AMERICAN JOURNAL of PHYSICS

Equivalent resistors of polyhedral resistive structures

F. J. van Steenwijk

Citation: American Journal of Physics **66**, 90 (1998); doi: 10.1119/1.18820
View online: http://dx.doi.org/10.1119/1.18820
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/66/1?ver=pdfcov
Published by the American Association of Physics Teachers

Equivalent resistors of polyhedral resistive structures

F. J. van Steenwijk
Afdeling Natuurkunde, Rijksuniversiteit Groningen, Nijenborgh 4, 9747 AG Groningen, the Netherlands

(Received 25 August 1997; accepted 5 September 1997)

The equivalent resistors of regular polyhedral resistive structures between any two of the vertices are calculated in terms of the characteristic properties of the structures. Some special cases are considered. © *1998 American Association of Physics Teachers.*

It is a well-known textbook exercise to determine the equivalent resistor between two vertices of a resistive structure formed by 12 equal resistors R arranged along the edges of a cube.[1–4] The didactic interest of the exercise is obvious: students should make use of the symmetry of the structure and recognize which vertices have the same potential, whereas they tend to consider the vertices separately and try to consider the connections between the leads in terms of parallel and series connections. Of course the same problem can be formulated for the other regular polyhedra. The equivalent resistor between opposite vertices of each of the regular polyhedra—except for the tetrahedron which does not have opposite vertices—is easy to evaluate if one recognizes the planes of equal potential through the vertices. Thus the equivalent resistor between opposite vertices is given by

$$R_e = \sum_{i=1}^{N} \left(\frac{1}{n_i}\right) R, \tag{1}$$

where i counts the planes through equipotential vertices. The number of resistors in parallel between the equipotential planes $i-1$ and i is denoted by n_i. Notice that the limiting values $i=0$ and $i=N$ pertain to the vertices that are directly connected to the leads.

At a certain level the problem has the characteristics of a brain twister. It was used at the University of Groningen as a prize game for secondary school students during open days of the Department of Physics. The structures were physically available and the contestants could check their answers by measurement.

To determine the equivalent resistor between any two vertices of a regular polyhedral structure with H vertices, we proceed as follows. Consider the situation where a current I is fed into one vertex while it is allowed to leave the structure at the remaining $H-1$ vertices in equal portions $I/(H-1)$. One may calculate the potential of the subsequent equipotential planes (cf. Fig. 1) perpendicular to the (single) axis of symmetry of this current distribution as

$$V_i = R\sum_{j=1}^{i} \frac{I_j}{n_j} = RI\sum_{j=1}^{i} \frac{1}{n_j}\left(1-\frac{1}{H-1}\sum_{k=1}^{j-1} q_k\right). \tag{2}$$

Here, I_j denotes the total current flowing between the equipotential planes $j-1$ and j, while q_k denotes the number of vertices in the equipotential plane k.

The equivalent resistor $R_e(i)$ between two vertices is found by superposition of the situation described above with a similar situation in which the full current I leaves the structure at one vertex while it is fed into the structure in $H-1$ equal portions at the other vertices. The superposition, which is schematically depicted in Fig. 2, leads to

$$\left(I+\frac{I}{H-1}\right)R_e(i) = 2V_i. \tag{3}$$

The argument i in $R_e(i)$ is equivalent to the smallest number of resistors that has to be traversed when going from the

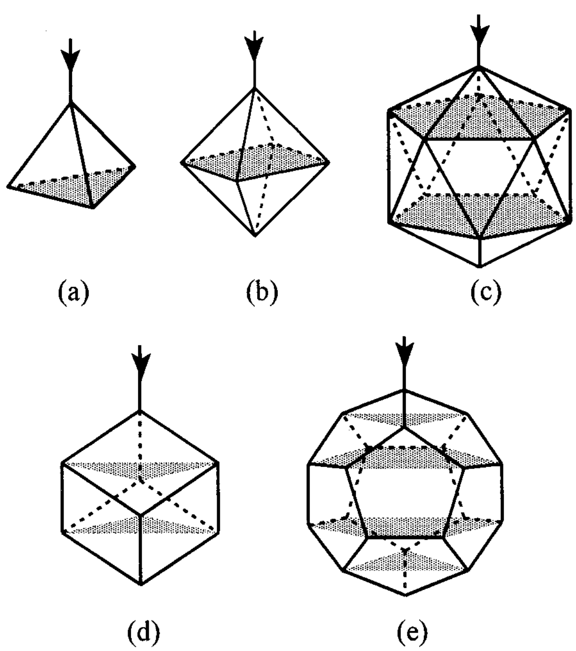

Fig. 1. Symmetric current distributions in polyhedral resistive structures: (a) tetrahedron, (b) octahedron, (c) icosahedredron, (d) cube, (e) dodecahedron. The current fed to one of the vertices (indicated) leaves the structure in equal portions at each of the remaining vertices (not indicated). Planes through equipotential vertices are shaded.

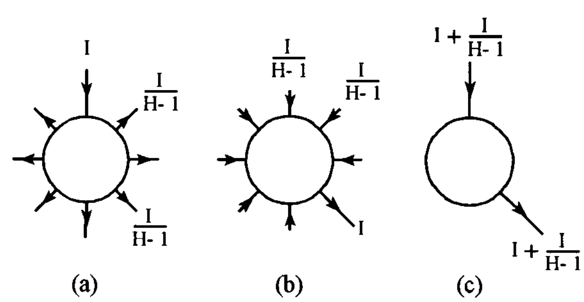

Fig. 2. Superposition of symmetric current distributions: (a) and (b) represent schematically the symmetric distributions described in the text (see also Fig. 1). The equivalent resistor is determined between the current carrying vertices in (c).

Table I. Characteristic numbers of polyhedra and polyhedral resistive structures.

	H	E	N	$\{n_1,\ldots,n_N\}$ $\{q_0,\ldots,q_N\}$	$R_e(1)$	$R_e(2)$	$R_e(3)$	$R_e(4)$	$R_e(5)$
Tetrahedron	4	6	1	{3} {1,3}	$\frac{1}{2}R$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Octahedron	6	12	2	{4,4} {1,4,1}	$\frac{5}{12}R$	$\frac{1}{2}R$	$\cdots$	$\cdots$	$\cdots$
Icosahedron	12	30	3	{5,10,5} {1,5,5,1}	$\frac{11}{30}R$	$\frac{7}{15}R$	$\frac{1}{2}R$	$\cdots$	$\cdots$
Cube	8	12	3	{3,6,3} {1,3,3,1}	$\frac{7}{12}R$	$\frac{3}{4}R$	$\frac{5}{6}R$	$\cdots$	$\cdots$
Dodecahedron	20	30	5	{3,6,6,6,3} {1,3,6,6,3,1}	$\frac{19}{30}R$	$\frac{9}{10}R$	$\frac{16}{15}R$	$\frac{17}{15}R$	$\frac{7}{6}R$

vertex of the incoming current to the vertex of the outgoing current. Rearranging and insertion of V_i gives

$$R_e(i)=\frac{2(H-1)V_i}{HI}=\frac{2R}{H}\sum_{j=1}^{i}\frac{1}{n_j}\left(H-1-\sum_{k=1}^{j-1}q_k\right). \tag{4}$$

By inclusion of $k=0$ in the sum over q_k one finds

$$R_e(i)=\frac{2R}{H}\sum_{j=1}^{i}\frac{1}{n_j}\left(H-\sum_{k=0}^{j-1}q_k\right). \tag{5}$$

The resulting R_e are tabulated together with the n's and the q's in Table I for each of the five regular polyhedra.

Some special cases may be noted.

For $R_e(1)$, the equivalent resistor between two adjacent vertices, Eq. (5) simplifies to

$$R_e(1)=\frac{2(H-1)}{Hn_1}R=\frac{(H-1)}{E}R. \tag{6}$$

Use is made of the well-known fact that the product of the number of vertices (H) and the number of edges coming together in one vertex (n_1) equals two times the number of edges (E) of a polyhedron.[5]

For $R_e(N)$, the equivalent resistor for opposite vertices, Eq. (5) should yield Eq. (1). This may not be immediately obvious, but can be seen by using the symmetry of the structures c.q. $n_j=n_{N-j+1}$ and $q_k=q_{N-k}$, the commutativity of addition, and the renaming of the indices:

$$\sum_{j=1}^{N}\sum_{k=0}^{j-1}\frac{q_k}{n_j}=\sum_{j=1}^{N}\sum_{k=0}^{j-1}\frac{q_{N-k}}{n_{N-j+1}}=\sum_{r=1}^{N}\sum_{k=0}^{N-r}\frac{q_{N-k}}{n_r}$$
$$=\sum_{r=1}^{N}\sum_{s=r}^{N}\frac{q_s}{n_r}=\sum_{j=1}^{N}\sum_{k=j}^{N}\frac{q_k}{n_j}$$
$$=\frac{1}{2}\left(\sum_{j=1}^{N}\sum_{k=0}^{j-1}\frac{q_k}{n_j}+\sum_{j=1}^{N}\sum_{k=j}^{N}\frac{q_k}{n_j}\right)$$
$$=\frac{1}{2}\sum_{j=1}^{N}\sum_{k=0}^{N}\frac{q_k}{n_j}=\frac{H}{2}\sum_{j=1}^{N}\frac{1}{n_j}. \tag{7}$$

By insertion of Eq. (7) into Eq. (5) one obtains Eq. (1), as required.

The difference between R_e for opposite vertices ($i=N$) and R_e for vertices one resistor different from opposite ($i=N-1$) is remarkable. This quantity is given by

$$\Delta R_e=R_e(N)-R_e(N-1)=\frac{2R}{H}\frac{1}{n_N}\left(H-\sum_{k=0}^{N-1}q_k\right)$$
$$=\frac{2R}{Hn_N}=\frac{R}{E}. \tag{8}$$

ACKNOWLEDGMENT

The author wishes to thank Professor L. Niesen for critically reading the manuscript.

[1]E. M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1965), 1st ed., p. 423.

[2]H. D. Young and R. A. Freedman, *University Physics* (Addison–Wesley, Reading, MA, 1996), 9th ed., p. 863.

[3]E. Hecht, *Physics* (Books Cole, Pacific Grove, CA, 1994), p. 717.

[4]D. C. Giancoli, *Physics for Scientists and Engineers* (Prentice-Hall, Englewood Cliffs, NJ, 1989), 2nd ed., p. 624.

[5]W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations & Essays* (University of Toronto Press, Toronto, 1974), 12th ed., pp. 131–132.