Nonlinear periodic solutions for isothermal magnetostatic atmospheres

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Abstract. The equations of magnetohydrostatic equilibria for a plasma in a gravitational field are investigated analytically. For equilibria with one ignorable spatial coordinate, these equations reduce to a single nonlinear elliptic equation for the magnetic vector potential $A$. This equation depends on an arbitrary function of $A$ that must be specified. In this paper analytical nonlinear periodic solutions of this elliptic equation are presented for the case of an isothermal atmosphere in a uniform gravitational field for different choices of the arbitrary function. These solutions are obtained using the generalized “tanh method”.

Key words: Magnetostatic equilibria, Nonlinear Evolution equations,

1 Introduction

The equations of magnetostatic (MS) equilibria have been used extensively to model solar magnetic structures\cite{1}-\cite{13}. The force balance in these models consists of the balance between the pressure gradient force, the Lorentz $\mathbf{J} \wedge \mathbf{B}$ force ($\mathbf{J}$ is the electric current density, $\mathbf{B}$ is the magnetic field) and the gravitational force. The temperature distribution in the atmosphere is, in general, determined from the energy transport equation. However, in many models, the temperature distribution is specified a priori, and direct reference to the energy equation is eliminated. The remaining equations for the system are an equation of state for the gas (e.g., the dependence of the gas pressure on density and temperature) and the steady-state Maxwell’s equations.

Many models of MS equilibria assume that one of the spatial coordinates is ignorable,\cite{5}-\cite{10},\cite{14}-\cite{20} leading to simple analytic models in terms of a linear or nonlinear equation for the magnetic potential $A$. Generalizations of this equation for steady-state magnetohydrodynamical (MHD) flows with one ignorable coordinate have been obtained by, Tsinganos \cite{11} and Low \cite{21} both having reduced the MS equilibrium problem to one of solving a single partial differential equation involving two scalar potentials describing the magnetic field. This development does not require the existence of an ignorable coordinate in the system and arises from a local compatibility condition for the magnetic field \cite{22}-\cite{24}. 

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The nonlinear MS equilibrium problem has been solved in several cases [5], [7]-[10], [20], [25]-[27]. Here we consider an isothermal atmosphere with one ignorable coordinate \( x \) of a rectangular Cartesian coordinate system \( xyz \) in which the gravitational force is directed in the negative \( z \)-direction. In this paper, we present a set of exact analytical periodic solutions [28] for the Liouville, sinh, and sine-Poisson equations modelling isothermal MS atmosphere using the generalized “hyperbolic tangent (Tanh) method”. This method has been used and developed for several years and provides a very effective algorithm in order to construct exact solutions for a large number of Nonlinear Evolution Equations.

2 The generalized Tanh Method

The key idea in this solution method is to use the solution of a Riccati equation to replace the \( \tanh \) function in the \( \tanh \) method. In what follows, the method will be reviewed briefly. Given a Nonlinear Evolution Equations (NLEE), with \( x \) and \( y \) independent variables,

\[
G(u, u_x, u_y, u_{xy}, \ldots) = 0.
\]

we wish to know whether it admits “travelling wave” (or stationary wave) solutions. The first step is to combine the independent variables \( x \) and \( y \) into a single variable by defining

\[
\zeta = x - \nu y, \quad \text{and} \quad u(x, y) = U(\zeta)
\]

so as to reduce Eq.(1) to an ordinary differential equation (ODE)

\[
G(U, U', U'', U''', \ldots) = 0.
\]

Our main goal is to derive exact or at least approximate solutions, for these ODE’s. For this purpose, we introduce a new variable

\[
\psi = \psi(\zeta),
\]

which is a solution of the Riccati equation

\[
\psi' = k + \psi^2.
\]

Then we propose the following series expansion as a solution of Eq.(1):

\[
u(x, y) = U(\zeta) = \sum_{i=0}^{m} a_i \psi^i.
\]

The parameter \( m \) is determined by balancing the linear term(s) of highest order with the nonlinear one(s). Normally \( m \) is a positive integer, so that an analytic solution in closed form may be obtained. Substituting Eq.(6) into Eq.(3) and using Eq.(5), by comparing the coefficients of each power of \( \psi \) on both sides, we
obtain an over-determined system of nonlinear algebraic equations with respect to \( k, a_0, a_1, \ldots \). This over-determined system of nonlinear algebraic equations is solved (e.g., with the help of “Mathematica”). With this procedure we can derive several types of solutions:

(i) for \( k < 0 \)
\[
\psi = -\sqrt{-k} \coth(\sqrt{-k}\zeta) = -\sqrt{-k} \tanh(\sqrt{-k}\zeta). \tag{7}
\]

(ii) for \( k = 0 \)
\[
\psi = -1/\zeta, \tag{8}
\]

(iii) for \( k > 0 \)
\[
\psi = -\sqrt{k} \cot(\sqrt{k}\zeta) = \sqrt{k} \tan(\sqrt{k}\zeta). \tag{9}
\]

Another advantage of using the Riccati Eq.(5) is that the sign of \( k \) can be chosen so as to obtain different types of travelling wave solutions of Eq.(1)

3 Isothermal magnetostatic atmospheres. Governing equations

The system of equations used to describe magnetostatic atmospheres consists of the force balance equation
\[
j \wedge B - \nabla P - \rho \nabla \Phi = 0, \tag{10}
\]
coupled to Maxwell’s equations
\[
\mu j = \nabla \wedge B, \quad \nabla \cdot B = 0, \tag{11}
\]
where \( P, \rho, \mu \) and \( \Phi \) are the gas pressure, the mass density, the magnetic permeability and the gravitational potential, respectively. We assume that the temperature is uniform in space and that the plasma is an ideal gas with equation of state \( P = \rho R_0 T_0 \), where \( R_0 \) is the gas constant and \( T_0 \) is the temperature.

Let us consider a system of Cartesian coordinates \((x, y, z)\), in which \( x \) is an ignorable coordinate and \( z \) measures height. Then the magnetic field \( B \) may be written as
\[
B = \nabla A \wedge e_x + B_x e_x = (B_x, \partial A/\partial z, -\partial A/\partial y), \tag{12}
\]
where \( A(y, z) \) and \( B_x(y, z) \) are the magnetic flux function (i.e. the \( x \)-component of the vector potential \( A \)) and the \( x \)-component of \( B \), respectively. Note that the form Eq.(12) for \( B \) ensures that \( \nabla \cdot B = 0 \). Since \( B \cdot \nabla A = 0 \), \( A(y, z) \) is constant along the magnetic lines of force. We restrict our attention to isothermal atmospheres in a uniform gravitational field \((\Phi = gz)\), in which \( B_x = 0 \) and use the ideal gas law to relate the pressure and the density to the uniform temperature \( T_0 \) of the atmosphere. Then Eq.(10) requires that the pressure and density be of the form [27, 39]
\[
P(y, z) = P(A)e^{-z/h}, \quad \rho(y, z) = [1/(gh)] P(A)e^{-z/h}, \tag{13}
\]
where $h = R_0T_0/g$ is the (constant) scale height, and $P(A)$ is an arbitrary function of one variable which describes the variation of the pressure across the magnetic lines at constant height. Substituting Eqs. (11, 12-13) into Eq. (10), we obtain [4, 15]

$$\nabla^2 A + f(A) e^{-z/h} = 0,$$

(14)

where

$$f(A) = \mu \frac{dP}{dA}.$$  

(15)

Subject to suitable boundary conditions on $A$, Eq. (14) may be solved for $A$ in a given domain if the functional form $P(A)$ is prescribed in some suitable manner [14, 15]. Equation (14) is Ampere’s law, which has been set in a form that relates the magnetic field to the plasma distribution through the equation for mechanical equilibrium.

The term $f(A)$ is, in general, nonlinear in $A$ raising nontrivial question of existence, uniqueness, and regularity of the solutions to the boundary value problems based on Eq. (14). Rigorous and general mathematical results on these questions in the nonlinear regime have been obtained and discussed [1, 40, 41]. The absence of a regular solution may be interpreted to imply the unavoidable occurrence of electric current sheets. Equation (15) gives

$$P(A) = P_0 + \frac{1}{\mu} \int f(A) dA.$$  

(16)

Substituting Eq. (16) into Eqs. (13) and (13), we obtain

$$P(y, z) = (P_0 + \frac{1}{\mu} \int f(A) dA) e^{-z/h},$$  

(17)

$$\rho(y, z) = \frac{1}{g \mu} (P_0 + \frac{1}{\mu} \int f(A) dA) e^{-z/h},$$  

(18)

where $P_0$ is constant. If we take the conformal transformation [25]

$$x_1 + i x_2 = e^{-z/l} e^{iy/l},$$  

(19)

Eq. (14) reduces to

$$\frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_2^2} + \ell^2 f(A) e^{(\ell^2 - \frac{1}{\ell^2})z} = 0.$$  

(20)

### 3.1 Liouville Equation

Let us assume $f(A)$ has the form [14, 15]:

$$f(A) = -\alpha^2 A_0 e^{-2A/A_0},$$  

(21)

where $\alpha^2$ and $A_0$ are constants. Hence

$$P(y, z) = (P_0 + \frac{\alpha^2 A_0^2}{2\mu} e^{-2A_0/A_0}) e^{-z/h},$$  

(22)
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The term involving $P_0$ represents a plane-stratified nonionized component of the atmosphere. Inserting Eq. (21) into Eq. (20) we obtain

$$\nabla^2 \frac{A}{A_0} = \alpha^2 l^2 e^{-2A/A_0 + \frac{2}{l} z},$$  \hspace{1cm} (23)

where $\nabla^2 = \frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_2^2}$. Solutions of Eq. (23) have been obtained previously [24, 25]. If we set $P_0 = 0$, removing the plane-parallel component of the atmosphere, we recover the well-known model for an infinite vertical sheet of diffuse plasma suspended by bowed magnetic field-lines.

We look for solutions of Eq. (25) where $A$ is periodic in $y$ with period $2\pi l$ [14, 15] $A(y + 2\pi l, z) = A(y, z)$, which corresponds to an array of plasma condensations or current filaments that are arranged periodically in the $y$-direction. These condensations must have a finite vertical extent. Hence in the far region $z \to \pm \infty$, the field is required to be horizontal and uniform. The following boundary conditions apply

$$\lim_{z \to \pm \infty} B = B_\pm \hat{y},$$  \hspace{1cm} (24)

where $B_\pm$ and $\hat{y}$ are constant field strengths and the unit vector in $y$ direction. Equation (23) is a nonlinear elliptic partial differential equation and one can not take for granted that the boundary conditions given by Eq. (24) admit a solution and, when such a solution exists, that it is unique. Let us set

$$A/A_0 = z/L + \omega(y, z),$$  \hspace{1cm} (25)

where $L$ is a constant. Then, Eq. (23) becomes

$$\nabla^2 \omega - \alpha^2 l^2 e^{-2\omega - \frac{2}{L} y} = 0.$$

Let us identify the period $l$ by:

$$2/l = 2/L + 1/h,$$

and take $l > 0$. Note that under the transformation (19) we have mapped the region $0 \leq y \leq 2\pi l$, $-\infty \leq z \leq \infty$ into the entire $x_1 - x_2$ plane with origin $x_1 = x_2 = 0$ corresponding to $z \to \infty$ and the circle $x_1^2 + x_2^2 \to 1$ corresponding to $z \to 0$. Note that in the limit of an infinite period $l$ Eq. (27) implies $2/L = -1/h$, and we recover the results of Refs. [24, 25].

Inserting Eq. (27) into Eq. (26) we obtain a Liouville type equation

$$\frac{\partial^2 \omega}{\partial x_1^2} + \frac{\partial^2 \omega}{\partial x_2^2} - \alpha^2 l^2 e^{-2\omega} = 0.$$  \hspace{1cm} (28)

Taking the transformation

$$e^{-2\omega} = u,$$  \hspace{1cm} (29)

Eq. (28) becomes

$$(u_{x_1})^2 + (u_{x_2})^2 - uu_{x_1}x_1 - uu_{x_2}x_2 - 2\alpha^2 l^2 u = 0.$$  \hspace{1cm} (30)
Using the wave variable
\[ \zeta = x_1 - \nu x_2, \quad u(x_1, x_2) = U(\zeta) \] (31)
the PDE (30) is transformed into the ODE
\[ (1 + \nu^2)U'' - 2a^2l^2U^3 = 0. \] (32)
Balancing the term \( U'' \) with the term \( U^3 \) we obtain \( m = 2 \), then
\[ U(\zeta) = \sum_{i=0}^{2} a_i \psi^i = a_0 + a_1 \psi + a_2 \psi^2, \quad \psi' = k + \psi^2. \] (33)
Substituting Eq.(33) into Eq.(32) and comparing the coefficients of each power of \( \psi \) on both sides, we obtain an over-determined system of nonlinear algebraic equations with respect to \( \nu, a_0, a_1, \) and \( k \). Solving this system we obtain
\[ a_0 = -\frac{k(1 + \nu^2)}{l^2a^2}, \quad a_2 = -\frac{(1 + \nu^2)}{l^2a^2} a_1 = 0 \] (34)
The general solution of Eq.(32) reads
\[ U = a_0[1 - \tanh^2 \sqrt{-k} \zeta], \quad k < 0. \] (35)
Substituting Eq.(35) into Eq.(25), yields
\[ A/A_0 = (1 - \frac{1}{2l})z - \frac{1}{2} \ln(a_0) - \ln \text{sech} \{\sqrt{-k}(\cos(\frac{y}{l}) - \nu \sin(\frac{y}{l})) e^{-z/l})\}. \] (36)
\[ B/A_0 = \{0, (1 - \frac{1}{2h}) - \frac{\sqrt{-k}}{l} e^{-z/l} (\cos(y/l) - \nu \sin(y/l)) \} \times \] (37)
\[ \tanh[e^{-z/l} \sqrt{-k}(\cos(y/l) - \nu \sin(y/l))], + \frac{\sqrt{-k}}{l} e^{-z/l} (\sin(y/l) + \nu \cos(y/l)) \times \] \[ \tanh[e^{-z/l} \sqrt{-k}(\cos(y/l) - \nu \sin(y/l))], \]
\[ P = P_0 e^{-z/h} + \frac{a_0^2 A_0^2}{2l} e^{-\frac{2z}{l}} \text{sech}^2[e^{-z/l} \sqrt{-k}(\cos(\frac{y}{l}) - \nu \sin(\frac{y}{l}))]. \] (38)
We consider the following subcases:

(i) \( l = 2h \). In this subcase, Eq.(37) gives \( B = 0 \) as \( z \to \infty \).

(ii) \( l > 2h \) (e.g. \( l = 3h \)). In this subcase, Eq.(37) gives \( B/A_0 = -(1/6h) \tilde{g} \) as \( z \to \infty \). Hence as \( l \) increases above \( 2h \), a negative field appears at \( z \to \infty \).

(iii) \( l < 2h \) (e.g. \( l = h \)). In this subcase, Eq.(37) gives \( B/A_0 = (1/2h) \tilde{g} \) as \( z \to \infty \). Hence as \( l \) decreases below \( 2h \), a positive field appears at \( z \to \infty \).
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3.2 sinh-Poisson Equation

Let us now assume that \( f(A) \) has the form

\[
f(A) = -\frac{\lambda^2}{4} \left( \frac{A_0}{h} \right) \sinh(\tilde{A}),
\]

where \( \tilde{A} = A/(hA_0) \), is a dimensionless form of \( A \), \( \lambda \) is a dimensionless constant. Equations (17) and (39) give

\[
P(y, z) = (P_0 - \frac{\lambda^2 A_0^2}{4\mu}) \cosh(\tilde{A}) e^{-z/h}.
\]

Using Eq.(20) and Eq.(39) we obtain

\[
\frac{\partial^2 \tilde{A}}{\partial x_1^2} + \frac{\partial^2 \tilde{A}}{\partial x_2^2} = \lambda^2 \sinh(\tilde{A}),
\]

where \( l = 2h \). Using the transformation

\[
e^{\tilde{A}} = u \quad \text{where} \quad \sinh(\tilde{A}) = (e^{\tilde{A}} - e^{-\tilde{A}})/2,
\]

Eq. (44) becomes

\[
2(u_{x_1})^2 + 2(u_{x_2})^2 - 2uu_{x_1}x_1 - 2uu_{x_2}x_2 + \lambda^2(u^3 - u) = 0.
\]

Using the wave variable

\[
\zeta = x_1 - \nu x_2, \quad u(x_1, x_2) = U(\zeta)
\]

the PDE (45) turns into the ODE

\[
2(1 + \nu^2)U'' - 2(1 + \nu^2)(U')^2 - \lambda^2(U^3 - U) = 0.
\]

Balancing the term \( UU'' \) with the term \( U^3 \) we obtain \( m = 2 \). Then, proceeding as in the previous case we obtain

\[
a_2 = 4(1 + \nu^2), \quad k = \frac{\lambda^2}{4(1 + \nu^2)} \quad \text{and} \quad a_0 = a_1 = 0
\]

The solution of Eq. (43) reads \( (k > 0) \)

\[
A/A_0 = 2h \ln \left[ \tan e^{-z/\sqrt{k}} (\cos(y/l) - \nu \sin(y/l)) \right].
\]

The associated magnetic field and pressure are given by

\[
\mathbf{B}/A_0 = \{0, -\sqrt{k} e^{-z/2h} \left[ (\cos(y/2h) - \nu \sin(y/2h)) \times \right. \sec[\sqrt{k} e^{-z/2h} (\cos(y/2h) - \nu \sin(y/2h))] \cosec[\sqrt{k} e^{-z/2h} (\cos(y/2h) - \nu \sin(y/2h))] \}
\]

\[
- \sqrt{k} e^{-z/2h} \left[ (-\sin(y/2h) - \nu \cos(y/2h)) \sec[\sqrt{k} e^{-z/2h} (\cos(y/2h) - \nu \sin(y/2h))] \times \cosec[\sqrt{k} e^{-z/2h} (\cos(y/2h) - \nu \sin(y/2h))] \right] \}
\]

\[
P = P_0 e^{-z/h} - \frac{\lambda^2 A_0^2}{8\mu} e^{-z/h} \times [\tan^2(\sqrt{k} e^{-z/2h} (\cos(y/2h) - \nu \sin(y/2h))) + 1].
\]
3.3 sine-Poisson Equation

If we now assume that \( f(A) \) has the form

\[
f(A) = -\frac{\lambda^2}{4} \left( \frac{A_0}{h} \right) \sin(\tilde{A}),
\]

where \( \tilde{A} = A/(hA_0) \), \( \lambda \) is a dimensionless constant, Eqs (17) and (50) give

\[
P(y, z) = (P_0 + \frac{\lambda^2 A_0^2}{4\mu} \cos(\tilde{A})) e^{-z/h}.
\]

Using Eqs. (20) and (50) we obtain

\[
\frac{\partial^2 \tilde{A}}{\partial x_1^2} + \frac{\partial^2 \tilde{A}}{\partial x_2^2} = \lambda^2 \sin(\tilde{A}),
\]

where \( l = 2h \). Taking the transformation

\[
e^{i\tilde{A}} = u \quad \text{where} \quad \sin(\tilde{A}) = (e^{i\tilde{A}} - e^{-i\tilde{A}})/(2i).
\]

and proceeding as for the \( \sinh \) case we obtain

\[
\frac{A}{A_0} = h \cos^{-1} \left[ \frac{1}{2} \tan^2[e^{-z/2h} \sqrt{k} (\cos(y/2h) - \nu \sin(y/2h))] + \frac{1}{2} \cot^2[e^{-z/2h} \sqrt{k} (\cos(y/2h) - \nu \sin(y/2h))] \right].
\]

4 SUMMARY AND DISCUSSION

In this paper we have investigated isothermal magnetostatic atmospheric models with one ignorable coordinate \( x \) of a Cartesian coordinate system \( xyz \) in which the distributed current is either an exponential or a trigonometric function of the \( x \) component \( A \) of the vector potential in the presence of a uniform gravity field. These models correspond to an array of plasma condensations or current filaments that are arranged periodically in the \( y \)-direction. These solutions have been obtained directly through a generalization of the \( \tanh \) method. In this paper this method has been worked out explicitly for three cases, but the solutions obtained can be extended to more general cases involving different combinations of exponential and trigonometric functions of the vector potential.

Acknowledgments. This investigation has been performed within the framework of the Programme of Cultural Co-operation between the Italian Republic and the Arab Republic of Egypt for the years 2008-2010, BSC project “Equilibrium, stability and dynamics of magnetically confined plasmas with flows”.

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References


