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Projectile motion with a drag force: were the Medievals right after all?

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Abstract
An educational and historical study of the projectile motion with drag forces dependent on speed shows, by simple results, that trajectories quite similar to those depicted before the Galilean era may be obtained with a realistic choice of quantities involved. Numerical simulations of the trajectory in space and velocity coordinates help us to understand the dynamics of motion.

Introduction

In the middle ages, the trajectory of a cannonball was quite often depicted as composed of an almost rectilinear segment followed by an abrupt descent (figure 1) [1]. At that time there was no way to observe the motion of a falling body to quantify its trajectory. What people saw with their eyes could be strongly influenced by what they expected to see—based on the Aristotelian description of natural and violent motions. The interpretation of the straight part of the motion was related to the initial momentum, which after some time was exhausted, giving rise to the second part of the trajectory characterized by a steep curvature, until the cannonball reached the ground. In his *Nova Scientia* (1537), Niccolò Tartaglia claimed that a body starts to lose velocity as soon as it escapes the propelling force. In his description of the projectile motion, the first part of the trajectory is a straight line; the second is a curve, while the third part is again a nearly vertical line associated with the projectile's natural motion. Tartaglia assumed that the curved part was the result of the body weight. However, he was forced to admit that the whole path was actually curved, but the curvature was so small as to be negligible, and impossible to perceive.

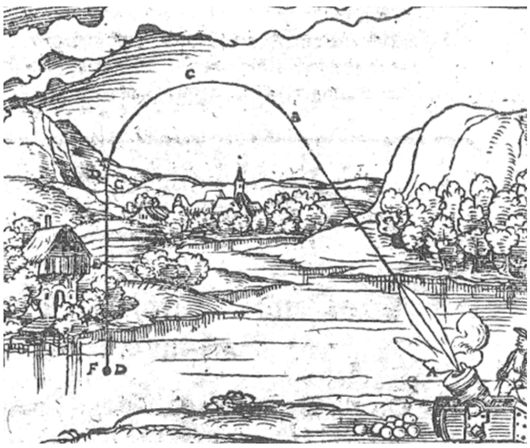

Figure 1. An illustration of the trajectory of a cannonball, as depicted in old documents before the Galilean era [1].

Only in the Galilean era was the parabolic nature of the projectile motion recognized as being the result of the composition of two independent motions along orthogonal axes.

Today, the study of parabolic motion, in the absence of any drag force, is a common example in

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introductory physics courses. Introducing friction forces into the study of the motion, however, gives rise to a problem that is difficult to solve analytically, except in a few particular cases.

Educational studies of projectile motion under the influence of a drag force have long been reported [2–6], addressing several aspects of the problem. For instance, a numerical study of projectile motion with quadratic dependence on projectile speed has been reported in [3]. The optimal launch conditions for a small body fired by hand under the influence of air drag have been discussed in [4]. The drag force has been considered in a variety of conditions, showing that the trajectory may be approximated under specific assumptions with cubic curves [6].

Simple simulation of the projectile motion with realistic drag forces shows, contrary to what most students believe, that the resulting trajectory may indeed be very similar to that originally represented in the middle ages, and directly seen with our eyes in a variety of situations in everyday life.

To stress such an aspect, we carried out basic calculations, which can be performed even by high school students with some knowledge of programming and graphics tools. The equations of motion along two orthogonal axes were integrated with simple first-order algorithms, comparing the results obtained with and without realistic, speed-dependent, drag forces. Projectile trajectories, together with plots of the components of the projectile velocity, allow us to demonstrate simple effects of the influence of the drag force on the resulting motion. Physical examples derived from phenomena which may be observed in everyday life are also discussed at the end of the article.

Equations of motion

The equations of motion, for a body of mass m subjected to a linear drag force

$$\mathbf{F} = -b\mathbf{v}$$

and launched from the origin (0, 0) with a speed v_0 and at an angle θ with respect to the horizontal are

$$m\,\mathrm{d}^2x(t)/\mathrm{d}t^2 = -bv_x$$

$$m\,\mathrm{d}^2y(t)/\mathrm{d}t^2 = -mg - bv_y$$

with the initial conditions

$$x(0) = 0 \qquad y(0) = 0$$

$$v_x(0) = v_0\cos\theta \qquad v_y(0) = v_{y0}\sin\theta.$$

Such equations admit the analytical solution [5]

$$x(t) = mv_0\cos\theta[1 - \exp(-bt/m)]/b$$

$$y(t) = m(mg/b + v_0\sin\theta)$$

$$[1 - \exp(-bt/m)]/b - mgt/b.$$

However, if the dependence on the speed is quadratic or even more complicated, one must make use of numerical methods for the solution of the problem.

From a physical point of view, drag forces proportional to the speed are generally expected only in the case of motion with low Reynolds numbers, such as the motion of bodies with very small sizes and low speed. One such case is the motion of a small sphere of radius r in a fluid of density ρ and viscosity η, where the coefficient $b = 6\pi\eta r$.

In most cases, however, quadratic speed dependence provides a more realistic description of the motion of bodies under drag forces. It is known that for body sizes and speeds of practical interest, a reasonable description of the drag force is provided by

$$\mathbf{F} = -\tfrac{1}{2}C_{\mathrm{D}}\rho A v^2(\mathbf{v}/v),$$

where ρ is the density of the medium, A is the transverse area of the body, v its speed and C_{D} the drag coefficient. The latter is a dimensionless quantity, whose value depends on the shape and orientation of the body with respect to the relative velocity vector between the body and the medium. For a sphere, it is approximately $C_{\mathrm{D}} = 0.5$, whereas for aerodynamic shapes C_{D} may reach values down to about 0.1.

The corresponding equations of motion are now:

$$m\,\mathrm{d}^2x(t)/\mathrm{d}t^2 = -cv_x(v_x^2 + v_y^2)^{1/2}$$

$$m\,\mathrm{d}^2y(t)/\mathrm{d}t^2 = -mg - cv_y(v_x^2 + v_y^2)^{1/2}$$

where the coefficient c is given by

$$c = \tfrac{1}{2}C_{\mathrm{D}}\rho A.$$

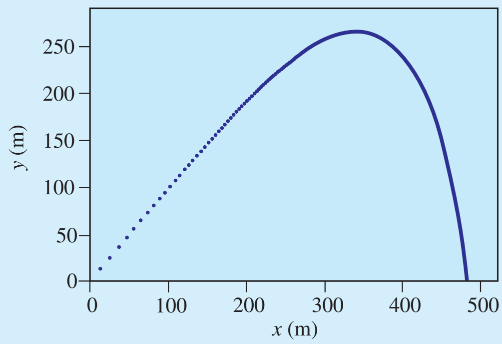

Figure 2. Simulated trajectory of a cannonball launched from the origin (0, 0) with an angle of 45° with respect to the horizontal, and an initial speed of 400 m s^{-1}. A drag force proportional to v^2 was assumed, with a value of the coefficient $c/m = 0.005$ m^{-1} (see text).

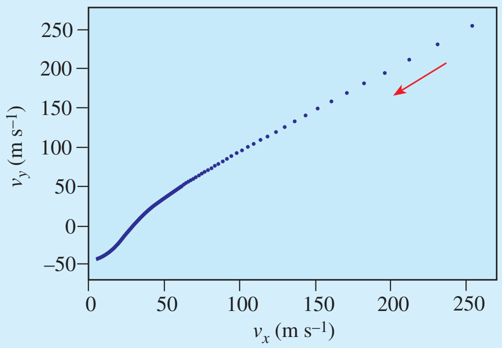

Figure 3. The components v_x, v_y of the projectile velocity are plotted for a body launched from the origin (0, 0) with the same parameters as in figure 2. The trajectory starts from the upper-right position.

Taking as an example the motion of a sphere of radius $r = 5$ cm in air, a value of $c = 0.0026$ kg m^{-1} is obtained.

The numerical solution of the previous equations may be obtained through various algorithms with different degrees of precision. For educational purposes, even the simple Euler method is good enough to get a precise set of (x, y) values which approximate the true trajectory:

$$x(t + \Delta t) = x(t) + v_x(t)\Delta t$$

$$y(t + \Delta t) = y(t) + v_y(t)\Delta t$$

$$v_x(t + \Delta t) = v_x(t) - cv_x(t)v(t)\Delta t/m$$

$$v_y(t + \Delta t) = v_y(t) - g\Delta t - cv_y(t)v(t)\Delta t/m.$$

Previous formulae may be easily implemented in any of the existing programming tools, setting the required parameters—initial position and velocity components, together with the value of the quantity c/m—to evaluate and plot the resulting trajectory. Ready-to-use packages, such as Modellus [7] or Matlab [8] may also be employed to produce such results. More refined algorithms, based on Runge–Kutta methods, may be employed if better precision is required.

Results

Figure 2 shows the simulated trajectory of a body launched from the origin (0, 0) with speed $v = 400$ m s^{-1} at an angle (with respect to the horizontal) $\theta = 45°$. A drag force proportional to v^2, with a coefficient equal to $c/m = 0.005$ m^{-1} was assumed. A time step of 0.05 s was employed in the numerical integration. Values of c/m in the range (0.001–0.005) m^{-1} roughly correspond to small cannonballs made of roughly spherical stones (5–10 cm diameter), launched in air. This situation was typical of old devices, whereas, starting from 1500, metal projectiles and larger sizes were introduced. The travelled distance for the trajectory depicted in figure 2 is about 500 m. As can be seen, the trajectory exhibits an almost rectilinear segment at the beginning of the motion, followed by a curved path, and it is quite similar to the old pictures found in historical documents (figure 1).

In the first part of the trajectory the speed of the projectile is relatively high, with a correspondingly high value of the drag force. The effect of gravity is small in comparison to the drag force and the trajectory is dominated by the original components of the velocity. When the speed is largely reduced, due to the drag force, the variation of the speed produced by gravity strongly modifies the trajectory, which is rapidly curved to reach the ground. An additional piece of information provided by the simulation is the combined plot of the two velocity components v_x, v_y (figure 3). The time history of the trajectory in this plot starts from the upper right position ($v_x = v_y = 282$ m s^{-1}) and progresses toward the left bottom corner. Most of the trajectory in this velocity space is a linear segment, showing that the damping factors for the two components are almost constant and nearly equal. The last part of the plot, however, exhibits a different behaviour:

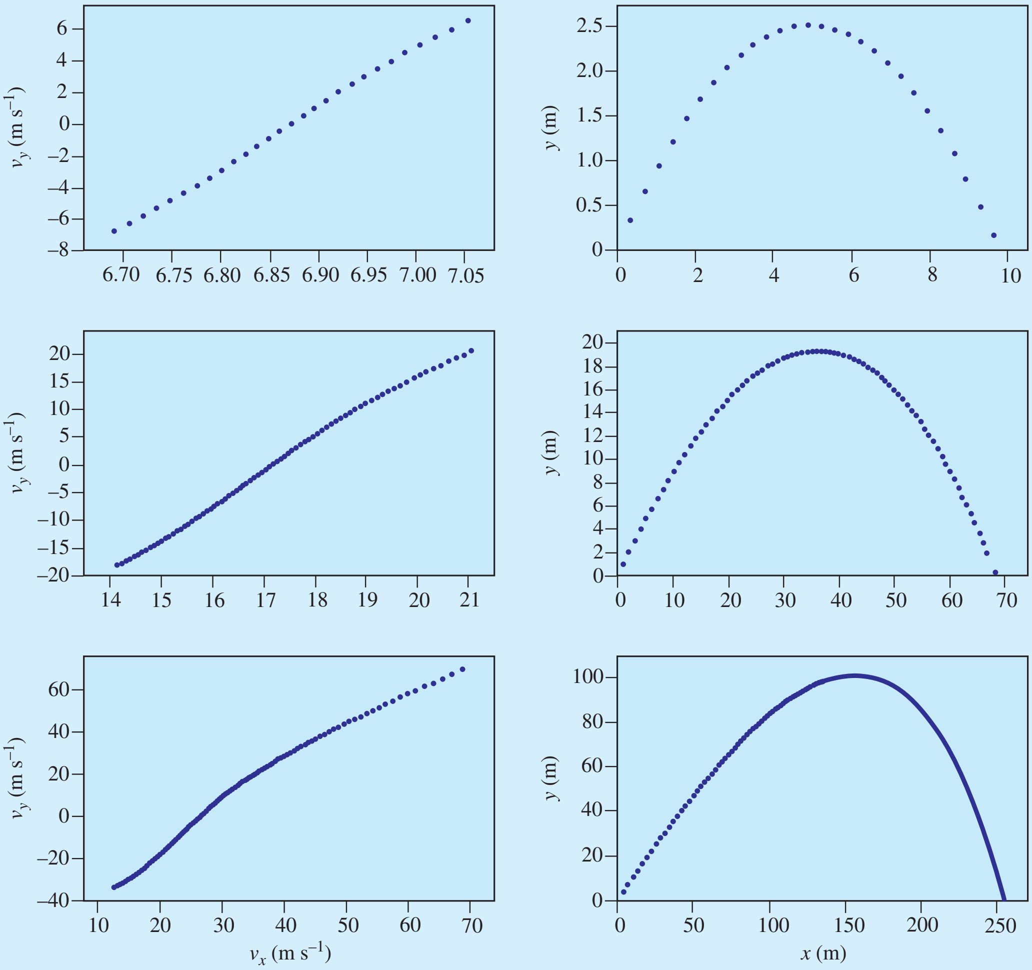

Figure 4. Trajectories (right) and velocity (left) plots obtained for different values of the initial speed (top: v = 10 m s^{-1}, middle: v = 30 m s^{-1}, bottom: v = 100 m s^{-1}). The body was assumed to be launched from the origin (0, 0) at 45° with respect to the horizontal, with a coefficient c/m = 0.005 m^{-1}.

the horizontal component is nearly damped to zero, while the vertical component decreases by the combined effect of the gravity and of the drag force, then inverts and increases its absolute value until it reaches a negative limiting value at the impact on the ground.

With realistic values of the drag coefficient in air, the terminal speed is of the order of 50 m s^{-1}. The measure of the terminal speed (for instance, of a small metal sphere) is an experiment frequently carried out in introductory laboratory courses, to obtain the viscosity of a fluid.

The combined values of the drag coefficient and of the initial speed determine how large the discrepancy is between the ideal (parabolic) trajectory and the real one. Figure 4 shows a set of space (right) and velocity (left) trajectories, obtained for increasing values of the initial speed, with the same fixed value of the coefficient c/m = 0.005 m^{-1}. For large values of the speed, the different damping effect in the two components is more clearly seen, and the trajectory resembles more and more that depicted in the illustration of figure 1.

The effect of a drag force in the modification of the ideal parabolic trajectory may be observed, although with some difficulty, in other phenomena of everyday life. For instance, when using a pipe to water the garden, the water flow from the pipe is only slightly curved at the beginning. The decrease

Figure 5. A water flow from a pipe in a garden exhibits an approximate parabolic shape. However, some deviation from the ideal trajectory may be observed, due to the drag force.

in the speed, resulting from the drag force, may slightly alter the last part of the trajectory. For instance, if the velocity at the exit of the hose has only a horizontal component (figure 5), the trajectory exhibits a nearly horizontal segment, followed by a curved part, which may be slightly different from the ideal parabola. The major effect of the drag force on large water jets (fountains, fire fighting jets, water jets from lakes, such as in Lake Geneva) is, however, the reduction of the travelled distance, for instance the height which a vertical jet may reach.

Several other phenomena which exhibit an approximate parabolic motion depending on the influence of drag forces may be discussed in this context. Lava fountains from volcanoes may show such an effect, with initial speeds of the order of few hundred m s^{-1}, reaching several hundred metres height. An additional example is provided by the trajectories seen in ski jumping, where a speed in excess of 30 m s^{-1} may be reached, with a travelled distance of the order of 100–200 m.

Conclusions

The present analysis allowed us to demonstrate, in a simple way, that the description of falling bodies provided by old historical pictures or even perceived nowadays by direct observation may correctly be reproduced by the physical laws. The influence of drag forces on the projectile motion may considerably alter its trajectory with respect to the parabolic shape expected in the absence of any force due to the medium. When looking at the motion of falling bodies, such as cannonballs, without the help of photographic tools which allow one to register and analyse the motion, people were further driven to describe and interpret the motion they expected to see. The parabolic motion is thus a concept, which comes at the end of the challenging work of interpretation and simplifying hypotheses, followed by experimental observations under controlled conditions.

Numerical studies of the projectile motion with various degrees of drag coefficients and different initial speeds may help to demonstrate the physics subtended by such phenomenon and extend the description to additional examples. The use of combined plots to show the resulting trajectory not only in the spatial coordinates but also in the velocity coordinates (or additional variables as well, such as the kinetic and potential energies) offers additional scope for the teaching of this topic.

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