Canonical and monophonic convexities in hypergraphs

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ABSTRACT

Known properties of “canonical connections” from database theory and of “closed sets” from statistics implicitly define a hypergraph convexity, here called canonical convexity (c-convexity), and provide an efficient algorithm to compute c-convex hulls. We characterize the class of hypergraphs in which c-convexity enjoys the Minkowski–Krein–Milman property. Moreover, we compare c-convexity with the natural extension to hypergraphs of monopohonic convexity (or m-convexity), and prove that: (1) m-convexity is coarser than c-convexity, (2) m-convexity and c-convexity are equivalent in conformal hypergraphs, and (3) m-convex hulls can be computed in the same efficient way as c-convex hulls.

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1. Introduction

Motivated by classical convexity theorems in Euclidean spaces, an axiomatic study of convexity was developed for the theory of abstract convexity structures [22]. Finite convexity structures are defined as follows. Given a finite nonempty set \( V \), an alignment on \( V \) [11] is a family \( \mathcal{L} \) of subsets of \( V \) that satisfies the two conditions:

(a) \( \mathcal{L} \) contains \( \emptyset \) and \( V \)
(b) \( \mathcal{L} \) is closed under intersection.

The pair \( (V, \mathcal{L}) \) defines a (finite) convex space. The members of \( \mathcal{L} \) are called convex sets and the smallest member of \( \mathcal{L} \) containing a set \( X \subseteq V \) is called the convex hull of \( X \). The operator \( \sigma \) that maps subsets of \( V \) to their hulls is a closure operator on \( \wp(V) \) [23] in that it enjoys the following three properties:

(1) \( X \subseteq \sigma(X) \)
(2) \( X \subseteq Y \) implies \( \sigma(X) \subseteq \sigma(Y) \)
(3) \( \sigma(\sigma(X)) = \sigma(X) \).

On the other hand, it is easy to see that, given a closure operator \( \sigma \) on \( \wp(V) \) such that \( \sigma(\emptyset) = \emptyset \), the set family \( \mathcal{L} = \{ X \in \wp(V) : \sigma(X) = X \} \) is an alignment on \( V \).

Let \( (V, \mathcal{L}) \) be a convexity space, and let \( X \) be a nonempty member of \( \mathcal{L} \). An element \( v \) of \( X \) is an extreme point of \( X \) if \( X \setminus \{ v \} \in \mathcal{L} \) [11]. A convexity space \( (V, \mathcal{L}) \) is geometric (or is a “convex geometry” [11] or is an “antimatroid” [13]) if it satisfies the following condition:

Minkowski–Krein–Milman property (MKM property): every convex set is the hull of the set of its extreme points.

Finite convexity theory [13,14] has been applied to graphs and related to path properties on graphs. Typically, a convexity in a connected graph is defined in terms of a “feasible” family of paths [4], which is a family that contains at least one path between any pair of vertices. Thus, the families of shortest paths [11], induced (or minimal) paths [8,11], triangle

Monophonic convexity (m-convexity, for short) can be naturally extended to hypergraphs. On the other hand, the database-theoretic “operator of canonical closure” [19] can be used to define another notion of convexity on hypergraphs, which we call canonical convexity (c-convexity, for short) and, then, c-convex sets come out to coincide with the vertex sets that in statistics are called “closed sets” [18]. Moreover, given a connected hypergraph \( H \) with \( m \) edges and \( n \) vertices and given a suitable superstructure of \( H \) (called the “compact hypergraph” of \( H \) and defined in Section 2.2), one can compute the c-convex hull of any subset of \( V(H) \) in \( O(m n) \) time.

A problem that naturally arises is to characterize hypergraphs with geometric c-convexity spaces. In this paper, we first prove that c-convexity spaces on “acyclic” hypergraphs (for their definition see Section 2.1) are geometric; thus, we endow acyclic hypergraphs with a geometric property which is added to their well-known properties proved in graph and hypergraph theory [1,5,10], in database theory [1,10,17] and in statistics [2,7,15,18]. In this paper, we characterize the whole class of hypergraphs with geometric c-convexity spaces. Moreover, we compare c-convexity with m-convexity and prove that:

1. m-convexity is coarser than c-convexity;
2. in conformal hypergraphs, c-convexity and m-convexity are equivalent;
3. given a connected hypergraph \( H \) with \( m \) edges and \( n \) vertices and given the compact hypergraph of \( H \), the m-convex hull of every subset \( V(H) \) can be computed in time \( O(m n) \).

The paper is organized as follows. In Section 2 we recall some more-or-less standard definitions of hypergraph and graph theory. In Section 3 we give the definition of c-convexity in terms of “closed sets” and “canonical connections” and re-state their known properties in convexity-theoretic terms. Section 4 contains the characterization of hypergraphs with geometric c-convexity spaces. In Section 5, c-convexity is compared with m-convexity and their equivalence in the case of conformal hypergraphs is proved; moreover, the algorithm for m-convex hulls is given. Finally, the Appendix contains the proofs of some propositions and lemmas, which are often used in the paper to prove some of the main results.

2. Hypergraphs

A hypergraph [9] is a (possibly empty) family \( H \) of nonempty sets; the members of \( H \) are called the (hyper) edges of \( H \) and their union is called the vertex set of \( H \), denoted by \( V(H) \). A hypergraph is trivial if it has exactly one edge. The dimension of a hypergraph \( H \) is the product \( |H| \cdot |V(H)| \).

A hypergraph is uniform of rank \( r \) if its edges are all of cardinality \( r \). In what follows, by a graph we mean a uniform hypergraph of rank 2.

An edge of a hypergraph is redundant if it is a subset of another edge. A hypergraph is simple (or reduced [1] or a “Sperner system” or a “clutter” or an “antichain” [9]) if it has no redundant edges. The simple reduction of a hypergraph is the simple hypergraph obtained by taking its maximal (w.r.t. inclusion) edges.

A partial edge of a hypergraph is any nonempty vertex set that is (properly or improperly) contained in some edge. Two distinct vertices \( u \) and \( v \) are adjacent if the pair \( \{u, v\} \) is a partial edge. A nonempty vertex set is a clique if either it is a singleton or its vertices are pairwise adjacent.

The clique hypergraph of a hypergraph \( H \) is the hypergraph having as its edges the maximal cliques of \( H \).

A hypergraph is conformal (or “graphical” [15]) if every clique is a partial edge. Of course, clique hypergraphs are all conformal hypergraphs and a hypergraph \( H \) is conformal if and only if the clique hypergraph of \( H \) equals the simple reduction of \( H \). Let \( H \) be a hypergraph. A subhypergraph of \( H \) is a hypergraph whose edges are all partial edges of \( H \). A cover of \( H \) is a simple hypergraph \( \mathcal{A} \) such that \( V(\mathcal{A}) = V(H) \) and \( \mathcal{A} \) is a subhypergraph of \( H \). A partial subhypergraph of \( H \) is a nonempty subset of \( H \). The subhypergraph of \( H \) induced by a subset \( X \) of \( V(H) \), denoted by \( H[X] \), is the simple reduction of the hypergraph \( \{X \cap A : A \in H \} \setminus \{\emptyset\} \). A path (of length \( k \)) in a hypergraph \( H \) is a sequence \( (v_0, A_1, v_1, A_2, \ldots, v_k - 1, A_k, v_k) \), \( k \geq 1 \), where the \( v_i \)’s (to be called the points of the path) are pairwise distinct vertices of \( H \), the \( A_i \)’s (to be called the steps of the path) are pairwise distinct edges of \( H \) and the vertex pair \( \{v_i, v_{i+1}\} \) is a subset of \( A_i \) for \( 1 \leq i \leq k \). We call the vertices \( v_0 \) and \( v_k \) the end-points of the path, the vertices \( v_1, \ldots, v_k - 1 \) its transit-points, and the edges \( A_1 \) and \( A_k \) its end-steps. In what follows, if \( H \) is a graph (i.e., a uniform hypergraph of rank 2), a path \( (v_0, A_1, v_1, A_2, \ldots, v_k - 1, A_k, v_k) \) in \( H \) will be denoted simply by \( (v_0, v_1, \ldots, v_k) \).

Two vertices (or edges) are connected if they are the end-points (the end-steps, respectively) of a path. A hypergraph is connected if every two vertices (or edges) are connected. The (connected) components of a hypergraph \( H \) are the maximal connected partial subhypergraphs of \( H \).

2.1. Acyclic hypergraphs

A vertex of a hypergraph is a leaf if it is contained in exactly one edge. A hypergraph is acyclic [1,17,21] (or “\( \alpha \)-acyclic” [5,10] or “decomposable” [2,12,15]) if the application of the following procedure reduces it to an empty hypergraph.

...
Several characterizations of acyclic hypergraphs exist [1], two of which will be recalled in Sections 2.2 and 2.3.

2.2. The compact hypergraph of a hypergraph

Two connected vertices of a hypergraph \( \mathcal{H} \) are separated by a nonempty subset \( X \) of \( V(\mathcal{H}) \) if they belong to different components of the subhypergraph of \( \mathcal{H} \) induced by \( V(\mathcal{H}) \setminus X \). A nonempty subset \( X \) of \( V(\mathcal{H}) \) is a compact set [19,20] of \( \mathcal{H} \) if every two vertices in \( X \) are connected and are separated by no partial edge of \( \mathcal{H} \). Note that a maximal compact set of \( \mathcal{H} \) is either an edge of \( \mathcal{H} \) or the union of three or more partial edges of \( \mathcal{H} \) [19]. The compact components of \( \mathcal{H} \) are the subhypergraphs of \( \mathcal{H} \) induced by maximal compact sets, and the compact hypergraph (or “compaction” [19,20]) of \( \mathcal{H} \) is the (simple) hypergraph having as its edges the maximal compact sets of \( \mathcal{H} \). An edge of the compact hypergraph of \( \mathcal{H} \) that is not an edge of \( \mathcal{H} \) is called compound; thus, a compact component of \( \mathcal{H} \) is a nontrivial hypergraph if and only if its vertex set is a compound edge of the compact hypergraph of \( \mathcal{H} \). Let \( \mathcal{H} \) be a connected and simple hypergraph with \( m \) edges and \( n \) vertices. The number of compact components of \( \mathcal{H} \) is not greater than \((3m - 1)/2\) so that the dimension of the compact hypergraph of \( \mathcal{H} \) is \( O(mn) \) [19]; moreover, the compact hypergraph of \( \mathcal{H} \) can be constructed in \( O(m^2n) \) time [19,20]. Finally, the compact hypergraph of a hypergraph \( \mathcal{H} \) enjoys the following two basic properties [19,20].

**Fact 1.** The compact hypergraph of \( \mathcal{H} \) is an acyclic cover of \( \mathcal{H} \) and coincides with the simple reduction of \( \mathcal{H} \) if and only if \( \mathcal{H} \) is acyclic.

**Fact 2.** The intersection of two edges of the compact hypergraph of \( \mathcal{H} \) is either the empty set or a partial edge of \( \mathcal{H} \).

**Example 1.** Consider the hypergraph \( \mathcal{H} = \{\{a, b\}, \{a, d\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}, \{c, f\}, \{d, e\}, \{e, f\}\} \). The compact hypergraph of \( \mathcal{H} \) is the acyclic hypergraph \( \{\{a, b, c, d, e\}, \{c, e, f\}\} \) and the intersection of its two edges is the (partial) edge \( \{c, e\} \) of \( \mathcal{H} \).

2.3. The two-section of a hypergraph

The two-section of a hypergraph \( \mathcal{H} \), denoted by \( \mathcal{H}_{[2]} \), is the graph with vertex set \( V(\mathcal{H}) \) where two vertices are adjacent if and only if they are adjacent in \( \mathcal{H} \). Note that \( \mathcal{H}_{[2]} \) also equals the two-section of the clique hypergraph of \( \mathcal{H} \).

A nonempty subset \( X \) of \( V(\mathcal{H}) \) is a prime set [6,16] of \( \mathcal{H}_{[2]} \) if every two vertices in \( X \) are connected and are separated by no clique of \( \mathcal{H}_{[2]} \). The prime hypergraph of \( \mathcal{H}_{[2]} \) is the (acyclic [6,16]) hypergraph having as its edges the maximal prime sets of \( \mathcal{H}_{[2]} \).

**Example 1 (continued).** Consider again the hypergraph \( \mathcal{H} = \{\{a, b\}, \{a, d\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}, \{c, f\}, \{d, e\}, \{e, f\}\} \). Since \( \mathcal{H} \) is a graph, trivially one has \( \mathcal{H}_{[2]} = \mathcal{H} \). The prime hypergraph of \( \mathcal{H}_{[2]} \) is the acyclic hypergraph \( \{\{a, b, d, e\}, \{b, c, d, e\}, \{c, e, f\}\} \).

Note that, since every partial edge of a hypergraph \( \mathcal{H} \) is a clique of \( \mathcal{H}_{[2]} \), every prime set of \( \mathcal{H}_{[2]} \) is a compact set of \( \mathcal{H} \); but the converse need not hold so that the compact hypergraph of \( \mathcal{H} \) is always a cover of the prime hypergraph of \( \mathcal{H}_{[2]} \) (see Example 1). However, if \( \mathcal{H} \) is conformal, then the prime sets of \( \mathcal{H}_{[2]} \) are exactly the compact sets of \( \mathcal{H} \) and the compact hypergraph of \( \mathcal{H} \) coincides with the prime hypergraph of \( \mathcal{H}_{[2]} \).

A cycle in \( \mathcal{H}_{[2]} \) is a vertex sequence \( (v_0, v_1, \ldots, v_k) \), \( k \geq 3 \), such that the vertices \( v_0, v_1, \ldots, v_{k-1} \) are pairwise distinct, the vertices \( v_{i-1} \) and \( v_i \) are adjacent for \( 1 \leq i \leq k \) and \( v_k = v_0 \). Two vertices \( v_i \) and \( v_j \) in the cycle are consecutive if \( j = i + 1 \mod k \) or \( j = i + 1 \mod k \). The graph \( \mathcal{H}_{[2]} \) is chordal (or “triangulated”) if every cycle of length greater than 3 contains two non-consecutive vertices that are adjacent in \( \mathcal{H}_{[2]} \).

A hypergraph is chordal [1] if its two-section is chordal.

**Fact 3 ([1,5]).** A hypergraph is acyclic if and only if it is conformal and chordal.

3. Canonical convexity

In this section, we recall the notions of “closed sets” and “canonical connections”, which were introduced in statistics [18] and database theory [17] respectively. Both implicitly define one notion of hypergraph convexity, here called “canonical convexity”. We refer the reader interested in the two applications to the work referenced.
3.1. Closed sets and canonical connections

Let $\mathcal{H}$ be a hypergraph and $X$ a (possibly empty) subset of $V(\mathcal{H})$. Two edges of $\mathcal{H}$ are $X$-connected if their intersection is not a subset of $X$ or they are the end-steps of a path $p$ of length at least 2 and no transit-point of $p$ belongs to $X$. The $X$-components of $\mathcal{H}$ are the maximal sets of pairwise $X$-connected edges. Note that, if $A$ is an edge of $\mathcal{H}$, then the trivial hypergraph $\{A\}$ is itself an $X$-component of $\mathcal{H}$ if and only if either $A \subseteq X$ or $A \setminus X$ is a set of leaves of $\mathcal{H}$. Of course, every $X$-component of $\mathcal{H}$ is connected and, hence, is a partial subhypergraph of one component of $\mathcal{H}$; moreover, if $X = \emptyset$ then the $X$-components of $\mathcal{H}$ are exactly the components of $\mathcal{H}$, and if $X = V(\mathcal{H})$ then there is one $X$-component of $\mathcal{H}$ for each edge of $\mathcal{H}$. Finally, the boundary of an $X$-component $\mathcal{H}'$ of $\mathcal{H}$ is the set $X \cap V(\mathcal{H}').$

A subset $X$ of $V(\mathcal{H})$ is closed [18] in $\mathcal{H}$ if the boundary of every $X$-component of $\mathcal{H}$ is either the empty set or a partial edge of $\mathcal{H}$. Trivially, $\emptyset$ and $V(\mathcal{H})$ are closed sets in $\mathcal{H}$. Note that, if $\mathcal{H}$ is connected, then $X$ is closed in $\mathcal{H}$ if and only if either $X = \emptyset$ or the boundary of every $X$-component of $\mathcal{H}$ is a partial edge of $\mathcal{H}$.

Example 2. Consider the hypergraph $\mathcal{H} = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$ (that is, $\mathcal{H}$ is a complete graph of order 4) and let $X = \{a, b, c\}$. The hypergraph $\mathcal{H}$ has four $X$-components: $\mathcal{H}_1 = \{\{a, b\}\}, \mathcal{H}_2 = \{\{a, c\}\}, \mathcal{H}_3 = \{\{b, c\}\}$ and $\mathcal{H}_4 = \{\{a, d\}, \{b, d\}, \{c, d\}\}$, and their boundaries are: $X \cap V(\mathcal{H}_1) = \{a, b\}, X \cap V(\mathcal{H}_2) = \{a, c\}, X \cap V(\mathcal{H}_3) = \{b, c\}$ and $X \cap V(\mathcal{H}_4) = \{a, b, c\}$. Since the boundary of $\mathcal{H}_4$ is not a partial edge of $\mathcal{H}$, $X$ is not closed in $\mathcal{H}$. Indeed, no subset of $V(\mathcal{H})$ of cardinality 3 is closed in $\mathcal{H}$.

Two main properties of closed sets are now recalled.

**Proposition 1** (See Corollary 5 in [18]). Let $\mathcal{H}$ be a hypergraph, and let $X$ and $Y$ be two subsets of $V(\mathcal{H})$ with $X \subseteq Y$. If $X$ is closed in $\mathcal{H}[Y]$ and $Y$ is closed in $\mathcal{H}$, then $X \cap Y$ is closed in $\mathcal{H}$ too.

**Proposition 2** (See Lemma 7 in [18]). Let $\mathcal{H}$ be a hypergraph, and let $X$ and $Y$ be two subsets of $V(\mathcal{H})$. If both $X$ and $Y$ are closed in $\mathcal{H}$, then $X \cap Y$ is closed in $\mathcal{H}$ too.

Let $\mathcal{H}$ be a connected hypergraph and let $L$ be the set family made up of subsets of $V(\mathcal{H})$ that are closed in $\mathcal{H}$. Since the empty set and $V(\mathcal{H})$ are both closed sets in $\mathcal{H}$ (see above), by Proposition 2 the pair $(V(\mathcal{H}), L)$ is a convexity space, which we call the canonical-convexity (c-convexity) space on $\mathcal{H}$. For a subset $X$ of $V(\mathcal{H})$, we denote the c-convex hull of $X$ in $\mathcal{H}$ by $X^c$. The c-convex hull of a nonempty subset $X$ of $V(\mathcal{H})$ is closely related to the notion of “canonical connection” for $X$ in $\mathcal{H}$ [17], whose properties are now re-stated in convexity-theoretic terms.

Let $\mathcal{H}$ be an arbitrary hypergraph and $X$ a nonempty subset of $V(\mathcal{H})$. A reduction of $\mathcal{H}$ with sacred set $X$ (an $X$-reduction of $\mathcal{H}$, for short) [19] is a mapping $f$ from $\mathcal{H}$ onto a partial subhypergraph of $\mathcal{H}$, denoted by $f(\mathcal{H})$, such that:

(a) $f(A) = A$ for every edge $A$ of $f(\mathcal{H})$;
(b) if $v$ is not a leaf of $\mathcal{H}$, then $f(A)$ contains $v$ for every edge $A$ containing $v$, or $f(A) = f(B)$ for every two edges $A$ and $B$ of $\mathcal{H}$ containing $v$;
(c) $A \cap X \subseteq f(A) \cap X$ for every edge $A$ of $\mathcal{H}$.

Note that the identity is an $X$-reduction of $\mathcal{H}$. Also note that, if $|X| = 1$ and $A$ is any edge of $\mathcal{H}$ containing $X$, then the mapping $f$ with $f(\mathcal{H}) = \{A\}$ is an $X$-reduction of $\mathcal{H}$.

**Remark 1.** If $Y \subseteq X \subseteq V(\mathcal{H})$ then every $X$-reduction of $\mathcal{H}$ is a $Y$-reduction of $\mathcal{H}$ too.

Let $f$ be an $X$-reduction of $\mathcal{H}$. It is easy to see that every path in $\mathcal{H}$ whose end-steps are edges of $f(\mathcal{H})$ is mapped to a path in $f(\mathcal{H})$ with the same end-steps. To see it, let $(v_0, A_1, v_1, A_2, \ldots, v_{k-1}, A_k, v_k)$ be a path in $\mathcal{H}$, where $A_1$ and $A_k$ are edges of $f(\mathcal{H})$. Consider the edges $f(A_1), f(A_2), \ldots, f(A_k)$ of $f(\mathcal{H})$. First of all, observe that by condition (a)

$$f(A_1) = A_1 \quad \text{and} \quad f(A_k) = A_k,$$

and by condition (b)

if $f(A_i) \neq f(A_{i+1})$ then, since $v_i \in A_i \cap A_{i+1}$, $v_i \in f(A_i) \cap f(A_{i+1})$.

Let $i(1)$ be the maximum of $i \in \{1, \ldots, k\}$ for which $f(A_i) = f(A_1)$. Note that, since $f(A_k) = A_k \neq A_1$, one has $1 \leq i(1) \leq k-1$. Moreover, since $f(A_{i(1)+1}) \neq f(A_{i(1)+1})$ and $v_{i(1)} \in A_{i(1)} \cap A_{i(1)+1}$, one has $v_{i(1)} \in f(A_{i(1)}) \cap f(A_{i(1)+1}) = A_1 \cap f(A_{i(1)+1})$. Let $i(2)$ be the maximum of $i \in \{i(1) + 1, \ldots, k\}$ for which $f(A_i) = f(A_{i(1)+1})$. Note that $i(1) + 1 \leq i(2) \leq k$. If $i(2) = k$ then, since $v_{i(1)} \in A_1 \cap f(A_{i(1)+1})$ and $f(A_{i(1)+1}) = f(A_k) = A_k$, one has that $v_{i(1)} \in A_1 \cap A_k$ and, hence, $(v_0, A_1, v_{i(1)}, A_k, v_k)$ is a path in $f(\mathcal{H})$. Otherwise (i.e., $i(2) < k$), let $i(3)$ be the maximum of $i \in \{i(2) + 1, \ldots, k\}$ for which $f(A_{i(2)+1}) = f(A_k) = A_k$. Note that $i(2) + 1 \leq i(3) \leq k$. If $i(3) = k$ then, since $v_{i(2)} \in f(A_{i(2)}) \cap f(A_{i(2)+1})$ and $f(A_{i(2)+1}) = f(A_k) = A_k$, one has that $v_{i(2)} \in f(A_{i(2)}) \cap A_k$ and, hence, $(v_0, A_1, v_{i(2)}, A_k, v_k)$ is a path in $f(\mathcal{H})$. So, the end-steps of every path in $\mathcal{H}$ are connected in $f(\mathcal{H})$. As an immediate consequence, the following holds.
**Fact 4.** Let $\mathcal{H}$ be a hypergraph and let $f$ be an $X$-reduction of $\mathcal{H}$. If $\mathcal{H}$ is connected then $f(\mathcal{H})$ is connected too.

Motivated by the following two properties of $X$-reductions which were stated in [18] (and whose proofs can be found in the Appendix), henceforth we call the partial subhypergraph $f(\mathcal{H})$ of $\mathcal{H}$, where $f$ is any $X$-reduction of $\mathcal{H}$, a convex connection for $X$ in $\mathcal{H}$.

**Proposition 3.** Let $\mathcal{H}$ be a connected hypergraph and $X$ a nonempty subset of $V(\mathcal{H})$. The vertex set of every convex connection for $X$ in $\mathcal{H}$ is $c$-convex in $\mathcal{H}$.

**Proposition 4.** Let $\mathcal{H}$ be a connected hypergraph and $X$ a nonempty subset of $V(\mathcal{H})$. A superset $Y$ of $X$ is $c$-convex in $\mathcal{H}$ if and only if there exists a convex connection $C$ for $X$ in $\mathcal{H}$ such that $Y \subseteq V(C)$ and $V(C) \setminus Y$ is a set of leaves of $C$.

A convex connection $C$ for $X$ in $\mathcal{H}$ is minimal if the identity is the unique $X$-reduction of $C$. As proven in [17], if $C_1$ and $C_2$ are two minimal convex connections for $X$ in $\mathcal{H}$ and $C_i$ ($i = 1, 2$) denotes the subhypergraph of $\mathcal{H}$ obtained from $C_i$ by deleting the leaves of $C_i$ that are not in $X$, then $C_1 = C_2$. The canonical connection for $X$ in $\mathcal{H}$ [17], here denoted by $CC(\mathcal{H}, X)$ as in [19], is the subhypergraph of $\mathcal{H}$ that is obtained from any minimal convex connection $C$ for $X$ in $\mathcal{H}$ by deleting the leaves of $C$ that are not in $X$. As proven in [17], $CC(\mathcal{H}, X)$ is an induced subhypergraph of $\mathcal{H}$; moreover, by **Fact 4**, if $\mathcal{H}$ is connected then $CC(\mathcal{H}, X)$ is connected too. Finally, as proven in [18], **Proposition 4** entails that the vertex set of $CC(\mathcal{H}, X)$ coincides exactly with the $c$-convex hull $X^c$ of $X$ in $\mathcal{H}$. (The proof of this result can be found in the Appendix.)

**Lemma 1.** Let $\mathcal{H}$ be a connected hypergraph and $X$ a nonempty subset of $V(\mathcal{H})$. The $c$-convex hull of $X$ in $\mathcal{H}$ equals the vertex set of $CC(\mathcal{H}, X)$.

**Example 2 (continued).** Consider again the hypergraph $\mathcal{H} = \{[a, b], [a, c], [a, d], [b, c], [b, d], [c, d]\}$. As we have seen above, the set $X = \{a, b, c\}$ is not $c$-convex in $\mathcal{H}$. We now show that $X^c = V(\mathcal{H})$. To this end, we first prove that the identity is the unique $X$-reduction of $\mathcal{H}$. First of all, the edge $[a, b]$ of $\mathcal{H}$ must be mapped to itself by condition ($c$), and the same holds for the edges $[a, c]$ and $[b, c]$ of $\mathcal{H}$. Consider now the edge $[a, d]$ of $\mathcal{H}$. It cannot be mapped to $[a, b]$ for, otherwise, by condition ($b$) also the edge $[c, d]$ of $\mathcal{H}$ should be mapped to $[a, b]$, which violates condition ($c$). On the other hand, the edge $[a, d]$ of $\mathcal{H}$ cannot be mapped to $[b, d]$ or $[c, b]$ by condition ($c$). Therefore, the edge $[a, d]$ of $\mathcal{H}$ must be mapped to itself, and the same holds for the edges $[b, d]$ and $[c, d]$ of $\mathcal{H}$.

**Corollary 1.** Let $\mathcal{H}$ be a connected hypergraph. A nonempty subset $X$ of $V(\mathcal{H})$ is $c$-convex in $\mathcal{H}$ if and only if $X$ equals the vertex set of $CC(\mathcal{H}, X)$.

### 3.2. Computing $c$-convex hulls

Let $\mathcal{H}$ be a connected hypergraph with $m$ edges and $n$ vertices. By **Lemma 1** the $c$-convex hull $X^c$ of $X$ in $\mathcal{H}$ can be obtained by taking the union of edges of the hypergraph $CC(\mathcal{H}, X)$ which can be constructed in $O(m^4n)$ time [17]. However, if $\mathcal{H}$ is acyclic then, as proven in [17], $CC(\mathcal{H}, X)$ (and, hence, $X^c$) can be obtained using the following algorithm, of which Tarjan and Yannakakis [21] gave a linear-time implementation.

<table>
<thead>
<tr>
<th>Graham reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repeatedly apply the following two operations until neither can be longer applied:</td>
</tr>
<tr>
<td>(Vertex Deletion)</td>
</tr>
<tr>
<td>(Edge Deletion)</td>
</tr>
</tbody>
</table>

Henceforth, as in [17] the resultant hypergraph of the algorithm above will be denoted by $GR(\mathcal{H}, X)$. Maier and Ullman [17] also proved that $GR(\mathcal{H}, X)$ is defined uniquely, that is, it is independent of the order in which vertices and edges are deleted. As a consequence, the following holds.

**Remark 2.** Let $\mathcal{H}$ be an acyclic hypergraph and $X$ a subset of $V(\mathcal{H})$. For every subset $Y$ of $X$, one has $GR(\mathcal{H}, Y) = GR(GR(\mathcal{H}, X), Y)$.

Note that $GR(\mathcal{H}, \emptyset)$ is the empty hypergraph. So, by **Lemma 1**, the following holds.

**Lemma 2.** Let $\mathcal{H}$ be an acyclic, connected hypergraph and $X$ a subset of $V(\mathcal{H})$. The $c$-convex hull of $X$ in $\mathcal{H}$ equals the vertex set of $GR(\mathcal{H}, X)$.

**Corollary 2.** Let $\mathcal{H}$ be an acyclic, connected hypergraph. A subset $X$ of $V(\mathcal{H})$ is $c$-convex in $\mathcal{H}$ if and only if $X$ is the vertex set of $GR(\mathcal{H}, X)$. 
In the general case, as proven in [19], once the compact hypergraph of $\mathcal{H}$ has been constructed (which can be done once and for all), the vertex set of $CC(\mathcal{H}, X)$ (and, hence, $X^c$) can be found in $O(nm)$ time using the following algorithm.

<table>
<thead>
<tr>
<th>canonical closure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1. Compute $GR(\mathcal{K}, X)$ and set $X'$ to the vertex set of $GR(\mathcal{K}, X)$.</td>
</tr>
<tr>
<td>Step 2. For every edge $A$ of $GR(\mathcal{K}, X)$, if $A$ is neither an edge of $\mathcal{K}$ nor a partial edge of $\mathcal{H}$, then set $X' := X' \cup B$ where $B$ is the edge of $\mathcal{K}$ containing $A$.</td>
</tr>
</tbody>
</table>

Note that for $X = \emptyset$, since $GR(\mathcal{K}, \emptyset)$ is the empty hypergraph, the output of the algorithm is the empty set. So, by Lemma 1, one has the following.

**Lemma 3** ([19]). Let $\mathcal{H}$ be a connected hypergraph and $X$ a subset of $V(\mathcal{H})$. The canonical closure algorithm correctly computes the $c$-convex hull of $X$ in $\mathcal{H}$.

For the sake of completeness, in the Appendix we report a part of the proof of Lemma 3, which emends a flaw contained in the original proof.

**Example 3.** Consider the hypergraph $\mathcal{H} = \{\{a, b\}, \{a, d\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}, \{d, e\}, \{e, f\}\}$ of Example 1, and the vertex set $X = \{c, d, e, f\}$. Recall that the compact hypergraph of $\mathcal{H}$ is $\mathcal{K} = \{\{a, b, c, d, e\}, \{c, e, f\}\}$. At Step 1 of the canonical closure algorithm, the hypergraph $GR(\mathcal{K}, X)$ resulting from the Graham reduction of $\mathcal{K}$ with sacred set $X$ has two edges: $\{c, d, e\}$ and $\{c, e, f\}$. At Step 2 one obtains $X' = V(\mathcal{H})$ so that, by Lemma 3, $V(\mathcal{H})$ is the $c$-convex hull of $\mathcal{H}$.

Finally, from Lemma 3 we derive the following characterization of $c$-convex sets, which generalizes Corollary 2.

**Corollary 3.** Let $\mathcal{H}$ be a connected hypergraph and $\mathcal{K}$ the compact hypergraph of $\mathcal{H}$. A subset $X$ of $V(\mathcal{H})$ is $c$-convex in $\mathcal{H}$ if and only if

1. $X$ is the vertex set of $GR(\mathcal{K}, X)$, and
2. every edge of $GR(\mathcal{K}, X)$ is either a partial edge of $\mathcal{H}$ or a compound edge of $\mathcal{K}$.

By Corollary 3, it is easy to see that every edge of the compact hypergraph $\mathcal{K}$ of $\mathcal{H}$ is $c$-convex in $\mathcal{H}$. Indeed, the following stronger result holds.

**Lemma 4** ([20]). Let $\mathcal{H}$ be a connected hypergraph. The compact hypergraph of $\mathcal{H}$ is the minimal acyclic cover of $\mathcal{H}$ whose edges are all $c$-convex sets in $\mathcal{H}$.

3.2. Characterizations of $c$-convex sets by paths

We shall state a characterization of $c$-convex sets in terms of paths. To this end, we need the following technical lemma.

**Lemma 5.** Let $\mathcal{H}$ be a connected hypergraph, $X$ a subset of $V(\mathcal{H})$ with $|X| \geq 2$, and $C$ a convex connection for $X$ in $\mathcal{H}$. If $C$ contains a non-leaf vertex that is not in $X$ and is the transit-point of no path in $C$ with end-points in $X$, then $C$ is not a minimal convex connection for $X$ in $\mathcal{H}$.

**Proof.** Let $v$ be a non-leaf vertex of $C$ that is not in $X$ and is the transit-point of no path in $C$ with end-points in $X$. Then, there exists an edge of $C$ that either is a step of every path in $C$ with one-endpoint $v$ and the other end-point in $X$, or contains a transit-point of every path in $C$ with one-endpoint $v$ and the other end-point in $X$. Therefore, there exists a partial edge $Y$ of $C$ that does not contain $v$ and separates $v$ from $X$. Let $C'$ be the $Y$-component of $C$ containing $v$. Since $v$ is a non-leaf vertex of $C$, $C'$ is not a trivial hypergraph. Let $B$ be an edge of $C$ that contains $Y$ and let $f$ be the mapping from $C$ onto $C'$ such that $f(A) = A$ for every edge $A$ of $C \setminus C'$ and $f(A) = B$ for every edge $A$ of $C'$. It is easy to see that $f$ is a non-identity $X$-reduction of $C$ and, hence, the convex connection $C$ is not minimal.

**Theorem 1.** Let $\mathcal{H}$ be a connected hypergraph. A subset $X$ of $V(\mathcal{H})$ is $c$-convex if and only if $X$ contains the points of every path with end-points in $X$ in every minimal convex connection for $X$ in $\mathcal{H}$.

**Proof.** Let $C$ be any minimal convex connection for $X$ in $\mathcal{H}$. By Fact 4, $C$ is connected. Moreover, $CC(\mathcal{H}, X)$ is obtained from $C$ by deleting the leaves of $C$ that are not in $X$.

(Only if) By hypothesis, $X$ is $c$-convex in $\mathcal{H}$. Let $p$ be a path in $C$ with end-points in $X$. Of course, the transit-points of $p$ cannot be leaves of $C$ and, hence, they are all vertices of $CC(\mathcal{H}, X)$. Since $X$ is $c$-convex, $X$ equals the vertex set of $CC(\mathcal{H}, X)$ by Corollary 1 and, hence, $X$ contains all the transit-points and, hence, all the points of $p$.

(If) By hypothesis, $X$ contains the points of every path with end-points in $X$ in $C$. Suppose, by contradiction, that $X$ is not $c$-convex. By Corollary 1, $X$ is a proper subset of the vertex set of $CC(\mathcal{H}, X)$. Let $v$ be a vertex of $CC(\mathcal{H}, X)$ that does not belong to $X$. Then, $v$ is a non-leaf vertex of $C$. Since $C$ is a minimal convex connection for $X$ in $\mathcal{H}$, by Lemma 5 the vertex $v$ is the transit-point of a path in $C$ with end-points in $X$, which contradicts the hypothesis since $v \not\in X$. Therefore, $X$ must be $c$-convex in $\mathcal{H}$. □
4. Geometric c-convexity spaces

A question that naturally arises about c-convexity is whether or not c-convexity spaces are geometric. The following example shows a hypergraph on which the c-convexity space is not geometric.

**Example 4.** Consider again the connected (hyper)graph $\mathcal{H} = \{[a, b], [a, c], [a, d], [b, c], [b, d], [c, d]\}$ of Example 2 and the c-convex set $V(\mathcal{H})$. As seen in Example 2, every subset $X$ of $V(\mathcal{H})$ with $|X| = 3$ is not c-convex so that $V(\mathcal{H})$ has no extreme points. Since $V(\mathcal{H})$ is not the c-convex hull of the set of its extreme points, the MKM property does not hold and the c-convexity space on $\mathcal{H}$ is not geometric.

We shall give a characterization of the whole class of connected hypergraphs with geometric c-convexity spaces. We begin by proving that the c-convexity space on an acyclic hypergraph is geometric. First of all, we state a characterization of extreme points of a c-convex set in an acyclic hypergraph.

**Lemma 6.** Let $\mathcal{H}$ be an acyclic, connected hypergraph, and let $X$ be a c-convex set in $\mathcal{H}$. A vertex in $X$ is an extreme point of $X$ if and only if it is a leaf of $GR(\mathcal{H}, X)$.

**Proof.** Since $X$ is c-convex, by Corollary 2 one has that $X$ is the vertex set of $GR(\mathcal{H}, X)$. Let $v$ be a vertex in $X$, and let $Y = X \setminus \{v\}$. By Corollary 2, $Y$ is c-convex in $\mathcal{H}$ if and only if $Y$ is the vertex set of $GR(\mathcal{H}, Y)$. By Remark 2, $GR(\mathcal{H}, Y) = GR(GR(\mathcal{H}, X), Y)$ so that $Y$ is the vertex set of $GR(\mathcal{H}, Y)$ if and only if $v$ is a leaf of $GR(\mathcal{H}, X)$. \qed

**Theorem 2.** The c-convexity space on an acyclic, connected hypergraph is geometric.

**Proof.** Let $\mathcal{H}$ be an acyclic, connected hypergraph, let $X$ be any c-convex set in $\mathcal{H}$ and let $Y$ be the set of the extreme points of $X$. In order to prove that the c-convexity space on $\mathcal{H}$ is geometric, we need to show that $X$ is the c-convex hull of $Y$.

By Corollary 2, $X$ is the vertex set of $GR(\mathcal{H}, X)$. By Lemma 6, $Y$ is the set of leaves of $GR(\mathcal{H}, X)$ so that no vertex can be deleted during the Graham reduction of $GR(\mathcal{H}, X)$ with sacred set $Y$; therefore, $GR(GR(\mathcal{H}, X), Y) = GR(\mathcal{H}, X)$ and, since $GR(\mathcal{H}, Y) = GR(GR(\mathcal{H}, X), Y)$ by Remark 2, $X$ is the vertex set of $GR(\mathcal{H}, Y)$ so that, by Lemma 6, $X$ is the c-convex hull of $Y$. \qed

Consider now the c-convexity space on any connected hypergraph. The following generalizes Lemma 6.

**Lemma 7.** Let $\mathcal{H}$ be a connected hypergraph, $\mathcal{K}$ the compact hypergraph of $\mathcal{H}$ and $X$ a c-convex set in $\mathcal{H}$. A vertex $v$ is an extreme point of $X$ if and only if

(a) $v$ is a leaf of $GR(\mathcal{K}, X)$, and
(b) if $A$ is the edge of $GR(\mathcal{K}, X)$ containing $v$, then $A \setminus \{v\}$ is a partial edge of $\mathcal{H}$.

**Proof.** By hypothesis, $X$ is c-convex in $\mathcal{H}$ so that, by Corollary 3, $X$ is the vertex set of $GR(\mathcal{K}, X)$ and every edge of $GR(\mathcal{K}, X)$ is either a partial edge of $\mathcal{H}$ or a compound edge of $\mathcal{K}$.

(If) Let $v$ be a leaf of $GR(\mathcal{K}, X)$ such that, if $A$ is the edge of $GR(\mathcal{K}, X)$ containing $v$, then $A \setminus \{v\}$ is a partial edge of $\mathcal{H}$. Let $Y = X \setminus \{v\}$. We now prove that $Y$ is c-convex in $\mathcal{H}$, which implies that $v$ is an extreme point of $X$. By Corollary 3, we need to prove that

(i) $Y$ is the vertex set of $GR(\mathcal{K}, Y)$, and
(ii) every edge of $GR(\mathcal{K}, Y)$ is either a partial edge of $\mathcal{H}$ or a compound edge of $\mathcal{K}$.

Proof of (i). Since $X$ is the vertex set of $GR(\mathcal{K}, X)$ and $v$ is the only leaf of $GR(\mathcal{K}, X)$ that is not in $Y$, the vertex $v$ is the only vertex that is deleted from $GR(\mathcal{K}, X)$ during the Graham reduction of $GR(\mathcal{K}, X)$ with sacred set $Y$. By Remark 2, $GR(\mathcal{K}, Y) = GR(GR(\mathcal{K}, X), Y)$ so that $Y$ is the vertex set of $GR(\mathcal{K}, Y)$.

Proof of (ii). Since $GR(\mathcal{K}, Y)$ is obtained from $GR(\mathcal{K}, X)$ by deleting the vertex $v$, the edges of $GR(\mathcal{K}, Y)$ are the edges of $GR(\mathcal{K}, X)$ other than $A$ with the addition of $A \setminus \{v\}$ whenever $A \setminus \{v\}$ is not a redundant edge. Since every edge of $GR(\mathcal{K}, X)$ is either a partial edge of $\mathcal{H}$ or a compound edge of $\mathcal{K}$, and since $A \setminus \{v\}$ is a partial edge of $\mathcal{H}$ by hypothesis, it is true that every edge of $GR(\mathcal{K}, Y)$ is either a partial edge of $\mathcal{H}$ or a compound edge of $\mathcal{K}$.

(Only if) Let $v$ be an extreme point of $X$ and let $Y = X \setminus \{v\}$. We now prove that both condition (a) and condition (b) hold.

Proof of (a). Since $v$ is an extreme point of $X$, $Y$ is c-convex in $\mathcal{H}$. By Corollary 3, $GR(\mathcal{K}, Y)$ has vertex set $Y$ and, since $GR(\mathcal{K}, Y) = GR(GR(\mathcal{K}, X), Y)$ by Remark 2, $v$ is the only vertex of $GR(\mathcal{K}, X)$ that is deleted during the Graham reduction of $GR(\mathcal{K}, X)$ with sacred set $Y$, which implies that $v$ is a leaf of $GR(\mathcal{K}, X)$.

Proof of (b). Let $A$ be the edge of $GR(\mathcal{K}, X)$ containing $v$ and let $A' = A \setminus \{v\}$. We need to prove that $A'$ is a partial edge of $\mathcal{H}$. Since every edge of $GR(\mathcal{K}, X)$ is either a partial edge of $\mathcal{H}$ or a compound edge of $\mathcal{K}$, $A$ is so. If $A$ is a partial edge of $\mathcal{H}$, then trivially $A'$ is a partial edge of $\mathcal{H}$. Consider now the case that $A$ is a compound edge of $\mathcal{K}$. Distinguish now two subcases depending on whether $A'$ is or is not an edge of $GR(\mathcal{K}, Y)$. If $A'$ is not an edge of $GR(\mathcal{K}, Y)$ then, since $GR(\mathcal{K}, Y) = GR(GR(\mathcal{K}, X), Y)$ by Remark 2, $A'$ has been deleted during the Graham reduction of $GR(\mathcal{K}, X)$ with sacred set $Y$ so that there exists an edge of $GR(\mathcal{K}, X)$ that contains $A'$ and, hence, there exists another edge $B$ of $\mathcal{K}$ that contains $A'$. Therefore, $A'$ equals the intersection $A \cap B$ of two edges of $\mathcal{K}$ and, by Fact 2, $A'$ is a partial edge of $\mathcal{H}$. If $A'$ is an edge of $GR(\mathcal{K}, Y)$ then, since $Y$ is c-convex and $A'$ is not an edge of $\mathcal{K}$, $A'$ must be a partial edge of $\mathcal{H}$ by Corollary 3. \qed
In order to characterize the whole class of hypergraphs with geometric c-convexity spaces, we now introduce another class of hypergraphs which are close to acyclic hypergraphs. Recall that, by Fact 1, a hypergraph is acyclic if and only if every compact component is a trivial hypergraph. A hypergraph \( \mathcal{H} \) is quasi-acyclic if it is not acyclic and every nontrivial compact component of \( \mathcal{H} \) is a uniform hypergraph of rank \( |V(\mathcal{H})| - 1 \). Let \( \mathcal{H} \) be a quasi-acyclic hypergraph and let \( \mathcal{H}' = \{ A_1, \ldots, A_k \} \), \( k \geq 3 \), be a nontrivial compact component of \( \mathcal{H} \). Since \( |A_i| = |V(\mathcal{H})| - 1 \), there exists a subset \( Z = \{ z_1, \ldots, z_k \} \) of \( V(\mathcal{H}) \) such that \( A_i = V(\mathcal{H}) \setminus \{ z_i \} \) for \( 1 \leq i \leq k \). Finally, note that, since \( k \geq 3 \), \( V(\mathcal{H}) \) is always a clique which implies that \( \mathcal{H} \) is always a chordal hypergraph. For example, the hypergraph with the three edges \( \{ a, b, d \} \), \( \{ a, c, d \} \) and \( \{ b, c, d \} \) is quasi-acyclic and \( Z = \{ a, b, c \} \).

**Theorem 3.** The c-convexity space on a quasi-acyclic, connected hypergraph is geometric.

**Proof.** Let \( \mathcal{H} \) be a quasi-acyclic, connected hypergraph. Consider any c-convex set \( X \) in \( \mathcal{H} \), and let \( Y \) be the set of extreme points of \( X \). By Lemma 3, it is sufficient to prove that \( X \) is exactly the output of the canonical closure algorithm with input \( \mathcal{H} \), \( \mathcal{K} \) and \( Y \), where \( \mathcal{K} \) is the compact hypergraph of \( \mathcal{H} \). Consider the two steps of the algorithm.

(Step 1) Since \( X \) is c-convex in \( \mathcal{H} \), by Corollary 3, \( GR(\mathcal{K}, X) \) has vertex set \( X \) and its edges are either partial edges of \( \mathcal{H} \) or compound edges of \( \mathcal{K} \) and, by Remark 2, \( GR(\mathcal{K}, Y) = GR(GR(\mathcal{K}, X), Y) \). We now prove that, after deleting the leaves of \( GR(\mathcal{K}, X) \) that are not in \( Y \), what remains of \( GR(\mathcal{K}, X) \) is exactly \( GR(\mathcal{K}, Y) \); that is, during the Graham reduction of \( GR(\mathcal{K}, X) \) with sacred set \( Y \) no set can be removed by the operation of edge deletion. Let \( A \) be an edge of \( GR(\mathcal{K}, X) \) containing leaves that are not in \( Y \) and assume (for the sake of simplicity) that we delete them first. Let \( v \) be such a vertex in \( A \). Since \( v \) is not in \( Y \), \( v \) is not an extreme point of \( X \) so that, since \( v \) is a leaf of \( GR(\mathcal{K}, X) \) but not an extreme point of \( X \), by Lemma 7 one has that \( A \setminus \{ v \} \) is not a partial edge of \( \mathcal{H} \) and, hence, \( A \) cannot be a partial edge of \( \mathcal{H} \). Since \( X \) is c-convex in \( \mathcal{H} \) and \( A \) is an edge of \( GR(\mathcal{K}, X) \), \( A \) must be a compound edge of \( \mathcal{K} \) by Corollary 3. Let \( \mathcal{H}' \) be the compact component of \( \mathcal{H} \) with vertex set \( A \). Since \( \mathcal{H} \) is quasi-acyclic, \( \mathcal{H}' \) is a uniform hypergraph of rank \( |A| - 1 \). Let \( Z \) be the subset of \( A \) such that \( \mathcal{H}' = \{ A \setminus \{ z \} : z \in Z \} \). Since \( A \setminus \{ v \} \) is not a partial edge of \( \mathcal{H} \), \( v \) does not belong to \( Z \) (for, otherwise, \( A \setminus \{ v \} \) would be an edge of \( \mathcal{H}' \) and, hence, a partial edge of \( \mathcal{H} \)). So, after deleting \( v \) and the other leaves of \( GR(\mathcal{K}, X) \), which are in \( A \setminus Y \), what remains of \( A \) is a superset of \( Z \) so that the residual part of \( A \) is not a partial edge of \( \mathcal{H} \) and, hence, is not a partial edge of \( \mathcal{H} \). By Remark 2, the residual part of \( A \) cannot be removed by the operation of edge deletion during the Graham reduction of \( GR(\mathcal{K}, X) \) with sacred set \( Y \). Since this argument can be repeated for each edge of \( GR(\mathcal{K}, X) \) containing leaves that are not in \( Y \), one has that \( GR(\mathcal{K}, Y) \) is simply the result of deleting the leaves of \( GR(\mathcal{K}, X) \) that are not in \( Y \). Therefore, every edge of \( GR(\mathcal{K}, Y) \) is either an edge of \( GR(\mathcal{K}, X) \) or the residual part of a compound edge of \( \mathcal{K} \) and, in the latter case, it is not a partial edge of \( \mathcal{H} \).

(Step 2) When the edges of \( GR(\mathcal{K}, X) \) are examined, for each edge \( A' \) of \( GR(\mathcal{K}, Y) \) that is not an edge of \( GR(\mathcal{K}, X) \), one has that \( A' \) is the residual part of an edge \( A \) of \( GR(\mathcal{K}, X) \) that is a compound edge of \( \mathcal{K} \) and, since \( A' \) is not a partial edge of \( \mathcal{H} \) (see above), the edge \( A \) of \( GR(\mathcal{K}, X) \) is added (by union) to the vertex set of \( GR(\mathcal{K}, Y) \). So, the output of the canonical closure algorithm with input \( \mathcal{H} \), \( \mathcal{K} \) and \( Y \) (and, hence, the c-convex hull of \( Y \)) is exactly the vertex set of \( GR(\mathcal{K}, X) \) and, hence, equals \( X \). \( \square \)

The following theorem characterizes the class of connected hypergraphs with geometric c-convexity spaces.

**Theorem 4.** The c-convexity space on a connected hypergraph is geometric if and only if the hypergraph is either acyclic or quasi-acyclic.

**Proof.** (If) By Theorems 2 and 3.

(Only if) By Theorems 2 and 3.

5. Monophonic convexity

In this section we first extend to hypergraphs the notion of m-convexity on graphs. Then, we show that m-convexity is coarser than c-convexity, but they are equivalent in the case of conformal hypergraphs. Finally, we prove that, given the compact hypergraph of a hypergraph \( \mathcal{H} \), the m-convex hull of any subset of \( V(\mathcal{H}) \) can be computed in an efficient way.

First of all, we introduce the notion of an \( X-X \) path in a hypergraph \( \mathcal{H} \) for a subset \( X \) of \( V(\mathcal{H}) \) with \( |X| \geq 2 \), and state a technical lemma. A path in \( \mathcal{H} \) is an \( X-X \) path if its end-points belong to \( X \) but no transit-point (if any) belongs to \( X \). Note that the steps of an \( X-X \) path are \( X \)-connected edges and, hence, lie all in one \( X \)-component of \( \mathcal{H} \).
Lemma 8. Let $\mathcal{H}$ be a hypergraph, and $X$ a subset of $V(\mathcal{H})$ with $|X| \geq 2$. Two distinct vertices are the end-points of an $X$–$X$ path in $\mathcal{H}$ if and only if they are the end-points of an $X$–$X$ path in the two-section of $\mathcal{H}$.

Proof. (Only if) Let $p = \{v_0, A_1, v_1, A_2, \ldots, v_{k-1}, A_k, v_k\}$ be an $X$–$X$ path in $\mathcal{H}$. In the two-section $\mathcal{H}(2)$ of $\mathcal{H}$, $v_{i-1}$ and $v_i$ are adjacent for $1 \leq i \leq k$ so that $\{v_0, v_1, \ldots, v_{k-1}, v_k\}$ is an $X$–$X$ path in $\mathcal{H}(2)$ with the same end-points as $p$.

(If) Let $p = \{v_0, v_1, \ldots, v_{k-1}, v_k\}$ be an $X$–$X$ path in $\mathcal{H}(2)$. Since $v_{i-1}$ and $v_i$ are adjacent in $\mathcal{H}(2)$, $\{v_{i-1}, v_i\}$ is a partial edge of $\mathcal{H}$ for $1 \leq i \leq k$. Let $A_i$ be an edge of $\mathcal{H}$ that contains both $v_{i-1}$ and $v_i$ ($1 \leq i \leq k$). Consider the sequence $\{v_0, A_1, v_1, A_2, \ldots, v_{k-1}, A_k, v_k\}$. Note that the $A_i$’s need not be pairwise distinct. Let $i(1)$ be the maximum of $i \in \{1, 2, \ldots, k\}$ for which $v_i$ belongs to the edge $A_{i(1)+1}$. If $i(1) = k$, then trivially $\{v_0, A_1, v_k\}$ is an $X$–$X$ path in $\mathcal{H}$ with the same end-points as $p$. Otherwise (that is, $i(1) < k$), let $i(2)$ be the maximum of $i \in \{i(1) + 1, \ldots, k\}$ for which $v_i$ belongs to the edge $A_{i(1)+1}$. If $i(2) = k$, then $\{v_0, A_{i(1)+1}, v_1, A_{i(1)+1}, v_2, \ldots, v_{k-1}, v_k\}$ is an $X$–$X$ path in $\mathcal{H}$ with the same end-points as $p$. Otherwise (that is, $i(2) < k$), let $i(3)$ be the maximum of $i \in \{i(2) + 1, \ldots, k\}$ for which $v_i$ belongs to the edge $A_{i(2)+1}$. If $i(3) = k$, then $\{v_0, A_{i(2)+1}, v_1, A_{i(2)+1}, v_2, \ldots, v_{k-1}, v_k\}$ is an $X$–$X$ path in $\mathcal{H}$ with the same end-points as $p$. And so on. Thus, we can construct an $X$–$X$ path in $\mathcal{H}$ with the same end-points as $p$. \hfill \QED

5.1. $m$-convexity in graphs

Let $\mathcal{G}$ be a connected graph. As usual, a path in $\mathcal{G}$ will be denoted simply by a vertex sequence $p = \{v_0, v_1, \ldots, v_{k-1}, v_k\}$, where $v_{i-1}$ and $v_i$ are adjacent for all $i$, $1 \leq i \leq k$. A path $\{v_0, v_1, \ldots, v_{k-1}, v_k\}$ in $\mathcal{G}$ is induced (or “minimal” or “chordless”) if $\mathcal{G}[\{v_0, v_1, \ldots, v_{k-1}, v_k\}] = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}\}$. A subset $X$ of $V(\mathcal{G})$ is $m$-convex in $\mathcal{G}$ if either $|X| \leq 1$ or $X$ contains every point of every induced path $p$ with end-points in $X$. We now recall two results: one characterizes $m$-convex sets in a graph and the other characterizes the graphs with geometric $m$-convexity spaces.

Lemma 9 ([8]). Let $\mathcal{G}$ be a connected graph. A subset $X$ of $V(\mathcal{G})$ is $m$-convex in $\mathcal{G}$ if and only if either $|X| \leq 1$ or the end-points of every $X$–$X$ path are adjacent.

Theorem 5 ([11]). The m-convexity space on a connected graph is geometric if and only if the graph is chordal.

5.2. $m$-convexity in hypergraphs

Let $\mathcal{H}$ be a connected hypergraph, and let $\mathcal{M}$ be the family of subsets $X$ of $V(\mathcal{H})$ such that either $|X| \leq 1$ or the end-points of every $X$–$X$ path in $\mathcal{H}$ are adjacent. By Lemmas 8 and 9, a vertex set belongs to $\mathcal{M}$ if and only if it is $m$-convex in $\mathcal{H}(2)$. Therefore, the pair $(V(\mathcal{H}), \mathcal{M})$ is a convexity space on $\mathcal{H}$, called the $m$-convexity space on $\mathcal{H}$. Moreover, by Theorem 5, the $m$-convexity space on a connected hypergraph is geometric if and only if the hypergraph is chordal.

We now state a characterization of $m$-convex sets, which will be used later on.

Lemma 10. Let $\mathcal{H}$ be a connected hypergraph. A nonempty subset $X$ of $V(\mathcal{H})$ is $m$-convex in $\mathcal{H}$ if and only if, for every $X$-component $\mathcal{H}'$ of $\mathcal{H}$, the set $X \cap V(\mathcal{H}')$ is a clique.

Proof. If $|X| = 1$, then the statement is trivial. Assume that $|X| \geq 2$.

(If) Let $u$ and $v$ be the end-points of an $X$–$X$ path $p$ in $\mathcal{H}$. Since the steps of $p$ are $X$-connected edges, they are the edges of one $X$-component of $\mathcal{H}$, say $\mathcal{H}'$, so that both $u$ and $v$ belong to $V(\mathcal{H}')$ and, hence, to $X \cap V(\mathcal{H}')$. By hypothesis, $X \cap V(\mathcal{H}')$ is a clique and, hence, $u$ and $v$ are adjacent.

(Only if) Suppose, by contradiction, that there is an $X$-component $\mathcal{H}'$ of $\mathcal{H}$ such that $X \cap V(\mathcal{H}')$ is not a clique. Then, there exist two vertices $u$ and $v$ in $X \cap V(\mathcal{H}')$ that are not adjacent. Since $\mathcal{H}'$ is connected, there exists a path $p$ in $\mathcal{H}'$ with end-points $u$ and $v$. Since the steps of $p$ are $X$-connected edges of $\mathcal{H}$, $p$ is an $X$–$X$ path in $\mathcal{H}$ whose end-points are not adjacent, which contradicts the $m$-convexity of $X$. \hfill \QED

5.3. $m$-convexity vs. $c$-convexity

In this subsection we compare $m$-convexity with $c$-convexity and prove that: (1) $m$-convexity is coarser than $c$-convexity, and (2) in conformal hypergraphs, $m$-convexity and $c$-convexity are equivalent.

Theorem 6. Every $c$-convex set is also an $m$-convex set.

Proof. Let $\mathcal{H}$ be a connected hypergraph and $X$ a $c$-convex set. It is sufficient to consider the case that $X \neq \emptyset$. By hypothesis, for every $X$-component $\mathcal{H}'$ of $\mathcal{H}$, the set $X \cap V(\mathcal{H}')$ is a partial edge of $\mathcal{H}$. Since every partial edge of $\mathcal{H}$ is a clique of $\mathcal{H}$, $X$ is $m$-convex by part (if) of Lemma 10. \hfill \QED

The following example shows an $m$-convex set that is not $c$-convex.

Example 5. Consider again the complete graph $\mathcal{H} = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$ of Example 2 and the set $X = \{a, b, c\}$. As seen in Section 3, $X$ is not $c$-convex. However, since $X$ is a clique, $X$ is trivially $m$-convex. \hfill \QED
Theorem 7. Let \( \mathcal{H} \) be a conformal, connected hypergraph \( \mathcal{H} \). A subset of \( V(\mathcal{H}) \) is c-convex if and only if it is m-convex.

Proof. Since \( \mathcal{H} \) is conformal, every clique of \( \mathcal{H} \) is a partial edge of \( \mathcal{H} \) so that, by Lemma 10, a nonempty subset \( X \) of \( V(\mathcal{H}) \) is m-convex if and only if, for every \( X \)-component \( \mathcal{H}' \) of \( \mathcal{H} \), the set \( X \cap V(\mathcal{H}') \) is a partial edge of \( \mathcal{H} \), that is, if and only if \( X \) is c-convex. \( \square \)

5.4. Computing m-convex hulls

Let \( \mathcal{H} \) be a connected hypergraph and let \( X \) be a subset of \( V(\mathcal{H}) \). We denote the m-convex hull of \( X \) in \( \mathcal{H} \) by \( X^m \). In this section, we present an efficient algorithm which, given the compact hypergraph of \( \mathcal{H} \), computes the m-convex hull of any subset of \( V(\mathcal{H}) \).

Lemma 11. Let \( \mathcal{H} \) be a connected hypergraph. A subset of \( V(\mathcal{H}) \) is m-convex in \( \mathcal{H} \) if and only if it is c-convex in the clique hypergraph of \( \mathcal{H}[2] \).

Proof. Let \( X \) be any subset of \( V(\mathcal{H}) \). Since \( X \) is m-convex in \( \mathcal{H} \) if and only if \( X \) is m-convex in \( \mathcal{H}[2] \) and since \( \mathcal{H}[2] \) coincides with the two-section of its clique hypergraph, \( X \) is m-convex in \( \mathcal{H} \) if and only if \( X \) is m-convex in the clique hypergraph of \( \mathcal{H}[2] \). Finally, since the clique hypergraph of \( \mathcal{H}[2] \) is a conformal hypergraph, by Theorem 7, \( X \) is m-convex in \( \mathcal{H} \) if and only if \( X \) is c-convex in the clique hypergraph of \( \mathcal{H}[2] \). \( \square \)

Let \( \mathcal{H} \) be the clique hypergraph of \( \mathcal{H}[2] \), let \( J \) be the compact hypergraph of \( \mathcal{H} \) and let \( X \) be a subset of \( V(\mathcal{H}) \). By Lemma 11, the m-convex hull \( X^m \) of \( X \) in \( \mathcal{H} \) equals the c-convex hull of \( X \) in \( \mathcal{H} \). Therefore, by Lemma 3, once the hypergraphs \( \mathcal{H} \) and \( J \) have been constructed (which can be done once and for all) \( X^m \) can be obtained by applying the canonical closure algorithm with input \( \mathcal{H} \) (instead of \( \mathcal{H} \)), \( J \) (instead of \( \mathcal{H} \)) and \( X \):

(1) Compute \( GR(J, X) \) and set \( X^m \) to the vertex set of \( GR(J, X) \).

(2) For every edge \( A \) of \( GR(J, X) \), if \( A \) is neither an edge of \( J \) nor a partial edge of \( \mathcal{H} \), then set \( X^m := X^m \cup B \) where \( B \) is the edge of \( J \) containing \( A \).

So, computing \( X^m \) as the same complexity as computing \( X^c \). Finally, we show that \( X^m \) can be computed without passing through the clique hypergraph \( \mathcal{H} \) of \( \mathcal{H}[2] \). First of all, since \( \mathcal{H} \) is a conformal hypergraph, the compact hypergraph \( J \) of \( \mathcal{H} \) coincides with the prime hypergraph of \( \mathcal{H}[2] = \mathcal{H}[2] \) and, hence, can be obtained from \( \mathcal{H}[2] \) using Leimer’s decomposition algorithm [16], which runs in \( O(nm) \) time if \( \mathcal{H}[2] \) has \( n \) vertices and \( m \) edges. Moreover, since \( \mathcal{H} \) is a conformal hypergraph, at Step (2) of the algorithm above we can change “if \( A \) is a partial edge of \( \mathcal{H} \)” to “if \( A \) is not a clique of \( \mathcal{H}[2] \).” To sum up, we can compute \( X^m \) also using the following algorithm with input \( \mathcal{H}[2] \), the prime hypergraph \( J \) of \( \mathcal{H}[2] \) and \( X \).

monophonic closure

(1) Compute \( GR(J, X) \) and set \( X^m \) to the vertex set of \( GR(J, X) \).

(2) For every edge \( A \) of \( GR(J, X) \), if \( A \) is neither an edge of \( J \) nor a clique of \( \mathcal{H}[2] \), then set \( X^m := X^m \cup B \) where \( B \) is the edge of \( J \) containing \( A \).

By Lemma 11, the following holds.

Theorem 8. Let \( \mathcal{H} \) be a connected hypergraph and \( X \) a subset of \( V(\mathcal{H}) \). The monophonic closure algorithm correctly computes the m-convex hull of \( X \) in \( \mathcal{H} \).

Appendix

Proposition 3. Let \( \mathcal{H} \) be a connected hypergraph and \( X \) a nonempty subset of \( V(\mathcal{H}) \). The vertex set of every convex connection for \( X \) in \( \mathcal{H} \) is c-convex in \( \mathcal{H} \).

Proof. Let \( C \) be a convex connection for \( X \) in \( \mathcal{H} \) and let \( Y \) be the vertex set of \( C \). In order to prove that \( Y \) is c-convex in \( \mathcal{H} \), we show that for every \( Y \)-component \( \mathcal{H}' \) of \( \mathcal{H} \) one has that

(1) \( f(A) = f(B) \) for every two edges \( A \) and \( B \) of \( \mathcal{H}' \), and

(2) \( Y \cap V(\mathcal{H}') \) is contained in the edge of \( C \) that all the edges of \( \mathcal{H}' \) are mapped to.

Proof of (1). The statement is trivially true if \( \mathcal{H}' \) is a trivial hypergraph. Otherwise, let \( A \) and \( B \) be two distinct edges of \( \mathcal{H}' \). Since \( \mathcal{H}' \) is a \( Y \)-component of \( \mathcal{H} \), \( A \) and \( B \) are the end-steps of a path \( (v_0, A_1, v_1, A_2, \ldots, v_{k-1}, A_k, v_k) \) in \( \mathcal{H}' \) whose transit-points do not belong to \( Y \). Since for all \( i \leq k \) one has \( v_i \in A_i \cap A_{i+1} \), \( v_i \) is not a leaf of \( \mathcal{H}' \) and, since \( v_i \notin Y \) and \( Y \) is the vertex set of \( C \), \( v_i \) is in neither \( f(A_i) \) nor \( f(A_{i+1}) \) so that \( f(A_i) = f(A_{i+1}) \) by condition (b). Therefore, one has \( f(A_1) = \cdots = f(A_k) \) and, hence, \( f(A) = f(B) \).

Proof of (2). Let \( E \) be the edge of \( C \) that all the edges of \( \mathcal{H}' \) are mapped to. It is sufficient to show that \( Y \cap A \subseteq E \) for every edge \( A \) of \( \mathcal{H}' \). Suppose, by contradiction, that there exists an edge \( A \) of \( \mathcal{H}' \) such that \( Y \cap A \) is not a subset of \( E \) and let \( v \) be a vertex in \( (Y \cap A) \setminus E \). Since \( v \in Y \), there exists an edge \( B \) of \( C \), \( B \not= E \), that contains \( v \). Since \( v \) belongs to both the edges \( A \)
and B of $\mathcal{H}$, $v$ is not a leaf of $\mathcal{H}$ and condition (b) must hold but, since $f(A) = E$ and $v \not\in E$, one should have $v \not\in f(A)$ and, hence, $f(A) = f(B)$ and a contradiction arises since $f(A) = E \neq B = f(B)$.

Finally, since conditions (1) and (2) hold for every $Y$-component of $\mathcal{H}$, one has that the boundary of every $Y$-component of $\mathcal{H}$ is a partial edge of $C$ and, hence, of $\mathcal{H}$ so that $Y$ is $c$-convex in $\mathcal{H}$.

**Proposition 4.** Let $\mathcal{H}$ be a connected hypergraph and $X$ a nonempty subset of $V(\mathcal{H})$. A superset $Y$ of $X$ is $c$-convex in $\mathcal{H}$ if and only if there exists a convex connection $C$ for $X$ in $\mathcal{H}$ such that $Y \subseteq V(C)$ and $V(C) \setminus Y$ is a set of leaves of $C$.

**Proof.** (1f) Let $Y$ be a superset of $X$ and let $C$ be a convex connection for $X$ in $\mathcal{H}$ such that $Y \subseteq V(C)$ and $V(C) \setminus Y$ is a set of leaves of $C$. Since there is one $Y$-component of $C$ for each edge of $Y$, the boundary of every $Y$-component of $C$ is a partial edge of $C$ and $Y$ is $c$-convex in $C$. On the other hand, $V(C)$ is $c$-convex in $\mathcal{H}$ by Proposition 3 so that, by Proposition 1, $Y$ is $c$-convex in $\mathcal{H}$.

(Only if) Let $Y$ be a $c$-convex superset of $X$. Since the boundary of every $Y$-component of $\mathcal{H}$ is a partial edge of $\mathcal{H}$, there exists a partial subhypergraph $C$ of $\mathcal{H}$ such that

1. for every $Y$-component $\mathcal{H}'$ of $\mathcal{H}$ there is an edge of $C$ that contains the boundary $Y \cap V(\mathcal{H}')$ of $\mathcal{H}'$,
2. for every two edges $A$ and $B$ of $C$, if $Y \cap A \subseteq Y \cap B$ then $A = B$.

We now prove that $C$ is a convex connection for $X$ in $\mathcal{H}$ and $V(C) \subseteq Y \subseteq V(C)$, where $C'$ is the subhypergraph of $C$ obtained by deleting the leaves that are not in $X$. First of all, note that by (2), for every $Y$-component $\mathcal{H}'$ of $\mathcal{H}$, there exists one edge of $C$ that contains the boundary $Y \cap V(\mathcal{H}')$ of $\mathcal{H}'$. Let $f$ be the mapping from $\mathcal{H}$ onto $C$ that, for every $Y$-component $\mathcal{H}'$ of $\mathcal{H}$, maps every edge of $\mathcal{H}'$ to the edge of $C$ that contains the boundary of $\mathcal{H}'$. We now prove that the mapping $f$ fulfils the three requirements (a), (b) and (c) that make $f$ a $Y$-reduction of $\mathcal{H}$.

Proof of (c). Let $A$ be an edge of $\mathcal{H}$ and let $\mathcal{H}'$ be the $Y$-component of $\mathcal{H}$ containing $A$. By construction of $C$, one has $Y \cap A \subseteq Y \cap V(\mathcal{H}') \subseteq Y \cap f(A)$.

Proof of (a). Let $A$ be an edge of $C$. By (c), one has $Y \cap A \subseteq Y \cap f(A)$. Since both $A$ and $f(A)$ are edges of $C$, by (2) one has $A = f(A)$.

Proof of (b). Let $v$ be a non-leaf vertex of $\mathcal{H}$. Distinguish two cases depending on whether or not $v$ is in $Y$. If $v$ is in $Y$, then $v \in Y \cap A$ for every edge $A$ of $\mathcal{H}$ containing $v$ and, since $Y \cap A \subseteq Y \cap f(A)$, $v \in Y \cap f(A)$ by (c) and, hence, $v$ belongs to $f(A)$. Otherwise, every two edges $A$ and $B$ of $H$ containing $v$ are in the same $Y$-component of $\mathcal{H}$ and, by construction of $C$, $f(A) = f(B)$.

So, $C$ is a convex connection for $X$ in $\mathcal{H}$ and, since $X$ is a subset of $Y$, $C$ is also a convex connection for $X$ in $\mathcal{H}$ by Remark 1. What remains to prove is that $V(C') \subseteq Y$, that is, $V(C) \setminus Y$ is a set of leaves of $C$. Let $v$ be a vertex of $C$ that is not in $Y$. Suppose, by contradiction, that $v$ is not a leaf of $C$. Then, $v$ is not a leaf of $\mathcal{H}$ and there is exactly one $Y$-component $\mathcal{H}'$ of $\mathcal{H}$ containing $v$. Since all the edges of $\mathcal{H}'$ are mapped to one edge of $C$, there is exactly one edge of $C$ that contains $v$ (contradiction).

**Lemma 1.** Let $\mathcal{H}$ be a connected hypergraph and $X$ a nonempty subset of $V(\mathcal{H})$. The $c$-convex hull of $X$ in $\mathcal{H}$ equals the vertex set of $CC(\mathcal{H}, X)$.

**Proof.** Let $X'$ be the $c$-convex hull of $X$ in $\mathcal{H}$. By Proposition 4, there exists a convex connection $C$ for $X$ in $\mathcal{H}$ such that $X' \subseteq V(C)$ and $V(C) \setminus X'$ is a set of leaves of $C$. Furthermore, since $X'$ is the minimal superset of $X$ that is $c$-convex in $\mathcal{H}$, $X'$ must equal $V(C)$ minus the set of leaves of $C$ that are not in $X$. At this point, what remains to prove is that $C$ is a minimal convex connection for $X$ in $\mathcal{H}$. Suppose, by contradiction, that there exists an $X$-reduction $f$ of $C$ other than the identity. Let $C_1 = f(C)$ and let $Y$ be the set $V(C_1)$ minus the set of leaves of $C_1$ that are not in $X$. By Proposition 4, $Y$ is a superset of $X$ that is $c$-convex in $\mathcal{H}$. On the other hand, since $f$ is not the identity, $V(C_1)$ is a proper subset of $V(C)$ and each leaf of $C$ that is a vertex of $C_1$ is also a leaf of $C$, so that $Y$ is a proper subset of $X'$ and, hence, $X'$ is not the minimal superset of $X$ that is $c$-convex in $\mathcal{H}$ (contradiction). □

**Lemma 3.** Let $\mathcal{H}$ be a connected hypergraph and $X$ a subset of $V(\mathcal{H})$. The canonical closure algorithm correctly computes the $c$-convex hull of $X$ in $\mathcal{H}$.

**Proof.** Let $X'$ be the $c$-convex hull of $X$ in $\mathcal{H}$. By Lemma 1, $X'$ equals the vertex set of $CC(\mathcal{H}, X)$. Let $Y$ be the output of the canonical closure algorithm with input $\mathcal{H}$, $\mathcal{K}$ and $X$ and the statement is proven by showing that $Y \subseteq X'$ and $X' \supseteq Y$ [19]. We now prove only the former inclusion since its original proof in [19] contained a flaw.

Proof of the inclusion $Y \subseteq X'$. Let $v \not\in X'$. We need to show that $v \not\in Y$. Since $v \not\in X'$ and $X'$ is the vertex set of $CC(\mathcal{H}, X)$, by Lemma 2 in [19] there is an edge $E$ of $CC(\mathcal{H}, X)$ that separates $v$ from $X$ in $\mathcal{H}$. Let $\mathcal{H}'$ be the $E$-component of $\mathcal{H}$ containing $v$. Since $CC(\mathcal{H}, X)$ is an induced subhypergraph of $\mathcal{H}$ and $E$ is an edge of $CC(\mathcal{H}, X)$, $E$ is a partial edge of $\mathcal{H}$ and, hence, of $\mathcal{K}$. Let $\mathcal{K}'$ be the $E$-component of $\mathcal{K}$ containing $v$; by Lemma 1 in [19], $V(\mathcal{K}') = V(\mathcal{K}')$ and, hence, $E$ separates $v$ from $X$ in $\mathcal{K}$. From Lemma 3 in [19] it follows that $v$ is not a vertex of $CC(\mathcal{K}, X)$. Since $\mathcal{K}$ is an acyclic hypergraph, one has $CC(\mathcal{K}, X) = GR(\mathcal{K}, X)$. At this point, in order to prove that $v$ does not belong to $Y$, it is sufficient to show (see Step 2 of the algorithm) that, for every edge $A$ of $GR(\mathcal{K}, X)$ that is not a partial edge of $\mathcal{H}$, $v$ does not belong to the edge $B$ of $\mathcal{K}$ containing $A$. Suppose, by contradiction, that there exists an edge $A$ of $GR(\mathcal{K}, X)$ that is not a partial edge of $\mathcal{H}$ and is such that $v$ belongs to the edge $B$ of $\mathcal{K}$ containing $A$. Since $E$ separates $v$ from $X$ and since $X'$ is the $E$-component of $\mathcal{K}$ containing $v$.
v, B is a subset of the union of E with V(κ’). Now, during the Graham reduction of κ with sacred set X, every edge of κ that is contained in the union of E with V(κ’) either is deleted or is reduced and its residual part is a subset of E. Of course, this holds for B. Therefore, B has been reduced and its residual part is a subset of E. Since the residual part of B is A, A must be a subset of E. Finally, since E is a partial edge of H, A should be a partial edge of H (contradiction).

References