the lemma above and constraints (4), we should have $\tilde{z}(v) > 0$ for all $v \in V$ with $|\delta^-(v)| \geq 1$. This and Lemma 46 will be used implicitly when we define a new solution from $\tilde{z}$.

In this section we may also assume that every sink node is also a pendant node, as shown in the next lemma.

**Lemma 47.** We may assume that every sink node $v$ in $G$ is a pendant node.

**Proof.** If $C$ is a $g$-odd $Y$-cycle then $v$ cannot be in $V(C)$. So the graph $G'$ as constructed in the proof of Lemma 16 has the same properties as $G$: it satisfies conditions (C1) and (C2) and it contains a $g$-odd $Y$-cycle. Hence the pair $G'$ and $z'$ as defined in the proof of Lemma 16 may be considered instead of $G$ and $\tilde{z}$. □

Let $v_k$ and $v_l$ be two nodes in $V(C)$. Call $P_1$ and $P_2$ the two chains in $C$ from $v_k$ to $v_l$. We are going to prove that if there is another chain between $v_k$ and $v_l$ whose internal nodes are not in $V(C)$, then this chain consists of just one arc and $v_k$ and $v_l$ should be consecutive in $C$. Assume the contrary, and let $P = v_k, b_1, u_1, \ldots, u_{k-1}, b_l, v_l$ be another chain between $v_k$ and $v_l$. Assume that all internal nodes of $P$ are not in $V(C)$. Notice that because of (C2) $P$ cannot have more than two arcs. We call $C_1$ (resp. $C_2$) the cycle defined by $P_1$ and $P$ (resp. $P_2$ and $P$).

**Lemma 48.** Assume that $v_k$ and $v_l$ are not consecutive in $C$ or $P$ contains two arcs, then if an arc of $P$ is directed into (resp. away from) $v_k$ (or $v_l$) then this node must be in $C$ (resp. $\hat{C} \cup \hat{C}$).

**Proof.** Suppose first that $b_1$ is directed into $v_k$, thus $b_1 = (u_1, v_k)$. Assume that $v_k$ and $v_l$ are not consecutive or that $P$ consists of two arcs.

Let $v_k \in C$. If $v_k \in \hat{C}$ (resp. $v_k \notin \hat{C}$) then $G$ contains $H_2$ (resp. $H_2'$) as a subgraph.
Now assume that $v_k \in \tilde{C}$. Let $(v_{k-1}, v_k)$ and $(v_k, v_{k+1})$ be the two arcs of $C$ incident to $v_k$. The node $v_{k+1}$ is not a sink node, so there is an arc $(v_{k+1}, u)$. If $u \in \{v_k, v_{k-1}, u_1\}$ (resp. $u \notin \{v_k, v_{k-1}, u_1\}$) then the graph defined by $(v_{k-1}, v_k)$, $(u_1, v_k)$, $(v_k, v_{k+1})$ and $(v_{k+1}, u)$ corresponds to $H_2$ or $H_4$ (resp. $H_1$). Therefore $v_k \notin \tilde{C}$.

- Suppose now that $b_1$ is directed away from $v_k$, thus $b_1 = (v_k, u_1)$. Suppose that $v_k \in \tilde{C}$, and $(v_k, v_{k-1})$ and $(v_{k+1}, v_k)$ are the two arcs of $C$ incident to $v_k$.

  Assume first that $P$ consists of two arcs.

  - Assume that $(u_1, v)$ is the second arc of $P$. If $v_1$ coincides with $v_{k+1}$ or $v_{k-1}$, then we have $H_3$ as a subgraph, otherwise we have $H_1$ as a subgraph.
  
  - Assume now that $(v, u_1)$ is the second arc of $P$. Since $|\delta^{-}(u_1)| \geq 2$, by Lemma 47 $u_1$ is not a sink node, so there is an arc $(u_1, u)$. If $u = v_k$ we have $H_4$ as a subgraph; if $u$ coincides with $v_{k-1}$ or $v_{k+1}$ we have $H_2$ as a subgraph; otherwise we have $H_1$ as a subgraph.

  Assume now that $P$ consists of one arc and $v_k$ and $v_1$ are not consecutive. So $u_1 = v_k$. Since $b_1$ is directed into $v_k$, we have seen above that $v_1$ must be in $\tilde{C}$. In this case we must have $H_1$ or $H_2$ as a subgraph. □

**Lemma 49.** If $v_k$ and $v_1$ are not consecutive in $C$, then $P$ cannot consist of just one arc.

**Proof.** Let $P = v_k, (v_k, v_1), v_1$. By Lemma 48, $v_1 \in \tilde{C}$ and $v_k \in \tilde{C} \cup \tilde{C}$. We then consider two cases: (a) $v_k \in \tilde{C}$ and (b) $v_k \in \tilde{C}$.

(a) $C_1$ and $C_2$ are both $Y$-cycles and exactly one of them is $g$-odd. The fact that $G$ satisfies (C2) implies that the $g$-even cycle contains three arcs. Let $C_1$ be the $g$-even cycle. Thus $C_1 = (v_k, (v_k, v_1), (v_1, v), (v, v), v_k)$, where $v \in \tilde{C}$. Since both nodes $v_k$ and $v_1$ are in $\tilde{C}$, there is an arc $(v, \tilde{v})$, where $\tilde{v}$ is a pendant node, $\tilde{v} \notin V(C)$. Therefore condition (C2) is violated by $C_2$ and $(v, \tilde{v})$.

(b) Let $(u, v_k)$ and $(v_k, v)$ be the two arcs in $A(C)$ incident to $v_k$. Notice that there is no arc from $v$ to $v_k$, otherwise $G$ contains $H_1$ or $H_2$ as a subgraph. Thus $C_1$ and $C_2$ are both $Y$-cycles. The parity of $C$ implies that exactly one of these cycles is $g$-odd. If one is $g$-odd the fact that $G$ satisfies (C2) implies that the other cycle must contain three arcs. So the $g$-odd cycle must be the one containing the arc $(u, v_k)$ and $C_2$. Let $C_1 = (v_k, (v_k, v_1), (v_1, v), (v, v), v_k)$. Since $C_1$ and $C_2$ are both $Y$-cycles, there is an arc $(v, \tilde{v})$, where $\tilde{v}$ is a pendant node, $\tilde{v} \notin V(C)$. Therefore condition (C2) is violated by $C_2$ and $(v, \tilde{v})$. □

**Lemma 50.** The chain $P$ cannot consist of two arcs.

**Proof.** Let $P = v_k, b_1, u_1, b_2, v_1$. We have to study three cases.

1. $b_1 = (u_1, v_k)$ and $b_2 = (u_1, v_1)$. By Lemma 48, both $v_k$ and $v_1$ are in $\tilde{C}$. Both $C_1$ and $C_2$ are $Y$-cycles and exactly one of them must be $g$-odd, otherwise $C$ is a $g$-even $Y$-cycle. Suppose that $C_1$ is $g$-odd. Then $C_2$ is $g$-even and must contain four arcs, otherwise $G$ does not satisfy (C2). Now it is easy to see that $|\hat{C}_2| + |\hat{C}_2| = 3$, a contradiction.

2. $b_1 = (v_k, u_1)$ and $b_2 = (u_1, v_1)$. The case where $b_1 = (u_1, v_k)$ and $b_2 = (u_1, u_1)$ may be treated by symmetry. By Lemma 48, $v_1 \in \tilde{C}$ and $v_k \in \tilde{C} \cup \tilde{C}$. So $v_k \in \tilde{C}$ or $v_k \in \tilde{C}$. If $v_k \in \tilde{C}$, then $C_1$ and $C_2$ are both $Y$-cycles. The parity of $C$ implies that exactly one of $C_1$ or $C_2$ is $g$-odd. As in the previous case we have that $|\hat{C}_2| + |\hat{C}_2| = 3$, a contradiction. Thus assume that $v_k \in \tilde{C}$. Let $(u, v_k)$ and $(v_k, v)$ be the two arcs of $C$ incident to $v_k$. Let $C_2$ be the cycle containing $u$ and $C_1$ the one containing $v$. It is easy to see that $u$ and $v$ are different from $v_1$. Let us see that $C_1$ and $C_2$ are $Y$-cycles. It is straightforward that $C_2$ is a $Y$-cycle. The only case that makes $C_1$ not a $Y$-cycle is that $v \in \tilde{C}$ and the only arc leaving $v$ is directed into $v_k$. But in this case $H_1$ is present.

Moreover, $C_2$ is a directed cycle of size four. In fact, the parity of $C$ implies that exactly one of the cycles $C_1$ or $C_2$ is $g$-odd. If $C_2$ is $g$-odd then, as in the previous case $|\hat{C}_2| + |\hat{C}_2| = 3$, which is impossible. So suppose that $C_1$ is $g$-odd. Then $C_2$ is a directed cycle of size four, $C_2 = (v_k, (v_k, v_1), (u_1, v_1), (v_1, v), u_1, (u_1, v_k), v_k)$, see Fig. 17.

Now we want to apply the labeling procedure of Section 2 to $C_2$, to obtain a solution that contradicts the fact that $\hat{z}$ is an extreme point of $P_2(G)$. To this end, we need the following additional fact: if there is an arc $(w, t)$ not in $A(C_2)$ directed into a node in $C_2$, then $\hat{z}(w, t) < \hat{z}(t)$.

If $w \notin V(C_2)$, then $G$ contains $H_1$; and if $w$ and $t$ are not consecutive in $C_2$, then $G$ contains $H_3$. So assume $(w, t) \in A \setminus A(C_2)$ and $t$ and $w$ are two consecutive nodes in $V(C_2)$. Let $C'_2$ be the cycle obtained from $C_2$ by adding $(w, t)$ and removing $(t, w)$. We have two sub-cases.

- Assume that $C'_2$ is a $g$-odd $Y$-cycle. This implies that $C'_1$ must be of size four, otherwise $G$ does not satisfy (C2). Thus the arcs $(v_k, v)$ and $(v, v_k)$ are in $A(C_1)$ and if $C_1$ is of size four, it was proved above that $v \in \hat{C}_1$. Let $(u, v) \in A$ with $\tilde{v} \notin V(C)$. If $u \neq u_1$, then the pair $C'_1$ and $(v, v)$ violates condition (C2) of Theorem 2. And if $\tilde{v} = u_1$, then the graph defined by $(v_1, u_1), (u_1, v), (v_1, v)$ and $(v_1, u_1)$ corresponds to $H_5$, which is not possible.

- The case when $C'_2$ is not a $Y$-cycle is obtained when $(w, t) = (v_1, u_1)$ or $(w, t) = (v_k, u)$ and in both cases $\delta^+(t) = \{t, w\}$. Suppose that $\hat{z}(w, t) = \hat{z}(t)$. Thus constraint (3) with respect to $t$ implies that

$$\hat{z}(t) + \hat{z}(w, t) = 1 = \hat{z}(t, w) + \hat{z}(w, t).$$

Since $w$ is one of the nodes $v_k$ or $v_1$, then there is an arc $(w, t')$ where $t'$ is another node in $C$ different from $t$. Lemma 46 implies that $\hat{z}(w, t') > 0$. Hence from constraint (3) with respect to $w$

$$\hat{z}(w) + \hat{z}(w, t) < 1.$$

Combining (30) with (31) we obtain, $\hat{z}(t, w) > \hat{z}(w)$. But this contradicts the validity of $\hat{z}$. 
Hence we may suppose that if there is an arc \((w, t)\) not in \(C_2\) directed into a node in \(C_2\), then \(\bar{z}(w, t) < \bar{z}(t)\). Assign labels to the nodes and arcs in \(C_2\) following the labeling procedure of a \(g\)-even cycle. Extend this labeling by assigning the label 0 to each node and arc with no label. Call this labeling \(I\). The constraints that are satisfied with equality by \(\bar{z}\) are also satisfied with equality by \(\bar{z}_i\). This contradicts the fact that \(\bar{z}\) is an extreme point of \(P_p(G)\). Notice that we do not need \(\bar{z}_i \in P_p(G)\).

(3) \(b_1 = (v_h, u_1)\) and \(b_2 = (v_l, u_1)\). Notice that by Lemma 47, \(u_1\) is not a sink node. Since \(G\) satisfies condition (C2), there is an arc \((u_1, t)\) with \(t \in V(C)\). If \(t\) is different from \(v_h\) and \(v_l\) then one can easily create a subgraph in \(G\) that is one of the subgraphs of Fig. 1. So \(t\) must coincide with \(v_h\) or \(v_l\), say \(t = v_l\). If we take the chain \(P' = v_h, (v_h, u_1), (u_1, v_l), v_l\), instead of \(P\), this reduces to Case (2) above. \(\square\)

**Lemma 51.** The node-set of any cycle of size at least three in \(G\) coincides with \(V(C)\).

**Proof.** The proof is straightforward from Lemmas 49 and 50 and condition (C2) of Theorem 2. \(\square\)

The following lemma permits the reduction to oriented graphs, the case where \(|\text{Pair}(G)| = 0\).

Let \((u, v)\) and \((v, u)\) be two arcs in \(A\). Denote by \(G(u, v)\) the graph obtained from \(G\) by removing the arc \((u, v)\) and adding a new arc \((u, t)\), where \(t\) is a new pendent node.

**Lemma 52.** Let \(G = (V, A)\) be a directed graph and \((u, v)\) and \((v, u)\) two arcs in \(A\). If \(P_p(G)\) admits a fractional extreme point \(\bar{z}\) with \(\bar{z}(v, u) > 0\), then \(P_p(G(u, v)) \neq \bar{p}\text{MP}(G(u, v))\), where \(\bar{p} = p + 1\).

**Proof.** Let \(\bar{z}\) be a fractional extreme point of \(P_p(G)\) with \(\bar{z}(v, u) > 0\). Suppose that \(P_p(G(u, v)) = \bar{p}\text{MP}(G(u, v))\). Define \(\bar{z} \in P_p(G(u, v))\) to be \(\bar{z}(u, v) = \bar{z}(u, v)\), \(\bar{z}(t) = 1\) and \(\bar{z}(r), \bar{z}(r, s) = \bar{z}(r, s)\) for all other nodes and arcs. The solution \(\bar{z}\) is fractional, so \(\bar{z}\) is not an extreme point of \(P_p(G(u, v))\). Since \(P_p(G(u, v))\) is integral, there must exist a 0–1 vector \(z^*\) satisfying the same constraints that are tight for \(\bar{z}\) and are also tight for \(z^*\). From \(z^*\) define \(z''\) as follows: \(z''(u, v) = z^*(u, t)\) and \(z''(t) = z''(r), z''(r, s) = z''(r, s)\) for all other nodes and arcs. All constraints that are tight for \(\bar{z}\) are also tight for \(z''\). To see this, it suffices to observe that \(z''(v) = z^*(v) = 0\) and \(z''(u, v) = z^*(u, t) = 0\). This contradicts the fact that \(\bar{z}\) is an extreme point of \(P_p(G)\). \(\square\)

4.1. The proof of Theorem 45 when \(|\text{Pair}(G)| = 0\)

Since \(G\) is oriented, then from Lemma 51 it contains exactly one \(g\)-odd \(Y\)-cycle. We can redefine \(G = (V, A)\) as follows.

Let \(C\) be the unique \(g\)-odd \(Y\)-cycle of \(G\). Let \(A'\) be the set of arcs directed from a node not in \(V(C)\) to a node in \(V(C)\). Let \(A''\) be the set of arcs directed from a node in \(V(C)\) to a node not in \(V(C)\). Let \(V'\) (resp. \(V'\)) be the node-set defined by the tails (resp. heads) of the arcs \(A'\) (resp. \(A''\)). The graph could also contain a set \(V''\) of isolated nodes. We have \(V = V(C) \cup V' \cup V'' \cup V''\) and \(A = A(C) \cup A' \cup A''\). Notice the following properties of \(G\):

- \(V' \cap V'' = \emptyset\). Otherwise \(C\) is not unique.
- The nodes in \(V''\) are sink nodes. This follows from condition (C2) and the fact that \(C\) is unique.
- The nodes in \(V'\) are adjacent to only nodes in \(\bar{C}\). Otherwise \(G\) contains \(H_{1}, H_{2}\) or \(H_{3}\) as a subgraph.
- Each node in \(V'\) is adjacent to exactly one node in \(\bar{C}\). Otherwise \(C\) is not unique.
- Each node in \(V(C)\) is adjacent to at most one node in \(V''\). Otherwise, suppose \(v \in V(C)\) and let \(u_1\) and \(u_2\) be two nodes in \(V''\) adjacent to \(v\). Define the following labels \(l(v, u_1) = +1\) and \(l(v, u_2) = -1\). Then all inequalities that are satisfied with equality by \(\bar{z}\) are also satisfied with equality by \(\bar{z}_i\); this contradicts the fact that \(\bar{z}\) is an extreme point of \(P_p(G)\).
- A node \(v \in \bar{C}\) can be adjacent to at most one node in \(V' \cup V''\). Otherwise, suppose that there is an arc \((v, w)\) with \(w \in V''\) and an arc \((u, v)\) with \(u \in V'\). Define \(l(u, v) = l(v) = +1\) and \(l(v, w) = l(u) = -1\). Then all constraints that are tight for \(\bar{z}\) are also tight for \(\bar{z}_i\). If there are two nodes in \(V'\) adjacent to \(v\), then we obtain the graph \(H_{1}\).

Clearly the nodes in \(V''\) can be ignored. An oriented graph with the above properties will be called an extended \(g\)-odd \(Y\)-cycle.
Lemma 53. For each arc \((u, v) \in A(C)\) we have \(\tilde{z}(u, v) = \tilde{z}(v, u)\).

Proof. Suppose \(z(u, v) < \tilde{z}(v, u)\). Consider the graph \(G\) obtained from \(G\) by removing the arc \((u, v)\) and adding the arc \((u, w)\), with \(w\) a new node. Let \(z'\) be defined as \(z'(u, v) = \tilde{z}(u, v)\), \(z'(w) = 1\) and \(z'(r) = \tilde{z}(r, s)\) for all other nodes and arcs. We have that \(z' \in P_{p+1}(G)\). The graph \(G\) satisfies (C1) and does not contain a g-odd Y-cycle. Theorem 10 implies that \(z'\) is not an extreme point of \(P_{p+1}(G)\). Hence, there exists a vector \(z^* \in P_{p+1}(G)\), \(z^* \neq z'\), such that all constraints that are tight for \(z'\) are also tight for \(z^*\). Define \(z''(u, v) = z^*(u, w)\) and \(z''(r) = z^*(r, s)\), \(z''(r, s) = z^*(r, s)\) for all other nodes and arcs of \(G\). Then \(z'' \neq z\) and all constraints that are tight for \(z\) are also tight for \(z''\). This is impossible since \(z\) is an extreme point of \(P_p(G)\). Notice that we do not need that \(z'' \in P_p(G)\). □

Now we may assume that \(G\) is an extended g-odd Y-cycle. In the following two sub-sections we establish several properties of the extreme points of \(P(G)\) and \(P_p(G)\). The last sub-section contains the end of the proof. Let \(C\) be the Y-cycle of \(G\).

4.1.1. Fractional extreme points of \(P(G)\)

Assume that \(z\) is a fractional extreme point of \(P(G)\).

Lemma 54. We have \(z(u, v) = z(v, u) > 0\), for all \((u, v) \in A(C)\).

Proof. Assume that \(z(u, r) < z(s)\) or that \(z(u, s) = 0\) for some arc \((r, s) \in A(C)\). Define the graph \(G'\) obtained from \(G\) by removing the arc \((r, s)\) and adding a new pendant node \(t\) and the arc \((r, t)\). Define a fractional solution \(z' \in P'(G)\) to be \(z'(u, r) = z(r, s), z'(r) = 1\) and \(z'(r, u) = z(u, r)\) and \(z'(r, t) = z(u, t)\) for all other arcs and nodes. Recall that \(G\) contains exactly one cycle, that is \(C\). It follows that \(G'\) does not contain any cycle. It was shown in [26] that if a graph \(G\) has no g-odd cycle then \(P(G)\) is integral. This implies that \(P(G')\) is integral and so \(z'\) is not extreme. There is an integer solution \(z'' \in P(G')\) with \(z'' \neq z'\) such that any constraint of \(P(G')\) that is tight for \(z''\) is also tight for \(z'\). Define \(z'' \in P(G)\) to be \(z''(r, s) = z''(r, t),\) and for all other nodes and arcs \(z''\) takes the same value as \(z''\). We have \(z''\) integer and so \(z'' \neq z\) and, from our assumptions, any constraint tight for \(z\) is also tight for \(z''\). This contradicts the fact that \(z\) is an extreme point of \(P(G)\). □

Lemma 55. If \(u \in \tilde{C}\), then \(z(u) = 0\).

Proof. Notice that \(z(u) = 1\) implies \(z(u, v) = z(u, t) = 0\) for the two arcs in \(C\) incident to \(u\); this contradicts Lemma 54. Thus suppose that \(1 > z(u) > 0\) and \(u \in \tilde{C}\). We give the label \(l(u) = -2\) to the node \(u\), then the label +1 to one of the arcs incident to \(u\) in \(C\) and extend the labels along \(C\) with the procedure of Section 2. If there is an arc \(a\) entering \(u\), with \(z(a) = z(u)\), we give the label \(l(a) = -2\) to \(a\), and the label +2 to the tail of \(a\). Also for each node \(w \in \tilde{C}\), we give the label \(-l(w)\) to the arc whose tail is \(w\). We set to zero the labels for all remaining nodes and arcs. With these labels we define a new vector \(z\) that satisfies with equality each constraint that \(z\) satisfies with equality.

In order to see that the labels around \(C\) are correct, we proceed as follows. Let \((u, v)\) and \((u, t)\) be the two arcs incident to \(u\) in \(C\). If we add extra node \(u'\) and replace \((u, t)\) by \((u, u')\) and \((u', t)\) we have a g-even cycle. The labeling procedure gives \(l(u, v) = -l(u, u') = l(u, t)\), therefore \(l(u, v) = l(u, t)\) in the original graph. □

Lemma 56. If \(u \in \tilde{C} \cup \bar{C}\) and \((u, v)\) is an arc where \(v\) is a sink node, then \(z(u, v) = 0\).

Proof. Suppose that \(u \in \tilde{C}\) and \(z(u, v) > 0\). Give the label \(-2\) to \((u, v)\), give the label \(+1\) to \(u\), and the label \(+1\) to the arc in \(C\) that leaves \(u\). Then extend the labels around \(C\) with the procedure of Section 2. Also for each node \(w \in \tilde{C}\), we give the label \(-l(w)\) to the arc whose tail is \(w\). We set to zero the labels for all remaining nodes and arcs. As before, these labels define a new vector that leads to a contradiction.

To see that the labels around \(C\) are correct, we do the following. Let \((u, w)\) and \((t, u)\) be the arcs incident to \(u\) in \(C\). We can add a new node \(w'\) and replace \((t, u)\) by \((t, u')\) and \((u', w)\). The new cycle is g-even, then \(l(u, w) = -l(u', u) = l(t, u)\).

Thus \(l(u, w) = l(t, u)\) in the original graph.

The case \(u \in \tilde{C}\) can be treated with a similar labeling, except that \(l(u) = 0\). □

Lemma 57. If \((u, v) \in A(C)\), then \(z(u, v) = 1/2\). Also \(z(u) = 1/2\), if \(u \in \tilde{C} \cup \bar{C}\).

Proof. From Lemmas 54–56, it follows that the values \(z(u, v)\), for \((u, v) \in A(C)\), are the solution of a system of equations like

\[
x(i) + x(i + 1) = 1, \quad \text{for } 0 \leq i \leq 2q, \quad x(2q + 1) = x(0),
\]

where \(2q + 1 = |\tilde{C}| + |ar{C}|\). □

4.1.2. Extreme points of \(P_p(G)\)

Here we show that several configurations cannot exist. We denote by \(A'_p\) the arcs in \(A''\) that are incident to a node in \(\tilde{C} \cup \bar{C}\). Also let \(\tilde{C}^+\) be the set of nodes \(v \in \tilde{C}\) with \(z(v) > 0\).

Lemma 58. \(|A'_p \cup A''| \leq 1\).
Proof. Consider the case when \( u \) and \( v \) are two nodes in \( \hat{C} \), and \( a_1 = (u, w) \) and \( a_2 = (v, t) \) are two arcs in \( \hat{A}' \). Let \( P_1 \) and \( P_2 \) be the two chains in \( C \) between \( u \) and \( v \). Let us add the arc \( a_3 = (t, w) \). Let \( C_1 \) and \( C_2 \) be the cycles defined by \( a_1, P_1, a_2, a_3 \) and \( a_1, P_2, a_2, a_3 \) respectively. One of them is g-even, \( C_1 \) say. We can apply the labeling procedure to \( C_1 \) and then remove \( a_1 \) and set the labels \( l(u) = l(t) = 0 \). Also any arc \( (r, s) \in A' \) such that \( r \) has a label, and \( r \in \hat{C} \), receives the label \( -l(r) \). We set to zero the labels for all remaining nodes and arcs. These labels define a new vector that satisfies with equality all constraints that were satisfied with equality by \( \hat{z} \).

The other cases are treated in a similar way. \( \square \)

Lemma 59. \( |\hat{C}^+| \leq 1 \).

Proof. Let \( u \) and \( v \) be two nodes in \( \hat{C} \), suppose that \( \hat{z}(u) > 0 \), \( \hat{z}(v) > 0 \), and \( A' = \emptyset \). Let \( P_1 \) and \( P_2 \) be the two chains in \( C \) between \( u \) and \( v \). We add a new node \( t \) and the arcs \( a_1 = (t, u) \) and \( a_2 = (t, v) \). Let \( C_1 \) and \( C_2 \) be the cycles defined by \( a_1, P_1, a_2 \) and \( a_1, P_2, a_2 \) respectively. One of them is g-even, \( C_1 \) say. We apply the labeling procedure to \( C_1 \). Then we remove \( t, a_1, a_2 \). Any arc \( (r, s) \in A' \) such that \( r \) has a label, and \( r \in \hat{C} \), receives the label \( -l(r) \). We set to zero the labels for all remaining nodes and arcs. Again we obtain a new vector that leads to a contradiction.

Now assume that \( u \) and \( v \) are two nodes in \( \hat{C} \), suppose that \( \hat{z}(u) > 0 \), \( \hat{z}(v) > 0 \), and \( A' = \{(u, u)\} \). From the preceding lemma we have \( |A| \leq 1 \). We add an arc \( (u, w) \) and the rest of the proof is as above. \( \square \)

Lemma 60. If \( |\hat{C}^+| = 1 \) then \( |A'_{\hat{C}}| = 0 \).

Proof. Let \( v \in \hat{C} \) with \( \hat{z}(v) > 0 \) and \( u \in \hat{C} \cup \hat{C}, u \neq v \). Assume that the arc \( a_1 = (u, t) \) is in \( A'_{\hat{C}} \). We add a node \( s \) and the arcs \( (t, s) \) and \( (s, v) \). The rest of the proof is as in the preceding lemmas.

If \( v \in \hat{C} \) with \( \hat{z}(v) > 0 \) and the arc \( a_1 = (v, t) \) is in \( A'_{\hat{C}} \), we give the label \(-1\) to \( v \), the label \(+1\) to one of the arcs in \( C \) incident to \( v \), we extend the labels around \( C \), and give the label \(-1\) to the arc \((v, t)\). Any arc \((r, s) \in A' \) such that \( r \) has a label, and \( r \in \hat{C} \), receives the label \(-l(r)\). We set to zero the labels for all remaining nodes and arcs. These labels define a new vector that leads to a contradiction. \( \square \)

4.1.3. Remainder of the proof of Theorem 45 when \( |\text{Pair}(G)| = 0 \)

We assume that \( \hat{z} \) is a fractional extreme point of \( P_{g}(G) \) for an extended g-odd Y-cycle \( G \), where \( C \) is the Y-cycle in \( G \). In the preceding section we have seen that several configurations can be eliminated. The remaining cases are

1. \( A' = \{(u, v)\} \). The lemmas above imply \( A'_{\hat{C}} = \emptyset \) and \( \hat{C}^+ = \{v\} \).
2. \( A'_{\hat{C}} = \{(u, v)\} \). The lemmas above imply \( A = \emptyset \) and \( \hat{C}^+ = \emptyset \).
3. \( \hat{C}^+ = \{v\} \). The lemmas above and Case (1) imply \( A' \cup A'_{\hat{C}} = \emptyset \).
4. \( \hat{C}^+ = \emptyset \). This implies \( A' = \emptyset \). Case (2) implies \( A'_{\hat{C}} = \emptyset \).

Lemma 61. \( \hat{z} \) cannot be an extreme point of \( P(G) \).

Proof. Assume that \( \hat{z} \) is an extreme point of \( P(G) \). By definition we have \( \hat{z}(v) = 1 \) for each node \( v \in V' \). From Lemma 55, \( \hat{z}(v) = 0 \) if \( v \in \hat{C} \), so by the definition of \( V' \) we have \( \hat{z}(v) = 1 \) for each \( v \in V' \). From Lemma 57 we have \( \hat{z}(v) = \frac{1}{2} \) if \( v \in \hat{C} \cup \hat{C} \). Hence \( \sum_{v \in V} \hat{z}(v) = |V'| + |V''| + \frac{|\hat{C}| + |\hat{C}'|}{2} \) but \( |\hat{C}| + |\hat{C}'| \) is odd, so \( \sum_{v \in V} \hat{z}(v) \) is not an integer, a contradiction. \( \square \)

From Observation 9 and Lemma 61, \( \hat{z} \) may be written as a convex combination of two different extreme points of \( P(G) \). Let \( \hat{z} \) and \( \tilde{z} \) be these two extreme points. We use them to study some of these cases.

Case (1) \( A' = \{(u, v)\}, A'_{\hat{C}} = \emptyset, \hat{C}^+ = \{v\} \).

Lemma 62. If \( \hat{z} \) is fractional and \( \hat{z} \) is integral then \( \hat{z} \) does not exist.

Proof. Lemma 55 implies that \( \hat{z}(v) = 0 \) and \( \hat{z}(u, v) = 0 \). Since \( \hat{z}(u, v) > 0 \) we have \( \hat{z}(u, v) = 1 \). This implies \( \hat{z}(u) = 0 \) and \( \hat{z}(v) = 1 \). Let \( a \) be one of the arcs in \( C \) incident to \( v \); we have \( \hat{z}(a) = 0 \). We can continue setting the values of the components of \( \hat{z} \) around the cycle, based on the following equations.

\[
\hat{z}(s, t) = \hat{z}(t) \quad \text{for every arc } (s, t) \in A(C),
\]
\[
\hat{z}(s) = 0, \quad \text{if } s \in \hat{C}, \ s \neq v,
\]
\[
\hat{z}(t, s) = 1 - \hat{z}(t, w) \quad \text{if } t \in \hat{C}, \ t \neq v, \ (t, s), \ (t, w) \in A(C),
\]
\[
\hat{z}(s, t) = 1 - \hat{z}(s), \quad \text{if } s \in \hat{C}, \ (s, t) \in A(C).
\]

This is similar to the labeling procedure, we just have to identify the value one with the label \(+1\) and the value zero with the label \(-1\), except for the nodes in \( \hat{C} \) that keep the value zero. To stress this analogy we proceed as follows. Add a node \( v' \). Let \( (v, r) \) and \( (v, n) \) be the two arcs in \( C \) incident to \( v \), replace \( (v, n) \) by \((v, v')\) and \((v', n)\). Let \( C' \) be this new cycle. It is g-even, so we can give the label \(-1\) to \((v, r)\) and extend the labels around the cycle.
Consider now the convex combination $\tilde{z} = \alpha \hat{z} + (1 - \alpha)\bar{z}$. We obtain $\tilde{z}(u) = \alpha, \tilde{z}(v) = 1 - \alpha$, and for all other nodes in $C$ we have the value $\alpha/2$ if its label is $-1$, $1 - \alpha/2$ if its label is $+1$, and $0$ if its label is zero. Let $S^+$ be the set of nodes with the label $+1$, not including the node $v'$. Let $S^-$ be the set of nodes with the label $-1$. We have that $|S^-| - |S^+| = 1$.

Thus $\sum_{r \in V} \tilde{z}(r) = q + \alpha/2$, where $q$ is an integer. Since $\sum_{r \in V} \tilde{z}(r)$ should be an integer, we have that $\alpha/2$ should be an integer, thus $\alpha = 0$, a contradiction. □

Lemma 63. If $\hat{z}$ and $\bar{z}$ are both integral then $\tilde{z}$ does not exist.

Proof. Suppose that $\tilde{z}(v) = 1$, then $\hat{z}(a_1) = \hat{z}(a_2) = 0$, where $a_1$ and $a_2$ are the arcs incident to $v$ in $C$. We should have $\hat{z}(v) = 0$, this implies $\hat{z}(a_1) = 1$, say, and $\hat{z}(a_2) = 0$. Therefore $\hat{z}(a_2) = 0$, a contradiction. □

Lemma 64. If $\hat{z}$ and $\bar{z}$ are both fractional then $\tilde{z}$ does not exist.

Proof. Lemma 55 implies $\tilde{z}(v) = \bar{z}(u, v) = 0$, a contradiction. □

Case (2) $A'_1 = \{(u, v)\}$, $A' = \emptyset, \hat{C}^+ = \emptyset$.

Lemma 65. If $\hat{z}$ is fractional and $\bar{z}$ is integral then $\tilde{z}$ does not exist.

Proof. Lemma 56 implies $\tilde{z}(u, v) = 0$. Since $\tilde{z}(u, v) > 0$, we have $\tilde{z}(u, v) = 1$. This implies $\hat{z}(u) = 0$ and $\hat{z}(v) = 1$. Let $\tilde{a}$ be an arc whose tail is $u$ in $C$; we have $\bar{z}(\tilde{a}) = 0$. We can continue setting the values of the components of $\tilde{z}$ around the cycle, based on the equations:

$$\tilde{z}(s, t) = \hat{z}(t) \quad \text{for every arc } (s, t) \in A(C),$$
$$\tilde{z}(s) = 0, \quad \text{if } s \in \hat{C},$$
$$\tilde{z}(t, s) = 1 - \tilde{z}(t, w) \quad \text{if } t \in \hat{C}, (t, s), (t, w) \in A(C),$$
$$\tilde{z}(s, t) = 1 - \tilde{z}(s), \quad \text{if } s \in \hat{C}, (s, t) \in A(C).$$

Again this is similar to the labeling procedure. Let $(u, t)$ be an arc incident to $u$ in $C$. We add a node $u'$ and replace $(u, t)$ by $(u, u')$ and $(u', t)$. Then we give the label $-1$ to $(u', t)$ and extend the labels.

Let $S^+$ be the set of nodes with the label $+1$, not including the node $u'$. Let $S^-$ be the set of nodes with the label $-1$. We have that $|S^-| - |S^+| = 1$. The rest of the proof is as in Lemma 62, we have $\sum_{r \in V} \tilde{z}(r) = q + \alpha/2$, where $q$ is an integer. Since $\sum_{r \in V} \tilde{z}(r)$ should be an integer we have $\alpha = 0$. □

Lemma 66. If $\hat{z}$ and $\bar{z}$ are both integral then $\tilde{z}$ does not exist.

Proof. Assume that $u \in \hat{C}$. Suppose that $\tilde{z}(u, v) = 1$, then $\hat{z}(a) = \hat{z}(u) = 0$, where $a$ is the arc whose tail is $u$ in $C$. We should have $\hat{z}(u) = 1$, this implies $\hat{z}(a) = 0$. Therefore $\hat{z}(a) = 0$, a contradiction. The proof when $u \in \hat{C}$ is similar. □

Lemma 67. If $\hat{z}$ and $\bar{z}$ are both fractional then $\tilde{z}$ does not exist.

Proof. Lemma 56 implies $\tilde{z}(u, v) = 0$, a contradiction. □

Case (3) $\hat{C}^+ = \{v\}$, $A' \cup A''_1 = \emptyset$.

Lemma 68. If $\hat{z}$ is fractional and $\bar{z}$ is integral then $\tilde{z}$ does not exist.

Proof. Lemma 55 implies $\tilde{z}(v) = 0$. Since $\tilde{z}(v) > 0$, we have $\tilde{z}(v) = 1$. Let $\tilde{a}$ be one of the arcs in $C$ incident to $v$; we have $\tilde{z}(\tilde{a}) = 0$. We can continue setting the values of the components of $\tilde{z}$ around the cycle, based on the equations:

$$\tilde{z}(s, t) = \hat{z}(t) \quad \text{for every arc } (s, t) \in A(C),$$
$$\tilde{z}(s) = 0, \quad \text{if } s \in \hat{C}, s \neq v,$$
$$\tilde{z}(t, s) = 1 - \tilde{z}(t, w) \quad \text{if } t \in \hat{C}, (t, s), (t, w) \in A(C),$$
$$\tilde{z}(s, t) = 1 - \tilde{z}(s), \quad \text{if } s \in \hat{C}, (s, t) \in A(C).$$

Again we can use the labeling procedure as follows. We add the node $v'$. Let $(v, r)$ and $(v, n)$ be the two arcs in $C$ incident to $v$; we replace $(v, r)$ by $(v, v')$ and $(v', r)$. Let $C'$ be this new cycle. It is $g$-even, so we can give the label $-1$ to $(v', r)$ and extend the labels around the cycle.

Consider now the convex combination $\tilde{z} = \alpha \hat{z} + (1 - \alpha)\bar{z}$. We obtain $\tilde{z}(v) = 1 - \alpha$, and for all other nodes in $C$ we have the value $\alpha/2$ if its label is $-1$, $1 - \alpha/2$ if its label is $+1$, and $0$ if its label is zero. Let $S^+$ be the set of nodes with the label $+1$, not including the node $v'$. Let $S^-$ be the set of nodes with the label $-1$. We have that $|S^-| - |S^+| = 1$.

Thus $\sum_{r \in V} \tilde{z}(r) = q - \alpha/2$, where $q$ is an integer. Since $\sum_{r \in V} \tilde{z}(r)$ should be an integer, we have that $\alpha/2$ should be an integer, thus $\alpha = 0$, a contradiction. □

Lemma 69. If $\hat{z}$ and $\bar{z}$ are both integral then $\tilde{z}$ does not exist.
Proof. Suppose that \( \bar{z}(v) = 1 \), then \( \bar{z}(a_1) = \bar{z}(a_2) = 0 \), where \( a_1 \) and \( a_2 \) are the arcs incident to \( v \) in \( C \). We should have \( \bar{z}(v) = 0 \), this implies \( \bar{z}(a_1) = 1 \), say, and \( \bar{z}(a_2) = 0 \). Therefore \( \bar{z}(a_2) = 0 \), a contradiction. \( \square \)

**Lemma 70.** If \( \bar{z} \) and \( \bar{z} \) are both fractional then \( \bar{z} \) does not exist.

**Proof.** Lemma 55 implies \( \bar{z}(v) = 0 \), a contradiction. \( \square \)

Case (4) \( \hat{C} = \emptyset, \hat{A} = \emptyset, \hat{A}' = \emptyset \).

In this case we have a vector that satisfies the hypothesis of Lemma 57. This implies \( \bar{z}(v) = 1/2 \) for all \( v \in \hat{C} \cup \hat{C}' \).

Then we have that \( \sum_{v \in V} \bar{z}(v) \) is a fractional number, a contradiction. This completes the proof of Theorem 45 in case where \( |\text{Pair}(G)| = 0 \).

### 4.2. The proof of Theorem 45 when \( |\text{Pair}(G)| \geq 1 \)

The proof is by induction on \( |\text{Pair}(G)| \). In the previous sub-section we have seen that the theorem is true when \( |\text{Pair}(G)| = 0 \).

Suppose that Theorem 45 is true for every simple directed graph \( H \) satisfying conditions (C1) and (C2) of Theorem 2, containing a \( g \)-odd \( Y \)-cycle and having \( |\text{Pair}(H)| \leq m, m \geq 0 \). Let \( G = (V, A) \) be a graph having the same properties as \( H \) with \( |\text{Pair}(G)| = m + 1 \). Assume that \( \bar{z} \) is a fractional number, a contradiction. This completes the proof of Theorem 45 in case where \( |\text{Pair}(G)| = 0 \).

### 5. Proof of Theorem 2

In this section we put all pieces together and prove Theorem 2, the main result of this paper.

**Necessity.** Let \( G = (V, A) \) be a directed graph. Let \( H \) be a subgraph of \( G \) that corresponds to one of the graphs \( H_1, H_2, H_3 \) or \( H_4 \) of Fig. 1. Define \( \bar{z} \) to be the solution obtained by extending the fractional extreme point associated with \( H \), defined in Fig. 1, as follows: \( \bar{z}(u) = 1 \) for each node \( u \) not in \( H \); \( \bar{z}(v) = 0 \) for each arc \( (u, v) \) not in \( H \). Then it is easy to check in all cases that \( \bar{z} \) is a fractional extreme point of \( P_{|V|-2}(G) \).

Now suppose that \( G \) contains a \( g \)-odd \( Y \)-cycle \( C \) with an arc \((t, w) \in A \setminus A(C)\), with \( t \) and \( w \) not in \( V(C) \). Define \( \bar{z} \) as follows: \( \bar{z}(t) = \frac{1}{2}, \bar{z}(t, w) = \frac{1}{2} \) and \( \bar{z}(w) = 1 \) for each node \( v \in \hat{C} \cup \hat{C} \); \( \bar{z}(v) = 0 \) for each node \( v \in \hat{C} \); \( \bar{z}(u, v) = \frac{1}{2} \) for each arc \((u, v) \in A(C)\); for each node \( v \in \hat{C}(i) \) by the definition of a \( Y \)-cycle there must exist an arc \((v, \hat{v}) \notin A(C)\) with \( \hat{v} \) a pendant node, so let \( \bar{z}(v, \hat{v}) = \frac{1}{2} \) and \( \bar{z}(\hat{v}) = 1 \); for each node \( v \in \hat{C} \setminus \hat{C}(i) \) the definition of a \( Y \)-cycle there must exist an arc \((u, \hat{u}) \) with \( \hat{u} \in \hat{C} \), so let \( \bar{z}(v, \hat{u}) = \frac{1}{2} \). For all other nodes \( v \) and arcs \((u, v) \), let \( \bar{z}(v) = 1 \) and \( \bar{z}(v, u) = 0 \).

It is straightforward and is left to the reader to see that \( \bar{z} \) is a fractional extreme point of \( P_{|V|}(G) \), where \( p = |V| - |\hat{C}| - \frac{(|\hat{C}| + |\hat{C}| + 1)}{2} \).

**Sufficiency.** It is straightforward from Theorems 10 and 45.

### 6. The bipartite case

Now we assume that \( V \) is partitioned into \( V_1 \) and \( V_2, A \subseteq V_1 \times V_2 \), and we deal with the system

\[
\sum_{v \in V_2} y(v) = p, \quad (33)
\]

\[
\sum_{(u, v) \in A} x(u, v) = 1 \quad \forall u \in V_1, \quad (34)
\]
If there is a path with an even number of fractional nodes, we label them before. This translates into a labeling in $\Pi_p(G)$ that implies that if $\delta^+(u) = 1$. Let $V_1$ be the set of nodes in $V_2$ that are adjacent to a node in $V_1$. It is clear that the variables associated with nodes in $V_2$ should be fixed, i.e., $y(v) = 1$ for all $v \in V_2$. Let $G$ be the graph induced by $V \setminus V_2$. Let $H$ be a graph with node-set $\{u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$ and arc-set
\[
\{(u_1, v_1), (u_2, v_2), (u_3, v_3), (u_1, v_4), (u_2, v_4), (u_3, v_4)\}.
\]
If the graph $G$ contains $H$ as a subgraph then we can construct a fractional extreme point as in Section 5. If $G$ contains a g-odd cycle and one extra node in $V_2 \setminus V_2$, we can also construct a fractional extreme point. Now we prove that these are the only configurations that should be forbidden in order to have an integral polytope.

**Theorem 71.** The polytope $\Pi_p(G)$ is integral for any integer $p$ if and only if

(P3) $G$ does not contain the graph $H$ as a subgraph, and

(P4) $G$ does not contain a g-odd cycle $C$ and one extra node in $V_2 \setminus V_2$.

So let $G$ be a graph such that $G$ does not contain these two configurations. We assume that $z$ is a fractional extreme point of $\Pi_p(G)$. As before, we may assume that $z(u, v) > 0$ for every arc $(u, v) \in A$.

**Lemma 72.** We may assume that $z(u, v) = z(v) = 0$ for each arc $(u, v)$ such that $v \in V_2 \setminus V_2$.

**Proof.** Suppose that $z(u, v) < z(v)$ for an arc $(u, v)$ and $v \in V_2 \setminus V_2$. We can add the nodes $u', v'$, the arcs $(u', v'), (u, v')$ and remove the arc $(u, v)$. Then define $z'(u', v') = z(v') = 1, z(u, v') = z(u, v)$, and $z(s, t) = z(s, t), z'(w) = z(w)$, for all other nodes and arcs. Let $G'$ be the new graph. Then $z'$ is an extreme point of $\Pi_{p+1}(G')$. The graph $G'$ satisfies the hypothesis of Theorem 71. □

The proof of Theorem 71 is divided into the following three cases.

1. If $G$ does not contain a g-odd cycle nor the graph $H$.
2. If $G$ does not contain $H$ and contains a g-odd cycle $C$ that includes all nodes in $V_2 \setminus V_2$, and $|V_2 \setminus V_2| \geq 5$.
3. If $G$ does not contain $H$ and contains a g-odd cycle $C$ that includes all nodes in $V_2 \setminus V_2$, and $|V_2 \setminus V_2| = 3$.

We treat these three cases below.

6.1. $G$ does not contain a g-odd cycle nor the graph $H$

**Lemma 73.** For all $u \in V_1$ we have $|\delta^+(u)| \leq 2$.

**Proof.** Since $G$ has no g-odd cycle, the polytope defined by (34)–(37) is integral, this was proved in [26]. This and Observation 9 show that $z$ is a convex combination of two integral vectors. Therefore $|\delta^+(u)| \leq 2$. □

Now we build an auxiliary undirected graph $G'$ whose node-set is $V_2 \setminus V_2$. For each node $u \in V_1$ such that $\delta^+(u) = \{(u, s), (u, t)\}$, we add an edge in $G'$ between $s$ and $t$. This could create parallel edges. Notice that any node $v$ in $G$ is adjacent to at most two other nodes. If $v$ was adjacent to three other nodes, we would have the subgraph $H$ in $G$.

**Lemma 72** implies that if $z(v) = 1$ for $v \in V_2 \setminus V_2$, then $v$ is not adjacent to any other node in $G$. A node $v \in V_2 \setminus V_2$ is called fractional if $0 < z(v) < 1$. So $G'$ consists of a set of isolated nodes, and a set of cycles and paths. We have to study the four cases below.

- If $G'$ contains a cycle, it should be even, because $G$ has no g-odd cycle. For a cycle in $G'$ we can label the nodes with $+1$ and $-1$ so that adjacent nodes in the cycle have opposite labels. This labeling translates into a labeling in $G$ as follows: if $s$ and $t$ have the labels $+1$ and $-1$ respectively, and the arcs $(u, s)$ and $(u, t)$ are in $G$, then $(u, s)$ receives the label $+1$ and $(u, t)$ receives the label $-1$. If $s$ has the label $l(s)$ and the arcs $(u, s)$ and $(u, t)$ are in $G$ with $t \in V_2$, then $(u, t)$ receives the label $l(s)$ and $(u, t)$ receives the label $-l(s)$. All other nodes and arcs receive the label 0. This defines a new vector that satisfies with equality the same constraints that $z$ satisfies with equality.

- If there is a path with an even number of fractional nodes we label them as before. This translates into a labeling in $G$ as follows. If $s$ and $t$ have the labels $+1$ and $-1$ respectively, and the arcs $(u, s)$ and $(u, t)$ are in $G$, then $(u, s)$ receives the label $+1$ and $(u, t)$ receives the label $-1$. If $s$ has the label $l(s)$ and the arcs $(u, s)$ and $(u, t)$ are in $G$ with $t \in V_2$, then $(u, t)$ receives the label $l(s)$ and $(u, t)$ receives the label $-l(s)$. All other nodes and arcs receive the label 0. This defines a new vector that satisfies with equality the same constraints that $z$ satisfies with equality.

- If $G'$ has two paths with an odd number of fractional nodes then again we can label the fractional nodes in these two paths and proceed as before. Here a path could consist of a single node.

- It remains the case where $G'$ contains just one path with an odd number of fractional nodes. Let $v_1, \ldots, v_{2q+1}$ be the ordered sequence of nodes in this path. We should have $z(v_i) = \alpha$ if $i$ is odd, and $z(v_i) = 1 - \alpha$ if $i$ is even, with $0 < \alpha < 1$. This implies $\sum_{v \in V_2} z(v) = r + \alpha$ where $r$ is an integer. We have then a contradiction.
6.2. $\tilde{G}$ does not contain $H$ and contains a $g$-odd cycle $C$ that includes all nodes in $V_2 \setminus \tilde{V}_2$, and $|V_2 \setminus \tilde{V}_2| \geq 5$

Here we use several transformations to obtain a new graph $\tilde{G}$ that satisfies conditions (C1) and (C2) of Theorem 2, and we use the fact that $P_p(\tilde{G})$ is an integral polytope.

Lemma 74. Let $u, v \in V(C)$, then there is no arc $(u, v) \notin A(C)$.

Proof. If such an arc exists, then the graph $H$ would be present. □

Lemma 75. A node $u \in (V_1 \setminus \tilde{V}_1)$ cannot be adjacent to more than one node in $\tilde{V}_2$.

Proof. Suppose that the arcs $(u, v_1)$ and $(u, v_2)$ exist with $v_1$ and $v_2$ in $\tilde{V}_2$. We can add and subtract $e$ to $z(u, v_1)$ and $z(u, v_2)$ to obtain a new vector that satisfies with equality the same constraints that $z$ does. □

Lemma 76. We may assume that $(V_1 \setminus \tilde{V}_1) \setminus V(C) = \emptyset$

Proof. Consider a node $u \in (V_1 \setminus \tilde{V}_1) \setminus V(C)$ and suppose that the arcs $(u, v_1)$ and $(u, v_2)$ exist with $v_1, v_2 \in V(C)$. If both chains in $C$ between $v_1$ and $v_2$ contain another node in $V_2$, then there is a $g$-odd cycle in $\tilde{G}$ and an extra node in $V_2 \setminus \tilde{V}_2$. If we may assume that there is a node $w \in V(C)$ and $(w, v_1), (w, v_2) \in A(C)$. If there is another node $v_3 \in V(C)$ such that the arc $(u, v_3)$ exists, then the graph $H$ is present, this is because $|V_2 \setminus \tilde{V}_2| \geq 5$. Thus $u$ cannot be adjacent to any other node in $V(C)$. Lemma 72 implies

$$z(u, v_1) = z(w, v_1),$$

$$z(u, v_2) = z(w, v_2).$$

Then we remove the node $u$ and study the vector $z'$ that is the restriction of $z$ to $G \setminus u$. If there is another vector $z''$ that satisfies with equality the same constraints that $z'$ does, we can extend $z''$ using equations Eqs. (38) and (39), to obtain a vector that satisfies with equality the same constraints that $z$ does.

If there is a node $u \in (V_1 \setminus \tilde{V}_1) \setminus V(C)$ that is adjacent to exactly one node $v \in V(C)$, then $u$ is adjacent also to a node $w \in V_2$. It follows from Lemma 75 that the node in $\tilde{V}_2$ is unique. Lemma 72 implies

$$z(u, v) = z(v),$$

and we also have

$$z(u, v) + z(u, w) = 1.$$ (41)

Then we remove the node $u$ and study the vector $z'$ that is the restriction of $z$ to $G \setminus u$. If there is another vector $z''$ that satisfies with equality the same constraints that $z'$ does, we can extend $z''$ using equations Eqs. (40) and (41), to obtain a vector that satisfies with equality the same constraints that $z$ does.

The resulting graph does not contain $H$ and contains the $g$-odd cycle $C$. □

Now consider a node $u \in \tilde{V}_1$ that is adjacent to $v \in \tilde{V}_2$. We should have $z(u, v) = 1$ and $z(v) = 1$. We remove $u$ from the graph and keep $v$ with $z(v) = 1$.

Finally we add slack variables to the inequalities Eq. (36) for each node in $V_2 \setminus \tilde{V}_2$. For that we add a node $v'$ and the arc $(v, v')$, for each node $v \in V_2 \setminus \tilde{V}_2$. Then we add the constraints

$$z(v) + z(v, v') = 1,$$

$$z(v, v') \leq z(v'),$$

$$z(v') = 1,$$

$$z(v, v') \geq 0.$$

Let $\tilde{G}$ be this new graph, and $\bar{p} = p + |V_2 \setminus \tilde{V}_2|$. It follows from Lemmas 74–76 that $\tilde{G}$ is an extended $g$-odd $Y$-cycle, defined in the last section. Here we have a face of $P_p(\tilde{G})$; because $z(v) = 0$ for all $v \in V_1$. Since $P_p(\tilde{G})$ is an integral polytope, there is an integral vector $\tilde{z}$ that satisfies with equality the same constraints that $z$ does. From $\tilde{z} \in P_p(\tilde{G})$ one can easily derive $\bar{z}' \in P_p(G)$ that satisfies with equality the same constraints that $z \in P_p(G)$ satisfies with equality.

6.3. $\tilde{G}$ does not contain $H$ and contains a $g$-odd cycle $C$ that includes all nodes in $V_2 \setminus \tilde{V}_2$, and $|V_2 \setminus \tilde{V}_2| = 3$

Let $p' = p - |\tilde{V}_2|$. If $p' = 3$, we should have $z(v) = 1$ for all $v \in V_2$. Then it is easy to see that we have an integral polytope. So we assume that $p' \leq 2$. Let $V_2 \setminus \tilde{V}_2 = \{v_1, v_2, v_3\}$.

Consider first $p' = 2$. If $z$ is fractional, then at most one variable $z(v_1)$ can take the value one, so assume that $z(v_1) = 1$, $1 > z(v_2) > 0$ and $1 > z(v_3) = 1 - z(v_2) > 0$. We give the label $l(v_2) = +1$ to $v_2$, the label $l(v_3) = -1$ to $v_3$, and $l(v) = 0$ for every other node in $V_2$. Then for each arc $(u, v)$ with $z(u, v) = z(v)$, we give it the label $l(u, v) = l(v)$. If there is a node
There is at most one arc $(u,v)$ incident to it that is labeled, pick another arc $(u,w)$ with $z(u,w) > 0$ and give it the label $l(u,w) = -l(u,v)$. For all the other arcs give the label 0. These labels define a new vector that satisfies with equality the same constraints that $z$ does.

Now suppose that $1 > z(v_1) > 0$, $1 > z(v_2) > 0$ and $1 > z(v_3) > 0$. Then for every node $u \in V_1$ there is at most one arc $(u,v)$ such that $z(u,v) = z(v)$. Otherwise there is a node $w \in V_2 \setminus V_1$ with $z(w) = 1$. Let us define a new vector $z'$ as follows. Start with $z' = 0$. Set $z'(v_1) = z'(v_2) = 1$, $z'(v_3) = 0$, and $z'(v) = 1$ for all $v \in V_2$. Then for each arc $(u,v_1)$ with $z(u,v_1) = z(v_1)$ set $z'(u,v_1) = 1$. Also for each arc $(u,v_2)$ with $z(u,v_2) = z(v_2)$ set $z'(u,v_2) = 1$. For each node $u$ with $\sum_{(u,v) \in E} z'(u,v) = 0$, pick an arc $(u,v)$ with $v \not= v_3$ and set $z'(u,v) = 1$. This new vector satisfies with equality all the constraints that $z$ does.

Finally suppose $p' = 1$ and $z(v_1) > 0$, $z(v_2) > 0$ and $z(v_3) > 0$. We define a new vector $z'$ as below. We set $z'(v_1) = 1$, $z'(v_2) = z'(v_3) = 0$, and $z'(v) = 1$ for $v \in V_2$. For each node $u \in V_1$, if the arc $(u,v_1)$ exists, we set $z(u,v_1) = 1$; otherwise there is a node $v \in V_2$ such that the arc $(u,v)$ exists, we set $z'(u,v) = 1$. We set $z'(s,t) = 0$ for every other arc. Every constraint that is satisfied with equality by $z$ is also satisfied with equality by $z'$.

7. Recognizing the graphs defined in Theorem 2

In this section we show how to decide if a graph satisfies conditions (C1) and (C2) of Theorem 2. Clearly condition (C1) can be tested in polynomial time. Thus we assume that we have a graph satisfying condition (C1). We are going to reduce the problem of finding a g-odd $Y$-cycle into finding a g-odd cycle. For that the first transformation consists of splitting every sink node as in Lemma 16; this is to avoid finding a g-odd cycle containing a sink node. Then we pick an arc $(u,v)$, we remove $u$ and $v$, and look for a g-odd $Y$-cycle in the new graph. We repeat this for every arc. It remains to show how to find a g-odd $Y$-cycle.

In [26] we gave a procedure that finds a g-odd cycle if there is any. We remind the reader that a cycle $C$ is g-odd if $|V(C)| + |C|$ is odd.

Since a g-odd cycle is not necessarily a $Y$-cycle, we need further transformations of the graph so that a g-odd cycle in the new graph gives a g-odd $Y$-cycle in the original graph. The main difficulty resides in how to deal with nodes that satisfy condition (ii) of Definition 6. Such a node should appear in a pair $\{(u,v),(v,u)\}$. Instead of working with such a pair we are going to work with a maximal bidirected chain $P = v_1, \ldots, v_q$. Notice that if the graph contains a bidirected cycle, then it is easy to derive a g-odd Y-cycle. So in what follows we assume that there is no bidirected cycle. The transformation is based on the following two observations.

**Observation 77.** There is at most one arc $(u,v_1), u \not\in P$, and at most one arc $(v,v_q), v \not\in P$. Otherwise the graph $H_4$ is present.

**Observation 78.** If the arc $(u,v_1)$ is in $A$, $u \not\in P$, and there is an arc $(v_1,w)$ also in $A$, $w \not\in P$, then $w$ is a sink node. Otherwise we obtain one of the graphs in Fig. 1.

Let $C$ be a $Y$-cycle that goes through $P$. We have three cases to study.

**Case 1.** $\delta^- (P) = \{(u,v_1),(v,v_q)\}$. In this case $C$ contains all nodes in $P$ and also the arcs $(u,v_1)$ and $(v,v_q)$. Since $C$ contains all nodes from $P$, the only variable that can change the parity of $C$ is the parity of $|C \cap P|$.

Notice that if $q \geq 5$ and if there is a $Y$-cycle going through $P$ then we can always change the parity of it if needed. In fact, we can always join the nodes $v_1$ and $v_q$ using arcs of $P$ in such a way that $|C \cap P| = 1$ as shown in Fig. 18(a), or $|C \cap P| = 2$ as shown in Fig. 18(b). It follows that if there is a cycle $C'$ going through $P$ then there is a cycle $C$ of the same parity, whose nodes in $|C \cap P|$ satisfy Definition 6(ii).

The only cases left to analyze are when $q \leq 4$. Two of them require the following transformation. The other two cases follow similar ideas.

- $q = 4$ and neither $v_1$ nor $v_q$ is adjacent to a sink node. In this case we should have $|C \cap P| = 1$. To impose that, when looking for a g-odd cycle, we replace $P$ by a bidirected chain with two nodes. See Fig. 19.

Let $P'$ the new bidirected chain. Any cycle $C'$ with $|C' \cap P'| = 1$ can be extended to a cycle $C$ with $|C \cap P| = 1$ and where the node in $C \cap P$ satisfies Definition 6(ii).
Theorem 2 (continued)

\[ q \geq 3 \quad \text{or} \quad q = 2 \quad \text{and} \quad v_1 \quad \text{adjacent to a sink node. If} \quad |C \cap P| \quad \text{is odd, we can assume that} \quad |C \cap P| = 1. \quad \text{Here no transformation is needed.} \]

\[ q = 2 \quad \text{and} \quad v_1 \quad \text{not adjacent to a sink node. Here we should have} \quad |C \cap P| = 0. \quad \text{To impose that, when looking for a} \quad g \quad \text{-odd cycle, we remove} \quad (v_2, v_1). \]

\[ \text{Case 3.} \quad \delta^{-}(P) = \emptyset. \quad \text{In this case} \quad C \quad \text{contains an arc} \quad (v_1, u), \quad u \notin P, \quad \text{all nodes in} \quad P, \quad \text{and an arc} \quad (v_q, v), \quad v \notin P. \quad \text{Again we have two cases to analyze.} \]

\[ q \neq 3 \quad \text{or} \quad q = 3 \quad \text{and} \quad v_2 \quad \text{adjacent to a sink node. If} \quad |C \cap P| \quad \text{is even, we can assume that} \quad |C \cap P| = 0. \quad \text{If} \quad |C \cap P| \quad \text{is odd, we can assume that} \quad |C \cap P| = 1. \quad \text{Here no transformation is needed.} \]

\[ q = 3 \quad \text{and} \quad v_2 \quad \text{not adjacent to a sink node. Here we should have} \quad |C \cap P| = 0. \quad \text{To impose that, when looking for a} \quad g \quad \text{-odd cycle, we remove} \quad (v_1, v_2) \quad \text{and} \quad (v_2, v_2). \]

\[ \text{After preprocessing the graph as in Cases 1–3, we look for a} \quad g \quad \text{-odd cycle; if there is one, it gives a} \quad g \quad \text{-odd} \quad Y \quad \text{-cycle in the original graph.} \]

8. Concluding remarks

We have characterized the directed graphs for which the system (2)–(6) defines an integral polytope. In some applications the p-median problem is associated with a complete undirected graph \( G = (V, E) \). The nodes are called locations and there is a cost \( c_{uv} \) between any pair of nodes \( u, v \in V \). The goal is to select \( p \) locations to be centers and then assign the other locations to these centers so as to minimize the sum of the assignment cost, (see [11] for instance). The linear programming relaxation of the p-median problem associated with \( G \) is exactly \( P_p(\vec{G}) \), where \( \vec{G} = (V, A) \) is the directed (symmetric) graph obtained from \( G \) by replacing each edge \( uv \in E \) by two arcs \( (u, v) \) and \( (v, u) \).

Now assume that \( \bar{z} \) is a fractional extreme point of \( P_p(\vec{G}) \). Denote by \( G^+ = (W, F), W \subseteq V, F \subseteq A \), the graph induced by the arcs \((u, v)\) in \( \vec{G}\) with \( \bar{z}(u, v) > 0 \). Let \( k = \sum_{e \in W} \bar{z}(v) \). Let \( z^* \) be the restriction of \( \bar{z} \) on \( G^+ \). It is easy to see that \( k \) is integer and that \( z^* \) is a fractional extreme point of \( P_k(G^+) \). Therefore, from Theorem 2 \( G^+ \) must contain one of the configurations of Fig. 1 or it contains a \( g \)-odd \( Y \)-cycle \( C \) with at least an arc \((u, v)\) where \( u, v \notin V(C) \). When \( G \) is not complete, we have the following.

Corollary 79. Let \( G \) be a connected undirected graph. Then \( P_p(\vec{G}) \) is integral for all \( p \) if and only if \( G \) is a chain or a cycle.

Proof. If \( G \) is a chain or a cycle, then \( \vec{G} \) satisfies conditions (C1) and (C2) of Theorem 2 and so \( P_p(\vec{G}) \) is integral.

Suppose \( G \) is not a chain nor a cycle. Then \( G \) contains a node of degree at least 3. Thus \( \vec{G} \) contains \( H_4 \) as a subgraph. Again Theorem 2 implies that \( P_p(\vec{G}) \) is not integral for all \( p \) (QED)
References


