A NEW CLASS OF FUSION RULES BASED ON T-CONORM AND T-NORM FUZZY OPERATORS

Albena TCHAMOVA, Jean DEZERT, and Florentin SMARANDACHE

Abstract: A new combination rule based on specified fuzzy T-Conorm/T-Norm operators is proposed and analyzed in this article - the TCN Rule of Combination. The rule does not belong to the general Weighted Operator Class. The advantages of the new rule could be defined as: very easy to implement, satisfying the impact of neutrality of Vacuous Belief Assignment, commutative, convergent to idempotence, reflecting majority opinion, and assuring an adequate data processing in case of total conflict. Several numerical examples and comparisons with the new advanced Proportional Conflict Redistribution Rules proposed recently by Florentin Smarandache and Jean Dezert within their theory of plausible and paradoxical reasoning are presented.

Keywords: Information Fusion, Combination Rules, Conjunctive Rule, Fuzzy Operators, Dezert-Smarandache Theory (DSmT), Proportional Conflict Redistribution Rules.

Introduction

There are many combination rules available for information fusion. However, none of them could satisfy the whole range of requirements associated with all possible applications. The main requirements the combination rules have to meet in temporal multiple target tracking relate especially to the way of adequate conflict processing/redistribution, ease of implementation, satisfaction of the impact of neutrality of Vacuous Belief Assignment (VBA), reflection of majority opinion, etc. In this work, the authors propose to connect the combination rules for information fusion with particular fuzzy operators: the Conjunctive rule is replaced with fuzzy T-norm operator and the Disjunctive rule with T-conorm operator, respectively. These rules originate from the T-norm and T-conorm operators in fuzzy logic, where the AND logical operator corresponds to the conjunctive rule in information fusion and the OR logical operator corresponds to the disjunctive rule. While the logical operators deal with degrees of truth, the fusion rules deal with degrees of belief of hypotheses. In this
work, the focus will be on the T-norm based conjunctive rule only as an analog of the ordinary conjunctive rule of combination. The reason is that the conjunctive rule is especially appropriate for identification problems, restricting the set of hypotheses under consideration.

**Fuzzy Inference for Information Fusion**

The main objective of information fusion is to produce a reasonably aggregated, refined and/or completed granule of data obtained from a single or multiple sources with a subsequent reasoning process. It means that the main problem here lies in the approach to aggregate correctly these sources of information, which, in general, are imprecise, uncertain, or/and conflicting. Actually, there is no single, unique rule to deal simultaneously with such kind of information specifics. Even more, there are a huge number of possible combination rules appropriate only for particular application conditions. Smarandache proposes an unification of fusion theories and a combination of fusion rules for solving different problems.\(^2\) The most suitable model for each considered application is selected. In this article, the case with a given Shafer’s model is considered.\(^3\) Let \(\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}\) be the frame of discernment for the problem under consideration, where \(\theta_1, \theta_2, \ldots, \theta_n\) are a set of \(n\) exhaustive and exclusive hypotheses. Within the applied model, Dempster-Shafer’s Power Set is described as: \(2^\Theta = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\}\).

The basic belief assignment (bba) \(m(\cdot) : 2^\Theta \to [0, 1]\), associated with a given information granule is defined with:

\[
m(\emptyset) = 0; \quad \sum_{X \in 2^n} m(X) = 1.
\]

Having given two basic belief assignments \(m_1(\cdot)\) and \(m_2(\cdot)\) and Shafer’s model, Dempster’s rule of combination \(^4\) appears to be the most frequently used combination rule. It is defined as:

\[
m_{12}(X) = \frac{\sum_{x_i \cap x_j \subseteq \emptyset} m_1(x_i) \cdot m_2(x_j)}{1 - \sum_{x_i \cap x_j \subseteq \emptyset} m_1(x_i) \cdot m_2(x_j)}
\]

The term \(k = \sum_{x_i \cap x_j \subseteq \emptyset} m_1(x_i) \cdot m_2(x_j)\) describes the degree of conflict between the sources of information. The normalization step (i.e., the division by \((1 - k)\)) in Dempster’s rule is definitely the most sensitive and weak point of the rule due to the fact that the fused result becomes a proper information granule only in the cases when \(k < 1\). The new advanced Proportional Conflict Redistribution rules proposed recently by Smarandache and Dezert,\(^5\) which are particular cases of the Weighted Operator, overcome successfully the main limitations of Dempster’s rule.

In this work, the authors’ objective is to propose a new, alternative combination rule,
interpreting the fusion in terms of fuzzy operators, a rule that avoids the Dempster’s rule limitation, possesses an adequate behavior in cases of total conflict, and has an easy implementation.

**Fusion Interpretation**

It is assumed that the relation between the two basic belief assignments (the information granules) \( m_1(\cdot) \) and \( m_2(\cdot) \) is considered a vague relation, characterized by the following two features:

- **The way of association** between the focal elements included in the basic belief assignments of the sources of information. It is a particular operation chosen among the operations union and intersection, respectively. These set operations correspond to the logic operations Conjunction and Disjunction.

- **The degree of association (interaction)** between the focal elements included in the basic belief assignments of the sources of information. It is obtained as a T-norm (for Conjunction) or T-conorm (for Disjunction) operators applied over the probability masses of the corresponding focal elements. There are multiple choices available to define T-norm and T-conorm operators.

In this article, as already mentioned, the authors will focus only on the T-norm based Conjunctive rule, more precisely the Minimum T-norm based Conjunctive rule as an analog of the ordinary conjunctive rule of combination. It will be demonstrated that it could give results very similar to the conjunctive rule, satisfying the principle of neutrality of VBA, reflecting the majority opinion, converging towards idempotence, and having adequate behavior in the presence of a total conflict. It is commutative, simple to apply, but not associative.

**Main Features of the T-Norm Function**

The \( T \) \( = norm : [0, 1]^2 \mapsto [0, 1] \) is a function defined in fuzzy set/logic theory in order to represent the intersection between two particular fuzzy sets and the AND fuzzy logical operator, respectively. If one extends the T-norm to data fusion, it will be a substitute for the conjunctive rule. The T-norm has to satisfy the following conditions:

- Associativity: \( T_{\text{norm}}(T_{\text{norm}}(x, y), z) = T_{\text{norm}}(x, T_{\text{norm}}(y, z)) \);

- Commutativity: \( T_{\text{norm}}(x, y) = T_{\text{norm}}(y, x) \);

- Monotonicity: if \( (x \leq a) \) and \( (y \leq b) \) then \( T_{\text{norm}}(x, y) \leq T_{\text{norm}}(a, b) \);

- Boundary Conditions: \( T_{\text{norm}}(0, 0) = 0; T_{\text{norm}}(x, 1) = x \).
**Functions, Satisfying the T-Norm Conditions**

There are many functions that satisfy the T-norm conditions:

- Zadeh’s (default) min operator:\[ m(X) = \min \{m_1(X_i), m_2(X_j)\}; \]
- Algebraic product operator: \[ m(X) = m_1(X_i) \cdot m_2(X_j); \]
- Bounded product operator: \[ m(X) = \max \{[m_1(X_i) + m_2(X_j)], 0\}. \]

A desirable characteristic is the chosen T-norm operator to satisfy the neutrality of VBA. From the functions described above, the default (min) and the algebraic product operators satisfy this condition. Considering this fact, the authors choose the default Minimum T-norm operator in order to define the degree of association between the focal elements of the information granules.

**Proof of the Vague min Set Operator**

The intersection \( X_i \cap X_j \) for crisp (ordinary) subsets of the universe \( U \) includes all elements of \( X_i \) and \( X_j \) such that:

\[
m(X) = 1, \quad \text{if } X \in X_i \text{ and } X \in X_j
\]

\[
m(X) = 0, \quad \text{if } X \notin X_i \text{ or } X \notin X_j.
\]

Let \( X_i \) and \( X_j \) are some vague subsets of \( U \). How do we define the conditions from above for the case of intersection \( X_i \cap X_j \):

- First condition \( X \in X_i \text{ and } X \in X_j \)

  It means that the following case exists: \( \{m(X \in X_i) = 1, m(X \in X_j) = 1\} \), for which: \( \min \{m(X \in X_i), m(X \in X_j)\} = 1; \)

- Second condition \( X \notin X_i \text{ or } X \notin X_j \)

  It means that one of the following cases exists:
  \[
  \begin{align*}
  \{m(X \in X_i) = 0, m(X \in X_j) = 0\} \text{ or } \\
  \{m(X \in X_i) = 1, m(X \in X_j) = 0\} \text{ or } \\
  \{m(X \in X_i) = 0, m(X \in X_j) = 1\}
  \end{align*}
  \]
  for which: \( \min \{m(X \in X_i), m(X \in X_j)\} = 0. \)

From these expressions it follows that \( \min \{m(X \in X_i), m(X \in X_j)\} \) provides the correct expression for intersection.
The T-conorm/T-norm (TCN) Combination Rule

Let us take a look at a general form of a fusion table, where the T-norm based interpretation of the ordinary conjunctive rule of combination for two given sources is considered (see Table 1). The frame of the fusion problem under consideration is $\Theta = \{\theta_1, \theta_2\}$ and the power set is: $\mathcal{P} = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$. The two basic belief assignments (sources of information) $m_1(.)$ and $m_2(.)$ are defined over $\mathcal{P}$. It is assumed that $m_1(.)$ and $m_2(.)$ are normalized bbs ($m(\emptyset) = 0$; $\sum_{X \in \mathcal{P}} m(X) = 1$).

**Step 1: Defining the min T-norm conjunctive consensus**

The min T-norm conjunctive consensus is based on the default min T-norm function. The way of association between the focal elements of the given two sources of information is defined as $X = X_i \cap X_j$, and the degree of association is as follows:

$$\tilde{m}(X) = \min \{m_1(X_i), m_2(X_j)\},$$

where $\tilde{m}(X)$ represents the mass of belief associated with the given proposition $X$ by using T-Norm based conjunctive rule.

| $m_1 | X_i$ | $m_2 | X_j$ | $m_1 | X_i \cap X_j$ | $m_2 | X_i \cap X_j$ | $m_1 | X_i \cup X_j$ | $m_2 | X_i \cup X_j$ |
|---|---|---|---|---|---|---|
| $m_1 | X_i$ | $X_i \cap X_j$ | $m_1 | X_i \cap X_j$ | $m_1 | X_i \cap X_j$ | $m_1 | X_i \cup X_j$ | $m_1 | X_i \cup X_j$ |
| $m_2 | X_j$ | $X_i \cap X_j$ | $m_2 | X_i \cap X_j$ | $m_2 | X_i \cap X_j$ | $m_2 | X_i \cup X_j$ | $m_2 | X_i \cup X_j$ |
| $m_1 | X_i \cup X_j$ | $X_i \cap X_j$ | $m_1 | X_i \cap X_j$ | $m_1 | X_i \cap X_j$ | $m_1 | X_i \cup X_j$ | $m_1 | X_i \cup X_j$ |
| $m_2 | X_i \cup X_j$ | $X_i \cap X_j$ | $m_2 | X_i \cap X_j$ | $m_2 | X_i \cap X_j$ | $m_2 | X_i \cup X_j$ | $m_2 | X_i \cup X_j$ |

The proposed T-conorm/T-norm based Combination rule, called by the authors TCN rule of combination, is defined in the framework of Dempster-Shafer Theory for $\forall X \in \mathcal{P}$ by the equation:

$$\tilde{m}(X) = \sum_{X_i \cap X_j \neq \emptyset} \min \{m_1(X_i), m_2(X_j)\}.$$  (1)

**Step 2: Distribution of the mass, assigned to the conflict**

The distribution of the mass assigned to the conflict follows to a certain degree the distribution of the conflicting mass in the DSmT Proportional Conflict Redistribution Rule 2, but the procedure here is based on fuzzy operators. Let us denote the two bbs
associated with the information sources in a matrix form as follows:

\[
\begin{bmatrix}
  m_{1}(.) \\
  m_{2}(.)
\end{bmatrix}
= \begin{bmatrix}
  m_{1}(\theta_{1}) & m_{1}(\theta_{2}) & m_{1}(\theta_{1} \cup \theta_{2}) \\
  m_{2}(\theta_{1}) & m_{2}(\theta_{2}) & m_{2}(\theta_{1} \cup \theta_{2})
\end{bmatrix}.
\]

The total conflicting mass is distributed proportionally among all non-empty sets with respect to the maximum (denoted here as \(x_{12}(X)\)) of the elements of the corresponding mass matrix’s columns, associated with element \(X\) of the power set. It means that the bigger mass is redistributed towards the element, involved in the conflict and contributing to the conflict, with the maximum specified probability mass. The fuzzy operator maximum is used to interpret the summation of the corresponding mass matrix’s columns associated with element \(X\) of the power set, as used in the DSmT Proportional Conflict Redistribution Rules.

\[
x_{12}(\theta_{1}) = \max(m_{1}(\theta_{1}), m_{2}(\theta_{1})) \\
x_{12}(\theta_{2}) = \max(m_{1}(\theta_{2}), m_{2}(\theta_{2}))
\]

One denotes by \(r(\theta_{1})\) and \(r(\theta_{2})\) the part of the conflicting mass distributed to the propositions \(\theta_{1}\) and \(\theta_{2}\). Then, one gets:

\[
\frac{r(\theta_{1})}{x_{12}(\theta_{1})} = \frac{r(\theta_{2})}{x_{12}(\theta_{2})} = \frac{r(\theta_{1}) + r(\theta_{2})}{x_{12}(\theta_{1}) + x_{12}(\theta_{2})} = \frac{k_{12}}{s_{12}}.
\]

In turn, the conflicting masses that have to be redistributed are:

\[
r(\theta_{1}) = x_{12}(\theta_{1}) \cdot \frac{k_{12}}{s_{12}}; \quad r(\theta_{2}) = x_{12}(\theta_{2}) \cdot \frac{k_{12}}{s_{12}}.
\]

Finally, the bba obtained as a result of the applied TCN rule with fuzzy-based Proportional Conflict Redistribution Rule 2, denoted here as \(\tilde{m}^{PC\cdot R2}(.)\), becomes:

\[
\tilde{m}^{PC\cdot R2}(\theta_{1}) = \tilde{m}(\theta_{1}) + x_{12}(\theta_{1}) \cdot \frac{k_{12}}{s_{12}} \\
\tilde{m}^{PC\cdot R2}(\theta_{2}) = \tilde{m}(\theta_{2}) + x_{12}(\theta_{2}) \cdot \frac{k_{12}}{s_{12}} \\
\tilde{m}^{PC\cdot R2}(\theta_{1} \cup \theta_{2}) = \tilde{m}(\theta_{1} \cup \theta_{2}),
\]

where \(k_{12}\) is the total conflict; \(x_{12}(X) = \max_{\theta_{1}, \theta_{2}}(m_{i}(X)) \neq 0\) and \(s_{12}\) is the sum of all non-zero maximum values of column’s masses assigned to non-empty sets. The conflict mass is redistributed only among the propositions involved in the conflict.

**Step 3: Normalization of the result**

The final step of the TCN rule concerns the normalization procedure:

\[
\bar{m}^{PC\cdot R2}(X) = \frac{\sum_{X \in \wp} \tilde{m}^{PC\cdot R2}(X)}{\sum_{X \in \wp} \tilde{m}^{PC\cdot R2}(X)}.
\]
Implementation of the TCN Combination Rule

Example 1

Assume problem frame $\theta = \{\theta_1, \theta_2\}$ and two independent sources of information with basic belief assignments as follows:

$$m_1(\theta_1) = 0.6 \quad m_1(\theta_2) = 0.2 \quad m_1(\theta_1 \cup \theta_2) = 0.2$$
$$m_2(\theta_1) = 0.4 \quad m_2(\theta_2) = 0.5 \quad m_2(\theta_1 \cup \theta_2) = 0.1$$

Applying the min T-norm based conjunctive consensus yields the results given in Table 2.

Table 2: Min T-norm based Interpretation of Conjunctive Rule.

<table>
<thead>
<tr>
<th>$m_1(\theta_1) = 0.6$</th>
<th>$m_1(\theta_2) = 0.2$</th>
<th>$m_1(\theta_1 \cup \theta_2) = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{m}(\theta_1) = \text{min}(0.6, 0.4) = 0.4$</td>
<td>$\tilde{m}(\theta_1 \cap \theta_2) = \text{min}(0.6, 0.5) = 0.5$</td>
<td>$\tilde{m}(\theta_1) = \text{min}(0.6, 0.1) = 0.1$</td>
</tr>
<tr>
<td>$m_1(\theta_2) = 0.2$</td>
<td>$m_2(\theta_2) = 0.5$</td>
<td>$m_2(\theta_1 \cup \theta_2) = 0.1$</td>
</tr>
<tr>
<td>$\tilde{m}(\theta_1 \cap \theta_2) = \text{min}(0.2, 0.4) = 0.2$</td>
<td>$\tilde{m}(\theta_1) = \text{min}(0.2, 0.5) = 0.2$</td>
<td>$\tilde{m}(\theta_1 \cup \theta_2) = \text{min}(0.2, 0.1) = 0.1$</td>
</tr>
<tr>
<td>$m_1(\theta_1 \cup \theta_2) = 0.2$</td>
<td>$m_2(\theta_1 \cup \theta_2) = 0.1$</td>
<td>$m_1(\theta_1 \cup \theta_2) = 0.2$</td>
</tr>
<tr>
<td>$\tilde{m}(\theta_1) = \text{min}(0.2, 0.4) = 0.2$</td>
<td>$\tilde{m}(\theta_2) = \text{min}(0.2, 0.5) = 0.2$</td>
<td>$\tilde{m}(\theta_1 \cup \theta_2) = \text{min}(0.2, 0.1) = 0.1$</td>
</tr>
</tbody>
</table>

Fusion with TCN Rule of Combination

Step 1: Obtaining min T-norm Conjunctive Consensus

Using Table 2 and applying Equation 1, the fusion result becomes:

$$\tilde{m}(\theta_1) = 0.4 + 0.2 + 0.1 = 0.7$$
$$\tilde{m}(\theta_2) = 0.2 + 0.1 + 0.2 = 0.5$$
$$\tilde{m}(\theta_1 \cap \theta_2) = 0.5 + 0.2 = 0.7$$
$$\tilde{m}(\theta_1 \cup \theta_2) = 0.1$$

Step 2: Redistribution of the conflict by using fuzzy-based PCR2

$$\frac{r(\theta_1)}{\max(m_1(\theta_1), m_2(\theta_1))} = \frac{r(\theta_2)}{\max(m_1(\theta_2), m_2(\theta_2))} =$$

$$\frac{r(\theta_1)}{\max(0.6, 0.4)} = \frac{r(\theta_2)}{\max(0.2, 0.5)} =$$

$$\frac{r(\theta_1) + r(\theta_2)}{\max(0.6, 0.4) + \max(0.2, 0.5)} = \frac{\tilde{m}(\theta_1 \cap \theta_2)}{0.6 + 0.5} = \frac{0.7}{1.1} = 0.636$$
\[ r(\theta_1) = 0.6 \cdot 0.636 = 0.3816; \quad r(\theta_2) = 0.5 \cdot 0.636 = 0.318 \]

Then, after conflict redistribution, the new masses become:
\[
\tilde{m}_{PCR2}(\cdot) = \{ \tilde{m}_{PCR2}(\theta_1) = 0.7 + 0.3816 = 1.0816; \tilde{m}_{PCR2}(\theta_2) = 0.5 + 0.318 = 0.818, \tilde{m}_{PCR2}(\theta_1 \cup \theta_2) = 0.1 \}.
\]

**Step 3: Normalization of the result**

After the application of the normalization procedure, the final information granule is obtained as follows:
\[
\tilde{m}_{PCR2}(\cdot) = \{ \tilde{m}_{PCR2}(\theta_1) = 0.54, \tilde{m}_{PCR2}(\theta_2) = 0.41, \tilde{m}_{PCR2}(\theta_1 \cup \theta_2) = 0.05 \}.
\]

**Fusion with Ordinary Conjunctive Rule**

The conjunctive consensus here is given by:
\[
m(\theta_1) = 0.38, \quad m(\theta_2) = 0.22, \quad m(\theta_1 \cap \theta_2) = k = 0.38 \quad m(\theta_1 \cup \theta_2) = 0.02
\]

The PCR2 rule\(^9\) is used to redistribute the resulting conflict:
\[
\frac{x}{0.4 + 0.6} = \frac{y}{0.5 + 0.2} = \frac{x + y}{1.7} = \frac{0.38}{1.7} = 0.224
\]

Then, the final masses of belief become:
\[
m_{PCR2}(\theta_1) = 0.38 + 1.0 \cdot 0.224 = 0.604
\]
\[
m_{PCR2}(\theta_2) = 0.22 + 0.7 \cdot 0.224 = 0.376
\]
\[
m_{PCR2}(\theta_1 \cup \theta_2) = 0.02
\]

**Table 3: Comparative Results.**

<table>
<thead>
<tr>
<th>Ordinary Conjunctive Rule with PCR2</th>
<th>TCN Rule with fuzzy based PCR2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_{PCR2}(\theta_1) = 0.604)</td>
<td>(\tilde{m}_{PCR2}(\theta_1) = 0.54)</td>
</tr>
<tr>
<td>(m_{PCR2}(\theta_2) = 0.376)</td>
<td>(\tilde{m}_{PCR2}(\theta_1) = 0.41)</td>
</tr>
<tr>
<td>(m_{PCR2}(\theta_1 \cup \theta_2) = 0.02)</td>
<td>(\tilde{m}_{PCR2}(\theta_1 \cup \theta_2) = 0.05)</td>
</tr>
</tbody>
</table>
Then, the normalization results in:

After considering:

Here, the min T-norm based conjunctive consensus yields the following result:

The partial conflicting masses will be redistributed to corresponding non-empty sets contributing to the particular partial conflicts by using fuzzy-based PCR3. According to $m(\{\theta_1 \cap \theta_2\}) = 0.99$:

Considering $m(\{\theta_1 \cap \theta_3\}) = 0.01$:

Considering $m(\{\theta_2 \cap \theta_3\}) = 0.01$:

After conflict redistribution using a fuzzy-based PCR3, the following result is obtained:

Then, the normalization results in:

Table 3 lists comparative results obtained using the ordinary conjunctive rule with PCR2 redistribution of conflicting mass and TCN rule with fuzzy-based PCR2.

**Zadeh’s Example**

Let us have $\theta = \{\theta_1, \theta_2, \theta_3\}$ and two independent sources of information with the corresponding bbas:

$$m_1(\theta_1) = 0.99 \quad m_1(\theta_2) = 0.0 \quad m_1(\theta_3) = 0.01$$

$$m_2(\theta_1) = 0.0 \quad m_2(\theta_2) = 0.99 \quad m_2(\theta_3) = 0.01$$

**Fusion with TCN Rule of Combination**

The partial conflicting masses will be redistributed to corresponding non-empty sets contributing to the particular partial conflicts by using fuzzy-based PCR3. According to $m(\theta_1 \cap \theta_2) = 0.99$:

Considering $m(\theta_1 \cap \theta_3) = 0.01$:

Considering $m(\theta_2 \cap \theta_3) = 0.01$:

After conflict redistribution using a fuzzy-based PCR3, the following result is obtained:

Then, the normalization results in:

$\tilde{m}_{PCR3}(\theta_1) = 0.945, \quad \tilde{m}_{PCR3}(\theta_2) = 0.945, \quad \tilde{m}_{PCR3}(\theta_3) = 0.01$.
Fusion with Ordinary Conjunctive Rule

The conjunctive consensus is given by:

\[ m(\theta_3) = 0.0001; \quad m(\theta_1 \cap \theta_2) = 0.98; \quad m(\theta_1 \cap \theta_3) = 0.0099; \quad m(\theta_2 \cap \theta_3) = 0.0099 \]

Applying the PCR3 rule to the partial conflicting masses, one gets as follows.

According to \( m(\theta_1 \cap \theta_2) = 0.98 \):

\[
\frac{x_1}{0.99 + 0.0} = \frac{y_1}{0.99 + 0.0} = \frac{x_1 + y_1}{1.98} = \frac{0.98}{1.98} = 0.495
\]

Considering \( m(\theta_1 \cap \theta_3) = 0.0099 \):

\[
\frac{x_2}{0.99} = \frac{z_1}{0.02} = \frac{x_2 + z_1}{1.01} = \frac{0.0099}{1.01} = 0.0098
\]

Considering \( m(\theta_2 \cap \theta_3) = 0.0099 \):

\[
\frac{y_2}{0.99} = \frac{z_2}{0.02} = \frac{y_2 + z_2}{1.01} = \frac{0.0099}{1.01} = 0.0098
\]

Finally, the result is given by:

\[
m_{PCR3}(\theta_1) = 0 + (0.99 \cdot 0.495) + (0.99 \cdot 0.0098) = 0.49975
\]

\[
m_{PCR3}(\theta_2) = 0 + (0.99 \cdot 0.495) + (0.99 \cdot 0.0098) = 0.49975
\]

\[
m_{PCR3}(\theta_3) = 0.0001 + (0.02 \cdot 0.0098) + (0.02 \cdot 0.0098) = 0.0005
\]

Table 4 lists the results of comparison.

<table>
<thead>
<tr>
<th>Ordinary Conjunctive Rule with PCR3</th>
<th>TCN Rule with fuzzy based PCR3</th>
</tr>
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<tr>
<td>( m_{PCR3}(\theta_1) = 0.49975 )</td>
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</tr>
<tr>
<td>( m_{PCR3}(\theta_2) = 0.49975 )</td>
<td>( m_{PCR3}(\theta_2) = 0.495 )</td>
</tr>
<tr>
<td>( m_{PCR3}(\theta_3) = 0.0005 )</td>
<td>( m_{PCR3}(\theta_3) = 0.01 )</td>
</tr>
</tbody>
</table>
Total Conflict Example

Let us consider a case with a problem frame $\theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and two independent sources of information:

\[
\begin{align*}
& m_1(\theta_1) = 0.3 \quad m_1(\theta_2) = 0.0 \quad m_1(\theta_3) = 0.7 \quad m_1(\theta_4) = 0.0 \\
& m_2(\theta_1) = 0.0 \quad m_2(\theta_2) = 0.4 \quad m_2(\theta_3) = 0.0 \quad m_2(\theta_4) = 0.6
\end{align*}
\]

Fusion with TCN Rule of Combination

In this case, the min T-norm conjunctive consensus yields the following result:

\[
\begin{align*}
& m_1(\theta_1 \cap \theta_2) = 0.3; \quad m_1(\theta_1 \cap \theta_4) = 0.3; \quad m_1(\theta_2 \cap \theta_3) = 0.4; \quad m_1(\theta_3 \cap \theta_4) = 0.6
\end{align*}
\]

Here one obtains the partial conflicting masses that will be redistributed using fuzzy-based PCR3.

According to the partial conflict $m_1(\theta_1 \cap \theta_2) = 0.3$:

\[
\frac{x_1}{\max(0, 0.3)} = \frac{x_1 + y_1}{\max(0, 0.4)} = \frac{0.3}{0.7} = 0.4285
\]

According to the partial conflict $m_1(\theta_1 \cap \theta_4) = 0.3$:

\[
\frac{x_2}{\max(0, 0.3)} = \frac{x_2 + h_1}{\max(0, 0.6)} = \frac{0.3}{0.9} = 0.3333
\]

According to the partial conflict $m_1(\theta_2 \cap \theta_3) = 0.4$:

\[
\frac{y_2}{\max(0, 0.4)} = \frac{y_2 + z_1}{\max(0, 0.7)} = \frac{0.4}{1.1} = 0.3636
\]

According to the partial conflict $m_1(\theta_3 \cap \theta_4) = 0.6$:

\[
\frac{x_2}{\max(0, 0.7)} = \frac{x_2 + h_2}{\max(0, 0.6)} = \frac{0.6}{1.3} = 0.4615
\]

After conflict redistribution, the result is given by:

\[
\begin{align*}
& m_{PCR3}(\theta_1) = 0.2275; \quad m_{PCR3}(\theta_2) = 0.3168; \\
& m_{PCR3}(\theta_3) = 0.5775; \quad m_{PCR3}(\theta_4) = 0.4768.
\end{align*}
\]

And finally, the normalization procedure yields the following result:

\[
\begin{align*}
& m_{PCR3}(\theta_1) = 0.1423, \quad m_{PCR3}(\theta_2) = 0.1982, \\
& m_{PCR3}(\theta_3) = 0.3612, \quad m_{PCR3}(\theta_4) = 0.2983.
\end{align*}
\]
Fusion with Ordinary Conjunctive Rule

The conjunctive consensus is given by:

\[ m(\theta_1 \cap \theta_2) = 0.12; \quad m(\theta_1 \cap \theta_4) = 0.18; \quad m(\theta_2 \cap \theta_3) = 0.28; \quad m(\theta_3 \cap \theta_4) = 0.42 \]

After applying the PCR3 rule to the partial conflicting masses one finally gets:

\[ m_{PCR3}(\theta_1) = 0.111; \quad m_{PCR3}(\theta_2) = 0.171; \]
\[ m_{PCR3}(\theta_3) = 0.404; \quad m_{PCR3}(\theta_4) = 0.314. \]

The comparative results are given in Table 5.

<table>
<thead>
<tr>
<th>Ordinary Conjunctive Rule with PCR3</th>
<th>TCN Rule with fuzzy based PCR3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_{PCR3}(\theta_1) = 0.111 )</td>
<td>( \tilde{m}_{PCR}(\theta_1) = 0.1423 )</td>
</tr>
<tr>
<td>( m_{PCR3}(\theta_2) = 0.171 )</td>
<td>( \tilde{m}_{PCR}(\theta_2) = 0.1982 )</td>
</tr>
<tr>
<td>( m_{PCR3}(\theta_3) = 0.404 )</td>
<td>( \tilde{m}_{PCR}(\theta_3) = 0.2612 )</td>
</tr>
<tr>
<td>( m_{PCR3}(\theta_4) = 0.314 )</td>
<td>( \tilde{m}_{PCR}(\theta_4) = 0.2983 )</td>
</tr>
</tbody>
</table>

**Example 5 (Convergence to Idempotence)**

Let us consider a case with a problem frame \( \theta = \{ \theta_1, \theta_2 \} \) and two independent sources of information:

\[
\begin{align*}
& m_1(\cdot) = \{ m_1(\theta_1) = 0.7; \quad m_1(\theta_2) = 0.3 \} \\
& m_2(\cdot) = \{ m_2(\theta_1) = 0.7; \quad m_2(\theta_2) = 0.3 \}
\end{align*}
\]

**Fusion with TCN Rule of Combination**

Here the min T-norm conjunctive consensus yields the following result:

\[ \tilde{m}(\cdot) = \{ \tilde{m}(\theta_1) = 0.7; \quad \tilde{m}(\theta_2) = 0.3; \quad \tilde{m}(\theta_1 \cap \theta_2) = 0.6 \}. \]

After conflict redistribution using fuzzy-based PCR2 one gets:

\[
\frac{x}{\max(0.7, 0.7)} = \frac{y}{\max(0.3, 0.3)} = \frac{0.6}{1.0} = 0.6
\]
\[ \tilde{m}_{PCR2}(\theta_1) = 1.12; \quad \tilde{m}_{PCR2}(\theta_2) = 0.48 \]

After normalization the final fused result becomes:

\[ \tilde{m}_{PCR2}(.) = \{ \tilde{m}_{PCR2}(\theta_1) = 0.7; \quad \tilde{m}_{PCR2}(\theta_2) = 0.3 \}. \]

**Fusion with Ordinary Conjunctive Rule**

The conjunctive consensus is given by:

\[ m(\theta_1) = 0.49; \quad m(\theta_2) = 0.09; \quad m(\theta_1 \cap \theta_2) = 0.42 \]

Finally, the vector of belief masses after applying the PCR2 rule to the partial conflicting mass becomes:

\[ m_{PCR2}(\theta_1) = 0.784; \quad m_{PCR2}(\theta_2) = 0.216 \]

The comparative results are given in Table 6.

<table>
<thead>
<tr>
<th>Ordinary Conjunctive Rule with PCR2</th>
<th>TCN Rule with fuzzy based PCR2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_{PCR2}(\theta_1) = 0.784 )</td>
<td>( \tilde{m}_{PCR2}(\theta_1) = 0.7 )</td>
</tr>
<tr>
<td>( m_{PCR2}(\theta_2) = 0.216 )</td>
<td>( \tilde{m}_{PCR2}(\theta_2) = 0.3 )</td>
</tr>
</tbody>
</table>

It is obvious that the fusion results obtained using the TCN rule of combination converge strongly towards idempotence.

**Example 6 (Majority Opinion)**

Let us consider a case with a problem frame \( \theta = \{ \theta_1, \theta_2 \} \) and two independent sources of information:

\[ m_1(.) = \{ m_1(\theta_1) = 0.8; \quad m_1(\theta_2) = 0.2 \} \]
\[ m_2(.) = \{ m_2(\theta_1) = 0.3; \quad m_2(\theta_2) = 0.7 \} \]

Assume that in the next time moment a third source of information is introduced with the following bba:

\[ m_3(.) = \{ m_3(\theta_1) = 0.3; \quad m_3(\theta_2) = 0.7 \}. \]
Fusion with TCN Rule of Combination

The TCN rule with fuzzy-based PCR2 yields the following normalized fusion result:

\[ \tilde{m}_{12,PCR2}(\theta_1) = 0.557; \quad \tilde{m}_{12,PCR2}(\theta_2) = 0.443 \]

Let us now combine \( \tilde{m}_{12,PCR2}(\cdot) \) with the gbba of the third source \( m_3(\cdot) \). Then the final fused result is obtained as:

\[ \tilde{m}_{12,3,PCR2}(\theta_1) = 0.417; \quad \tilde{m}_{12,3,PCR2}(\theta_2) = 0.583 \]

It is evident from this result that the final bba \( \tilde{m}_{12,3,PCR2}(\cdot) = [0.417 \quad 0.583] \) starts to reflect the majority opinion; it means that \( \tilde{m}_{12,3,PCR2}(\theta_1) < \tilde{m}_{12,3,PCR2}(\theta_2) \). If a fourth source is considered with a probability mass vector supporting the majority opinion, i.e. \( m_4(\cdot) = \{m_4(\theta_1) = 0.3; \quad m_4(\theta_2) = 0.7\} \), then the final probability mass vector becomes:

\[ \tilde{m}_{(12,3),4,PCR2}(\theta_1) = 0.348; \quad \tilde{m}_{(12,3),4,PCR2}(\theta_2) = 0.652 \]

The new fused vector \( \tilde{m}_{(12,3),4,PCR2}(\cdot) = [0.348 \quad 0.652] \) reflects again the majority opinion since \( \tilde{m}_{(12,3),4,PCR2}(\theta_1) \) decreases more and more and, at the same time, \( \tilde{m}_{(12,3),4,PCR2}(\theta_2) \) increases in the same manner.

Fusion with Ordinary Conjunctive Rule

The conjunctive consensus between sources 1 and 2 is given by:

\[ m_{12}(\cdot) = \{m_{12}(\theta_1) = 0.24; \quad m_{12}(\theta_2) = 0.14; \quad m_{12}(\theta_1 \cap \theta_2) = 0.62\} \]

After applying the PCR2 rule to the partial conflicting mass \( m_{12}(\theta_1 \cap \theta_2) = 0.62 \), the final probability mass vector becomes:

\[ m_{12,PCR2}(\theta_1) = 0.58; \quad m_{12,PCR2}(\theta_2) = 0.42 \]

Let us now combine \( m_{12,PCR2}(\cdot) \) with the bba of the third source \( m_3(\cdot) \).

Then, after applying PCR2 to the obtained conjunctive consensus, the final probability mass vector becomes:

\[ m_{12,3,PCR2}(\theta_1) = 0.408; \quad m_{12,3,PCR2}(\theta_2) = 0.592 \]

It is evident from this result that the final bba \( m_{12,3,PCR2}(\cdot) = [0.408 \quad 0.592] \) starts to reflect the majority opinion; it means that \( m_{12,3,PCR2}(\theta_1) < m_{12,3,PCR2}(\theta_2) \). If a fourth source is considered with a probability mass vector supporting the majority opinion, i.e. \( m_4(\cdot) = \{m_4(\theta_1) = 0.3; \quad m_4(\theta_2) = 0.7\} \), the final probability mass vector becomes:

\[ m_{(12,3),4,PCR2}(\theta_1) = 0.286; \quad m_{(12,3),4,PCR2}(\theta_2) = 0.714 \]
The new fused vector $m_{(12,3)}^{APCR2}() = [0.286 \quad 0.714]$ reflects the majority opinion since $m_{(12,3)}^{APCR2}(\theta_1)$ decreases more and more and, at the same time, $m_{(12,3)}^{APCR2}(\theta_2)$ increases in the same manner.

The comparative results are given in Table 7.

<table>
<thead>
<tr>
<th></th>
<th>Ordinary Conjunctive Rule with PCR2</th>
<th>TCN Rule with fuzzy based PCR2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{(12,3)}^{APCR2}(\theta_1)$</td>
<td>0.286</td>
<td>$\tilde{m}_{(12,3)}^{APCR2}(\theta_1)$ = 0.348</td>
</tr>
<tr>
<td>$m_{(12,3)}^{APCR2}(\theta_2)$</td>
<td>0.714</td>
<td>$\tilde{m}_{(12,3)}^{APCR2}(\theta_2)$ = 0.652</td>
</tr>
</tbody>
</table>

The new TCN combination rule with fuzzy-based PCR2 reflects the majority opinion slower than the PCR2.

**Example 7 (Neutrality of VBA)**

Let us consider a case with a problem frame $\theta = \{\theta_1, \theta_2\}$ and two independent sources of information:

$m_1(.) = m_1(\theta_1) = 0.4; \quad m_1(\theta_2) = 0.5 \quad m_1(\theta_1 \cup \theta_2) = 0.1$
$m_2(.) = m_2(\theta_1) = 0.0; \quad m_2(\theta_2) = 0.0 \quad m_2(\theta_1 \cup \theta_2) = 1.0$

The second source is characterized with vacuous gbba.

The TCN rule yields the following result:

$\tilde{m}(.) = \{\tilde{m}(\theta_1) = 0.4; \quad \tilde{m}(\theta_2) = 0.5; \quad \tilde{m}(\theta_1 \cup \theta_2) = 0.1\}$.

From the obtained result it is evident that TCN rule satisfies the principle of neutrality of the vacuous belief assignment (VBA). The min T-norm operator will always give a result that is equal to the non-vacuous bba $m_1(.)$, because in either case the probability masses assigned to their corresponding propositions will always be lower or equal to the probability mass assigned to the full ignorance in $m_2(.) = m_2(\theta_1 \cup \theta_2) = 1.0$.

It means that according to the way of obtaining the degree of association between the focal elements in $m_1(.)$, and $m_2(.)$, $(\tilde{m}(X) = \min \{m_1(X_i), m_2(X_j)\})$, the resulting bba will become equal to the non-vacuous $m_1(.)$.

**Main Characteristics of the TCN Combination Rule**

Although the TCN rule is not associative (like most of the fusion rules except Dempster’s rule and the conjunctive rule on free-DSm model), it presents the following advantages:
The rule is simple and very easy to implement;

- It reflects the majority opinion;

- The rule is convergent toward idempotence in cases when there are no intersections and unions between the elementary hypotheses;

- It reflects the effect of neutrality of vacuous belief assignment;

- It leads to adequate solutions in case of a total conflict between the sources of information.

Conclusions

In this article, a new combination rule (the TCN combination rule) based on fuzzy T-conorm/T-norm operators is proposed and analyzed. It does not belong to the general Weighted Operator Class. It overcomes the main limitations of Dempster’s rule related to the normalization in cases of high conflict and the counter-intuitive fusion results. The advantages of the new rule could be summarized as: very easy to implement, satisfying the impact of neutral Vacuous Belief Assignment, commutative, convergent to idempotence, reflecting majority opinion, and assuring adequate data processing in case of a partial or total conflict between the information granules. It is suitable for the requirements of temporal multiple target tracking. The main drawback of this rule is related to the lack of associativity, which is not a major issue in temporal data fusion applications such as those involved in target type tracking and classification.

Acknowledgement

This work is partially supported by the Bulgarian National Science Fund- grants MI-1506/05, EC FP6 funded project - BIS21++ (FP6-2004-ACC-SSA-2).
Notes:


5. Smarandache and Dezert, “Proportional Conflict Redistribution Rules for Information Fusion.”


7. We introduce in this paper the over-tilded notation for masses to specify that the masses of belief are obtained with fuzzy T-norm and T-conorm operators.

8. Smarandache and Dezert, “Proportional Conflict Redistribution Rules for Information Fusion.”


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JEAN DEZERT see page 48

FLORENTIN SMARANDACHE see page 49