\ell_1\text{-summability of higher-dimensional Fourier series}

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Abstract

It is proved that the maximal operator of the \(\ell_1\)-Fejér means of a \(d\)-dimensional Fourier series is bounded from the periodic Hardy space \(H_p(\mathbb{T}^d)\) to \(L_p(\mathbb{T}^d)\) for all \(d/(d+1) < p \leq \infty\) and, consequently, is of weak type \((1, 1)\). As a consequence we obtain that the \(\ell_1\)-Fejér means of a function \(f \in L_1(\mathbb{T}^d)\) converge \(\text{a.e.}\) to \(f\). Moreover, we prove that the \(\ell_1\)-Fejér means are uniformly bounded on the spaces \(H_p(\mathbb{T}^d)\) and so they converge in norm \((d/(d+1) < p < \infty)\). Similar results are shown for conjugate functions and for a general summability method, called \(\theta\)-summability. Some special cases of the \(\ell_1-\theta\)-summation are considered, such as the Weierstrass, Picard, Bessel, Fejér, de la Vallée Poussin, Rogosinski and Riesz summations.

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1. Introduction

It is known that Carleson’s theorem holds for higher dimensions. More exactly,

\[ s_k f(x) := \sum_{j \in \mathbb{Z}^d, |j| \leq k} \hat{f}(j)e^{ij \cdot x} \rightarrow f(x) \quad \text{for a.e. } x \in \mathbb{T}^d \text{ as } k \rightarrow \infty \]

if \(f \in L_p(\mathbb{T}^d)\) \((1 < p < \infty)\), where \(| \cdot | = \| \cdot \|_1\) or \(| \cdot | = \| \cdot \|_\infty\) (see [5,8,11]). This is false for \(p = 1\). However, in the one-dimensional case the Fejér, Riesz, Weierstrass, Abel, etc. summability means \(\sigma_n f\) of \(f\) converge to \(f\) almost everywhere if \(f \in L_1(\mathbb{T})\) (see [25,4,16] or [19]).

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In this paper we generalize the summation results to dimensions \( d \geq 2 \). We consider the \( \ell_1 \)-Fejér means

\[
\sigma_n f(x) := \sum_{j \in \mathbb{Z}^d, |j| \leq n} \left( 1 - \frac{|j|}{n} \right) \hat{f}(j)e^{j \cdot x}.
\]

For \(| \cdot |\) denoting the \( \| \cdot \|_\infty \) norm, the summation was investigated in [12,24,20]; for the \( \| \cdot \|_2 \) norm, the summation was investigated in [16,6,11]. In this paper we consider the triangular or \( \ell_1 \)-summability, i.e., where \(| \cdot | = \| \cdot \|_1\) (see [1,2,23] and more recently [18]). Because of the complexity of the kernel function, this case is rarely investigated in the literature. Since the kernel functions are very different for each norm, we need essentially different ideas. Berens et al. [1] proved for the Fourier transform that \( \sigma_T f \to f \) in \( L_p(\mathbb{R}^d) \) norm and a.e. as \( T \to \infty \), when \( f \in L_p(\mathbb{R}^d) \) (1 \( \leq p < \infty \)) (for the norm convergence see also [14]).

We will give a sharp estimation for the Fejér kernel function and for its derivative. Using this we generalize the results just mentioned. We prove that \( \sigma_n f \to f \) in \( B \)-norm, where \( B \) is a homogeneous Banach space, which includes the norm convergence in \( L_p(\mathbb{T}^d) \) (1 \( \leq p < \infty \)) and in \( C(\mathbb{T}^d) \). Next we obtain that the maximal operator \( \sigma_n \) is bounded from the Hardy space \( H_p(\mathbb{T}^d) \) to \( L_p(\mathbb{T}^d) \) for all \( d/(d+1) < p \leq \infty \). This implies by interpolation that \( \sigma_n \) is of weak type \((1,1)\). As a consequence we get the a.e. convergence of \( \sigma_n f \to f \), whenever \( f \in L_1(\mathbb{T}^d) \). Moreover, we prove that the Fejér means \( \sigma_n \) are uniformly bounded on the spaces \( H_p(\mathbb{T}^d) \) and so they converge in norm \((d/(d+1) < p < \infty)\).

A general method of summation, the so called \( \theta \)-summation method, which is generated by a single function \( \theta \) and which includes all summations mentioned above, is also considered. Similar results are shown for the \( \theta \)-summation and for conjugate functions and means.

2. Hardy spaces

Let us fix \( d \geq 2 \), \( d \in \mathbb{N} \). For a set \( \mathbb{Y} \neq \emptyset \) let \( \mathbb{Y}^d \) be its Cartesian product \( \mathbb{Y} \times \cdots \times \mathbb{Y} \) taken with itself, \( d \) times. For \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) and \( u = (u_1, \ldots, u_d) \in \mathbb{R}^d \) set

\[
u \cdot x := \sum_{k=1}^d u_k x_k, \quad \| u \| := \left( \sum_{k=1}^d |x_k|^p \right)^{1/p}, \quad |x| := \| x \|_1.
\]

We briefly write \( L_p(\mathbb{T}^d) \) instead of the \( L_p(\mathbb{T}^d, \lambda) \) space equipped with the norm (or quasi-norm) \( \| f \|_p := \left( \int_{\mathbb{T}^d} |f|^p \, d\lambda \right)^{1/p} \), \( 0 < p \leq \infty \), where \( \mathbb{T} := [-\pi, \pi] \) is the torus and \( \lambda \) the Lebesgue measure. We use the notation \( |f| \) for the Lebesgue measure of the set \( I \). The weak \( L_p \) space, \( L_{p,\infty}(\mathbb{T}^d) \) \( 0 < p < \infty \), consists of all measurable functions \( f \) for which

\[
\| f \|_{L_{p,\infty}} := \sup_{\rho > 0} \rho \lambda(|f| > \rho)^{1/p} < \infty.
\]

Note that \( L_{p,\infty}(\mathbb{T}^d) \) is a quasi-normed space (see [3]). It is easy to see that for each \( 0 < p < \infty \),

\[
L_p(\mathbb{T}^d) \subset L_{p,\infty}(\mathbb{T}^d) \quad \text{and} \quad \| \cdot \|_{L_{p,\infty}} \leq \| \cdot \|_p.
\]

The space of continuous functions with the supremum norm is denoted by \( C(\mathbb{T}^d) \).

The Hardy space \( H_p(\mathbb{T}^d) \) \( 0 < p \leq \infty \) consists of all distributions \( f \) for which

\[
\| f \|_{H_p} := \sup_{0 < t} \| f * P^d_t \|_p < \infty.
\]
where
\[ P^d_t(x) := \sum_{m \in \mathbb{Z}^d} e^{-t\|m\|^2} e^{im \cdot x} \quad (x \in \mathbb{T}^d, \ t > 0) \]
is the \(d\)-dimensional Poisson kernel. It is known that the Hardy spaces \(H_p(\mathbb{T}^d)\) are equivalent to the \(L_p(\mathbb{T}^d)\) spaces when \(1 < p < \infty\) (see e.g. [15] or [20]).

The atomic decomposition is a useful characterization of Hardy spaces. A bounded function \(a\) is an \(H_p(\mathbb{T}^d)\)-atom if there exists a cube \(I \subset \mathbb{T}^d\) such that

(i) \(\text{supp } a \subset I\),
(ii) \(\|a\|_{\infty} \leq |I|^{-1/p}\),
(iii) \(\int_I a(x)x^k d\lambda(x) = 0\) for all multi-indices \(k = (k_1, \ldots, k_d)\) with \(|k| = M\), where \(M \geq |d(1/p - 1)|\); note that \([x]\) denotes the integer part of \(x \in \mathbb{R}\).

The atomic decomposition theorem can be found e.g. in [15] or [20].

3. The kernel functions

For a distribution \(f\) the \(n\)th Fourier coefficient is defined by \(\hat{f}(n) := f(e_{-n})\), where \(e_n(x) := e^{im \cdot x} (n \in \mathbb{Z}^d)\) (see e.g. [7, p. 67]). If \(f\) is an integrable function then
\[
\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x)e^{-im \cdot x} \, dx, \quad (t = \sqrt{-1}).
\]

For \(f \in L_1(\mathbb{T}^d)\) the \(k\)th triangular or \(\ell_1\)-partial sum \(s_k f\) is introduced by
\[
s_k f(x) := \sum_{j \in \mathbb{Z}^d} 1_{|j| \leq k} \hat{f}(j)e^{ij \cdot x} = \int_{\mathbb{T}^d} f(x - u)D_k(u) \, du \quad (k \in \mathbb{N}),
\]
where
\[
D_k(u) := \sum_{j \in \mathbb{Z}^d} 1_{|j| \leq k} e^{ij \cdot u}
\]
is the Dirichlet kernel. It is easy to see that \(|D_k| \leq Ck^d\). Recently Szili and Vértesi [18] verified that \(|D_k|_1 \sim (\log k)^d\).

For \(n \geq 1\) the \(\ell_1\)-Fejér means of a function \(f \in L_1(\mathbb{T}^d)\) are defined by
\[
\sigma_n f(x) := \sum_{j \in \mathbb{Z}^d, |j| \leq n} \left(1 - \frac{|j|}{n}\right) \hat{f}(j)e^{ij \cdot x} = \int_{\mathbb{T}^d} f(x - u)K_n(u) \, du,
\]
where the Fejér kernel is given by
\[
K_n(u) := \sum_{j \in \mathbb{Z}^d, |j| \leq n} \left(1 - \frac{|j|}{n}\right) e^{ij \cdot u} = \sum_{|j| \leq n} \sum_{k=0}^{n-1} \frac{1}{n} e^{ij \cdot u} = \frac{1}{n} \sum_{k=0}^{n-1} D_k(u).
\]

Then
\[
\sigma_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} s_k f(x).
\]

Observe that
\[
|K_n| \leq Cn^d. \quad (1)
\]
We will estimate the kernel functions \( D_k \) and \( K_n \). The \( n \)th divided difference of a function \( f \) at the (pairwise distinct) knots \( x_1, \ldots, x_n \in \mathbb{R} \) is introduced inductively as

\[
[x_1] f := f(x_1), \quad [x_1, \ldots, x_n] f := \frac{[x_1, \ldots, x_{n-1}] f - [x_2, \ldots, x_n] f}{x_1 - x_n}.
\]

One can see that the difference is a symmetric function of the knots. It was proved by Berens and Xu [2,23] that

\[
D_n(x) = [\cos x_1, \ldots, \cos x_d] G_n, \quad (x \in \mathbb{T}^d),
\]

where

\[
G_n(t) := (-1)^{(d-1)/2}2 \cos(t/2)(\sin t)^{d-2} \text{soc}(n + 1/2)t
\]

and

\[
\text{soc} t := \begin{cases} 
\cos t, & \text{if } d \text{ is even}; \\
\sin t, & \text{if } d \text{ is odd}.
\end{cases}
\]

If we apply the inductive definition of the divided difference in (2) to \( D_n \), then in the denominator we have to choose the factors from the following table:

\[
\begin{array}{cccccc}
\cos x_1 - \cos x_d \\
\cos x_1 - \cos x_{d-1} & \cos x_2 - \cos x_d \\
\vdots \\
\cos x_1 - \cos x_{d-k+1} & \cos x_2 - \cos x_{d-k+2} & \cdots & \cos x_k - \cos x_d \\
\vdots \\
\cos x_1 - \cos x_2 & \cos x_2 - \cos x_3 & \cdots & \cos x_{d-1} - \cos x_d.
\end{array}
\]

Observe that the \( k \)th row contains \( k \) terms and the differences of the indices in the \( k \)th row are equal to \( d - k \); more precisely, if \( \cos x_{i_k} - \cos x_{j_k} \) is in the \( k \)th row, then \( j_k - i_k = d - k \).

We choose exactly one factor from each row. First we choose \( \cos x_1 - \cos x_d \) and then from the second row \( \cos x_1 - \cos x_{d-1} \) or \( \cos x_2 - \cos x_d \). If we have chosen the \( (k - 1) \)th factor from the \( (k - 1) \)th row, say \( \cos x_j - \cos x_{j+d-k+1} \), then we have to choose the next one from the \( k \)th row which is below the \( (k - 1) \)th factor (it is equal to \( \cos x_j - \cos x_{j+d-k} \) or the right neighbor (it is equal to \( \cos x_{j+1} - \cos x_{j+d-k+1} \)). More exactly, we introduce a set \( \mathcal{I} \) of sequences of integer pairs \((i_n, j_n); n = 1, \ldots, d - 1\). Let \( i_1 = 1, j_1 = d \), \((i_n)\) be non-decreasing and \((j_n)\) be non-increasing. If \((i_n, j_n)\) is given then let \( i_{n+1} = i_n \) and \( j_{n+1} = j_n - 1 \) or \( i_{n+1} = i_n + 1 \) and \( j_{n+1} = j_n \). If the sequence \((i_n, j_n)\) has these properties then we say that it is in \( \mathcal{I} \). Observe that the difference \( \cos x_{i_k} - \cos x_{j_k} \) is in the \( k \)th row of the table \((k = 1, \ldots, d - 1)\). So the factors that we have just chosen can be written as \( \prod_{l=1}^{d-1} (\cos x_{i_l} - \cos x_{j_l}) \). In other words,

\[
D_n(x) = \sum_{(i_l, j_l) \in \mathcal{I}} (-1)^{d-1-1} \prod_{l=1}^{d-2} (\cos x_{i_l} - \cos x_{j_l})^{-1} [\cos x_{i_{d-1}}, \cos x_{j_{d-1}}] G_n
\]

\[
= (-1)^{d-1-1} \prod_{l=1}^{d-1} (\cos x_{i_l} - \cos x_{j_l})^{-1} (G_n(\cos x_{i_{d-1}}) - G_n(\cos x_{j_{d-1}}))
\]

\[
=: D_{n,(i_l,j_l)}(x).
\]

(3)
Then

\[ K_n(x) = \sum_{(i, j) \in I} \frac{(-1)^{d-1} n}{l!} \prod_{l=1}^{d-1} (\cos x_i - \cos x_j)^{-1} \sum_{k=0}^{n-1} (G_k(\cos x_{id-1}) - G_k(\cos x_{jd-1})) \]

\[ =: \sum_{(i, j) \in I} K_{n,(i, j)}(x). \]

We may suppose that \( \pi > x_1 > x_2 > \cdots > x_d > 0 \). We will need the following sharp estimations of the kernel functions.

**Lemma 1.** For all \( 0 < \beta < \frac{2}{d-1} \),

\[ |K_{n,(i, j)}(x)| \leq C \prod_{l=1}^{d-1} (x_{il} - x_{jl})^{-1 - \beta} x_{jd-1}^{\beta(d-1)-2} 1_{(x_{id-1} \leq \pi/2]} n^{-1} \]

\[ + C \prod_{l=1}^{d-1} (x_{il} - x_{jl})^{-1 - \beta} (\pi - x_{jd-1})^{\beta(d-1)-2} 1_{(x_{jd-1} > \pi/2]} n^{-1}. \]

**Proof.** Using the formulas

\[ \sum_{k=0}^{n-1} \cos(k + 1/2)t = \frac{\sin(nt)}{2 \sin(t/2)}, \quad \sum_{k=0}^{n-1} \sin(k + 1/2)t = \frac{1 - \cos(nt)}{2 \sin(t/2)}, \]

we conclude that

\[ |K_{n,(i, j)}(x)| \leq \prod_{l=1}^{d-1} \frac{(\sin x_{id-1})^{d-2} (\sin x_{id-1}/2)^{-1} + (\sin x_{jd-1})^{d-2} (\sin x_{jd-1}/2)^{-1}}{n \sin((x_{il} - x_{jl})/2) \sin((x_{il} + x_{jl})/2)}. \]

If \( x_{jd-1} \leq \pi/2 \) then \((x_{il} + x_{jl})/2 \leq 3\pi/4\) and so

\[ |K_{n,(i, j)}(x)| \leq C \prod_{l=1}^{d-1} (x_{il} - x_{jl})^{-1} (x_{il} + x_{jl})^{-1} (x_{id-1}^{d-3} + x_{jd-1}^{d-3}) n^{-1}. \]

Since \( x_{il} + x_{jl} > x_{il} - x_{jl} \) and \( x_{il} + x_{jl} > x_{jd-1} > x_{id-1} \) we can see that

\[ |K_{n,(i, j)}(x)| \leq C \prod_{l=1}^{d-1} (x_{il} - x_{jl})^{-1 - \beta} (x_{id-1}^{d-3 + (\beta-1)(d-1)} + x_{jd-1}^{d-3 + (\beta-1)(d-1)}) n^{-1} \]

\[ \leq C \prod_{l=1}^{d-1} (x_{il} - x_{jl})^{-1 - \beta} x_{jd-1}^{\beta(d-1)-2} n^{-1} \]

for all \( 0 < \beta < \frac{2}{d-1} \).

If \( x_{jd-1} > \pi/2 \) then \((x_{il} + x_{jl})/2 > \pi/4\) and

\[ |K_{n,(i, j)}(x)| \leq C \prod_{l=1}^{d-1} (x_{il} - x_{jl})^{-1} (2\pi - x_{il} - x_{jl})^{-1} ((\pi - x_{id-1})^{d-3} \]

\[ + (\pi - x_{jd-1})^{d-3}) n^{-1}. \]
Observe that $2\pi - x_i - x_j > x_i - x_j$ and $2\pi - x_i - x_j > \pi - x_j > \pi - x_{jd-1} > \pi - x_{id-1}$.
Thus
\[
|K_{n,(i,j)}(x)| \leq C \prod_{l=1}^{d-1} (x_i - x_j)^{1-\beta} \left((\pi - x_{i,j,l})^{d-3} + (\beta - 1)(d-1)n^{-1}\right)
\]
\[
\leq C \prod_{l=1}^{d-1} (x_i - x_j)^{1-\beta} (\pi - x_{i,j,l})^{\beta(d-2)-2} n^{-1},
\]
if $0 < \beta < \frac{2}{d-1}$. \(\square\)

**Lemma 2.** For all $0 < \beta < \frac{2}{d-2}$,
\[
|K_{n,(i,j)}(x)| \leq C \prod_{l=1}^{d-2} (x_i - x_j)^{1-\beta} x_{i,j,l}^{\beta(d-2)-2} 1_{(x_{i,j,l}\leq \pi/2)}
\]
\[
+ C \prod_{l=1}^{d-2} (x_i - x_j)^{1-\beta} (\pi - x_{i,j,l})^{\beta(d-2)-2} 1_{(x_{i,j,l}\geq \pi/2)}. \tag{6}
\]

**Proof.** The Lagrange theorem and (3) imply that there exists $x_{i,j,l} > \xi > x_{i,j,l-1}$, such that
\[
D_{k,(i,j)}(x) = (-1)^{d-1} \prod_{l=1}^{d-1} (\cos x_i - \cos x_j)^{-1} H_k'((\xi)^{(x_{i,j,l}-x_{i,j,l-1}))},
\]
where
\[
H_k(t) = (-1)^{(d-1)/2} 2 \cos(t/2)(\sin t)^{d-2} \cos (k + 1/2)t.
\]
Then
\[
|K_{n,(i,j)}(x)| \leq C \prod_{l=1}^{d-1} \frac{(\sin \xi)^{d-2} + n(\sin \xi)^{d-2}}{n \sin((x_i - x_j)/2) \sin((x_i + x_j)/2) \sin(\xi/2)} (x_{i,j,l} - x_{i,j,l-1})
\]
\[
+ C \prod_{l=1}^{d-1} \frac{(\sin \xi)^{d-3}}{n \sin((x_i - x_j)/2) \sin((x_i + x_j)/2)} (x_{i,j,l} - x_{i,j,l-1}).
\]
Besides (5) we have used that $|\sum_{k=0}^{n-1} \cos (k + 1/2)t| \leq n$. If $x_{i,j,l} \leq \pi/2$,
\[
|K_{n,(i,j)}(x)| \leq C \prod_{l=1}^{d-1} (x_i - x_j)^{-1} (x_i + x_j)^{-1} (x_{i,j,l} - x_{i,j,l-1})^{\xi(d-3)}
\]
\[
\leq C \prod_{l=1}^{d-2} (x_i - x_j)^{-1-\beta \xi^{d-4} + (\beta - 1)(d-2)}
\]
\[
\leq C \prod_{l=1}^{d-2} (x_i - x_j)^{-1-\beta \xi^{d-2} - 2}
\]
for all $0 < \beta < \frac{2}{d-2}$. 
Similarly, if \( x_{jd-1} > \pi / 2 \) then \((x_{it} + x_{jt}) / 2 > \pi / 4\) and

\[
|K_{n,(i_j,j_j)}(x)| \leq C \prod_{l=1}^{d-1} (x_{it} - x_{jt})^{-1} (2\pi - x_{it} - x_{jt})^{-1} (x_{id-1} - x_{jd-1}) (\pi - \xi)^{d-3}
\]

\[
\leq C \prod_{l=1}^{d-2} (x_{it} - x_{jt})^{-1-\beta} (\pi - \xi)^{d-4+(\beta-1)(d-2)}
\]

\[
\leq C \prod_{l=1}^{d-2} (x_{it} - x_{jt})^{-1-\beta} (\pi - x_{id-1})^{\beta(d-2)-2},
\]

if \( 0 < \beta < \frac{2}{d-2} \). □

In the next lemma we estimate the partial derivatives of the kernel function.

**Lemma 3.** If \( 0 < \beta < \frac{2}{d-1} \) then for all \( q = 1, \ldots, d \),

\[
|\partial_q K_{n,(i_j,j_j)}(x)| \leq C \prod_{l=1}^{d-1} (x_{it} - x_{jt})^{-1-\beta} x_{jd-1}^{\beta(d-1)-2} 1_{\{x_{jd-1} \leq \pi/2\}}
\]

\[
+ C \prod_{l=1}^{d-1} (x_{it} - x_{jt})^{-1-\beta} (\pi - x_{id-1})^{\beta(d-1)-2} 1_{\{x_{jd-1} > \pi/2\}}.
\] (7)

**Proof.** Let \( m_l = 0, 1 \) and \( \delta_{m_1,\ldots,m_d} = 0, \pm 1 \) be suitable numbers. (3) implies that the partial derivative of \( D_{k,(i_j,j_j)} \) is

\[
\partial_q D_{k,(i_j,j_j)}(x) = (-1)^{id-1-1} \sum_{m_1,\ldots,m_d=1} \delta_{m_1,\ldots,m_d}
\]

\[
\times \prod_{l=1}^{d-1} \partial_q ((\cos x_{it} - \cos x_{jt})^{-1}) \partial_q (G_k(\cos x_{id-1}) - G_k(\cos x_{jd-1})).
\]

If we differentiate the first \( (d-1) \) factors then we get

\[
\sum_{m_1,\ldots,m_d=1} \prod_{l=1}^{d-1} (\cos x_{it} - \cos x_{jt})^{-1-m_l} (-\sin y_l)^{m_l} (G_k(\cos x_{id-1}) - G_k(\cos x_{jd-1}))
\]

\[
= \sum_{m_1,\ldots,m_d=1} \prod_{l=1}^{d-1} (\cos x_{it} - \cos x_{jt})^{-1-m_l} (-\sin y_l)^{m_l} H_k'(\xi)(x_{id-1} - x_{jd-1}),
\]

where \( y_l = x_{it} \) or \( y_l = -x_{jt} \). If \( x_{jd-1} \leq \pi / 2 \) then \( |\sin y_l| \leq |y_l| \leq x_{it} + x_{jt} \) and, like in the proof of Lemma 2,

\[
\left| \frac{1}{n} \sum_{k=0}^{n-1} \prod_{l=1}^{d-1} (\cos x_{it} - \cos x_{jt})^{-1-m_l} (\sin y_l)^{m_l} (G_k(\cos x_{id-1}) - G_k(\cos x_{jd-1})) \right|
\]

\[
\leq C \prod_{l=1}^{d-1} (x_{it} - x_{jt})^{-1-m_l} (x_{it} + x_{jt})^{-1} \left| \frac{1}{n} \sum_{k=0}^{n-1} H_k'(\xi)(x_{id-1} - x_{jd-1}) \right|
\]
\[ \leq C \sum_{m_1 + \cdots + m_{d-1} = 1}^{d-1} \prod_{l=1}^{d-1} (x_{ij} - x_{ji})^{-1} (x_{il} + x_{ji})^{-1} \xi^{d-3} \]

\[ \leq C \prod_{l=1}^{d-1} (x_{ij} - x_{ji})^{-1-\beta} \xi^{(\beta-1)(d-1)+d-3} \]

\[ \leq C \prod_{l=1}^{d-1} (x_{ij} - x_{ji})^{-1-\beta} x_{jd-1}^{-\beta(d-1)-2}, \]

whenever \( 0 < \beta < \frac{2}{d-1} \).

If \( m_d = 1 \) and, say \( i_d-1 = q \), then

\[ \prod_{l=1}^{d-1} (\cos x_{il} - \cos x_{ji})^{-1} \partial_q (G_k(\cos x_{i_d-1}) - G_k(\cos x_{j_d-1})) \]

\[ = \prod_{l=1}^{d-1} (\cos x_{il} - \cos x_{ji})^{-1} H_k'(\cos x_{i_d-1}) \]

and

\[ \left| \frac{1}{n} \sum_{k=0}^{n-1} \prod_{l=1}^{d-1} (\cos x_{il} - \cos x_{ji})^{-1} \partial_q (G_k(\cos x_{i_d-1}) - G_k(\cos x_{j_d-1})) \right| \]

\[ \leq C \prod_{l=1}^{d-1} (x_{il} - x_{ji})^{-1} (x_{il} + x_{ji})^{-1} \left| \frac{1}{n} \sum_{k=0}^{n-1} H_k'(\cos x_{i_d-1}) \right| \]

\[ \leq C \prod_{l=1}^{d-1} (x_{il} - x_{ji})^{-1} (x_{il} + x_{ji})^{-1} \xi^{d-3} \]

\[ \leq C \prod_{l=1}^{d-1} (x_{il} - x_{ji})^{-1-\beta} x_{jd-1}^{-\beta(d-1)-2}. \]

Consequently,

\[ |\partial_q K_n(i_j)(x)| = \left| \frac{1}{n} \sum_{k=0}^{n-1} \partial_q D_k(i_j)(x) \right| \leq C \prod_{l=1}^{d-1} (x_{il} - x_{ji})^{-1-\beta} x_{jd-1}^{-\beta(d-1)-2}, \]

if \( x_{jd-1} \leq \pi/2 \) and \( 0 < \beta < \frac{2}{d-1} \). The case \( x_{jd-1} > \pi/2 \) can be proved similarly. \( \square \)

4. Norm convergence of the summability

First we show that the \( L_1 \)-norms of the kernel functions are uniformly bounded.

**Theorem 1.** We have

\[ \int_{\mathbb{T}^d} |K_n(x)| \, dx \leq C, \quad (n \in \mathbb{N}). \]

**Proof.** We may suppose again that \( \pi > x_1 > x_2 > \cdots > x_d > 0 \). If \( x_1 \leq 16/n \) or \( \pi - x_d \leq 16/n \) then (1) implies
\[
\int_{\{16/n \geq x_1 > x_2 > \cdots > x_d > 0\}} |K_n(x)| \, dx + \int_{\{\pi > x_1 > x_2 > \cdots > x_d \geq \pi - 16/n\}} |K_n(x)| \, dx \leq C.
\]
Hence it is enough to integrate over
\[S := \{ x \in \mathbb{T}^d : \pi > x_1 > x_2 > \cdots > x_d > 0, x_1 > 16/n, x_d < \pi - 16/n \}.
\]
For a sequence \((i_1, j_1) \in \mathcal{I}\) let us define the set \(S_{(i_1, j_1), k}\) by
\[
S_{(i_1, j_1), k} := \begin{cases}
    x & x_i - x_j > 4/n, l = 1, \ldots, k - 1, x_k - x_j \leq 4/n, \quad \text{if } k < d; \\
    x & x_i - x_j > 4/n, l = 1, \ldots, d - 1, \quad \text{if } k = d
\end{cases}
\]
and
\[
S_{(i_1, j_1), k, 1} := \begin{cases}
    x & x_j > 4/n, x_{j_{d-1}} \leq \pi/2, \quad \text{if } k < d; \\
    x & x_j > 4/n, x_{j_{d-1}} \leq \pi/2, \quad \text{if } k = d,
\end{cases}
\]
\[
S_{(i_1, j_1), k, 2} := \begin{cases}
    x & x_j \leq 4/n, x_{j_{d-1}} \leq \pi/2, \quad \text{if } k < d; \\
    x & x_j \leq 4/n, x_{j_{d-1}} \leq \pi/2, \quad \text{if } k = d,
\end{cases}
\]
\[
S_{(i_1, j_1), k, 3} := \begin{cases}
    x & x_j - x_i > 4/n, x_{j_{d-1}} > \pi/2, \quad \text{if } k < d; \\
    x & x_j - x_i > 4/n, x_{j_{d-1}} > \pi/2, \quad \text{if } k = d,
\end{cases}
\]
\[
S_{(i_1, j_1), k, 4} := \begin{cases}
    x & x_j - x_i \leq 4/n, x_{j_{d-1}} > \pi/2, \quad \text{if } k < d; \\
    x & x_j - x_i \leq 4/n, x_{j_{d-1}} > \pi/2, \quad \text{if } k = d.
\end{cases}
\]
Then
\[\int_{\mathbb{T}^d} |K_n(x)| 1_S(x) \, dx \leq \sum_{k=1}^d \sum_{m=1}^4 \int_{\mathbb{T}^d} |K_n(i_1, j_1)(x)| 1_{S_{(i_1, j_1), k, m}}(x) \, dx.
\]
From now on the proof is similar to that for Fourier transforms (see [22]), so we do not give a full version of the proof. We outline rather the differences and we consider the set \(S_{(i_1, j_1), k, 1}\), only.

First let \(1 \leq k \leq d - 1\). Since \(x_{j_{d-1}} - x_{j_{d-1}} \leq x_{i_1} - x_j\), (6) implies
\[
\int_{\mathbb{T}^d} K_{n, (i_1, j_1)}(x) 1_{S_{(i_1, j_1), k, 1}}(x) \, dx \leq C \int_{\mathbb{T}^d} \prod_{l=1}^{k-1} (x_{i_l} - x_j)^{-\beta} \prod_{l=k}^{d-2} (x_{i_l} - x_j)^{-\beta + 1/(d-k)} \times (x_{i_{j_{d-1}}} - x_{j_{d-1}})^{1/(d-k) - 1} x_{j_{d-1}}^{(d-2)-2} 1_{S_{(i_1, j_1), k, 1}}(x) \, dx.
\]
First we choose the indices \(j_{d-1} (=i_{d-1}')\), \(i_{d-1} (=i_{d-1}')\) and then \(i_{d-2}\) if \(i_{d-2} \neq j_{d-2}\). (Exactly one of these two cases is satisfied.) If we repeat this process then we get an injective sequence \((i_1', l = 1, \ldots, d)\). We integrate the term \(x_{i_1} - x_{j_1}\) in \(x_{i_1}'\), the term \(x_{i_2} - x_{j_2}\) in \(x_{i_2}'\), and finally the term \(x_{i_{d-1}} - x_{j_{d-1}}\) in \(x_{i_{d-1}}'\) and \(x_{j_{d-1}}\) in \(x_{j_{d-1}}'\). Since \(x_{i_l} - x_{j_l} > 4/n(l = 1, \ldots, k - 1), x_{i_l} - x_{j_l} \leq 4/n(l = k, \ldots, d - 1), x_{j_{d-1}} \geq x_{j_k} > 4/n\) and we can choose \(\beta\) such that \(\beta < 1/(d-1)\), we have
\[
\int_{\mathbb{T}^d} K_{n, (i_1, j_1)}(x) 1_{S_{(i_1, j_1), k, 1}}(x) \, dx \\
\leq C \prod_{l=1}^{k-1} \frac{1}{(n)^{\beta}} \prod_{l=k}^{d-2} \frac{1}{(n)^{\beta + 1/(d-k)}} \frac{1}{(n)^{1/(d-k)}} \frac{1}{(n)^{\beta (d-2) - 1}} \leq C.
\]
For $k = d$ we use (4) to obtain
\[
\int_{\mathbb{T}^d} |K_n, (i_j, j_i) (x)| 1_{S(i_j, j_i), d, 1} (x) \, dx \leq C n^{-1} \int_{\mathbb{T}^d} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \beta x_{j_d}^{\beta(d-1) - 2} 1_{S(i_l, j_l), d, 1} (x) \, dx
\]
\[
\leq C n^{-1} \prod_{l=1}^{d-1} (1/n)^{-\beta} (1/n)^{\beta(d-1) - 1} \leq C,
\]
if $\beta < 1/(d - 1)$, which proves the theorem. □

A Banach space $B$ consisting of Lebesgue measurable functions on $\mathbb{T}^d$ is called a homogeneous Banach space if $\|f\|_1 \leq C \|f\|_B$, $(f \in B)$ and

(i) for all $f \in B$ and $x \in \mathbb{T}^d$, $T_x f := f (-x) \in B$ and $\|T_x f\|_B = \|f\|_B$,

(ii) the function $x \mapsto T_x f$ from $\mathbb{T}^d$ to $B$ is continuous for all $f \in B$.

It is easy to see that the spaces $L_p(\mathbb{T}^d)$ ($1 \leq p < \infty$), $C(\mathbb{T}^d)$, Lorentz spaces $L_{p,q}(\mathbb{T}^d)$ ($1 < p < \infty$, $1 \leq q < \infty$) and Hardy space $H_1(\mathbb{T}^d)$ are homogeneous Banach spaces.

We can extend the definition of the $\ell_{1-\theta}$-means to distributions as follows:

$$\sigma_n f := f \ast K_n \quad (n \in \mathbb{N}).$$

Indeed, $\sigma_n f$ is well defined for all $f \in H_p(\mathbb{T}^d)$ ($0 < p \leq \infty$), for all $f \in L_p(\mathbb{T}^d)$ ($1 \leq p \leq \infty$) and $f \in B$, where $B$ is a homogeneous Banach space (cf. [15]).

**Theorem 2.** If $B$ is a homogeneous Banach space on $\mathbb{T}^d$ then

$$\|\sigma_n f\|_B \leq \|f\|_B \|K_n\|_1, \quad (n \in \mathbb{N})$$

and $\sigma_n f \to f$ in $B$, for all $f \in B$ as $n \to \infty$.

**Proof.** The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is defined by

$$\hat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi i x \cdot t} \, dt \quad (x \in \mathbb{R}^d).$$

We have proved in [9] that

$$\sigma_n f (x) = n^d \int_{\mathbb{R}^d} f(x - t) \hat{\theta}_0(nt) \, dt,$$

where $\theta_0(u) := \max(1 - |u|, 0)(u \in \mathbb{R}^d)$. Since $\hat{\theta}_0 \in L_1(\mathbb{R}^d)$ (see [22]), we conclude that

$$\sigma_n f (x) - f(x) = \int_{\mathbb{R}^d} \left( f \left( x - \frac{t}{n} \right) - f(x) \right) \hat{\theta}_0(t) \, dt$$

and

$$\|\sigma_n f - f\|_B = \int_{\mathbb{R}^d} \left\| T_{\hat{\theta}_0/n} f - f \right\|_B |\hat{\theta}_0(t)| \, dt.$$

The theorem follows from (i) and (ii) of the definition of the homogeneous Banach spaces and from the Lebesgue dominated convergence theorem. □
5. $\ell_1$–$\theta$-summability and Hardy spaces

To investigate the almost everywhere convergence we introduce the maximal operator

$$\sigma_n f := \sup_{n \geq 1} |\sigma_n f|.$$  

**Theorem 3.** We have

$$\|\sigma_n f\|_p \leq C_p \|f\|_{H^p} \quad (f \in H^p(\mathbb{T}^d))$$

for all $\frac{d}{d+1} < p < \infty$. In particular, if $f \in L_1(\mathbb{T}^d)$ then

$$\lambda(\sigma_n f > \rho) \leq \frac{C}{\rho} \|f\|_1 \quad (\rho > 0).$$

**Proof.** We have to show that

$$\int_{\mathbb{T}^d} |\sigma_n a(x)|^p \, dx = \int_{\mathbb{T}^d} \sup_{n \geq 1} \left| \int_I a(u) K_n(x-u) \, du \right|^p \, dx \leq C_p$$

for every $p$-atom $a$, where $\frac{d}{d+1} < p < 1$ and $I$ is the support of the atom (see [20]). Without loss of generality we can suppose that $a$ is a $p$-atom with support $I = I_1 \times \cdots \times I_d$ and

$$[-2^{-K-2}, 2^{-K-2}] \subset I_j \subset [-2^{-K-1}, 2^{-K-1}] \quad (j = 1, \ldots, d)$$

for some $K \in \mathbb{N}$. By symmetry we can assume that $\pi > x_1 - u_1 > x_2 - u_2 > \cdots > x_d - u_d > 0$. If $0 < x_1 - u_1 \leq 2^{-K+4}$ then $-2^{-K-1} < x_1 \leq 2^{-K+5}$ and if $\pi - 2^{-K+4} \leq x_1 - u_1 < \pi$ then $\pi - 2^{-K+5} \leq x_1 < \pi + 2^{-K-1}$. By the definition of the atom and by **Theorem 1**, we get the same inequality as in the last equality if we integrate over the set $\{\pi > x_1 - u_1 > x_2 - u_2 > \cdots > x_d - u_d \geq \pi - 2^{-K+4}\}$. Hence, instead of (8) it is enough to prove that

$$\int_{\mathbb{T}^d} \sup_{n \geq 1} \left| \int_I a(u) K_n(x-u) \right|^p \, dx \leq C_p 2^K d 2^{-Kd}.$$

where

$$S := \{x \in \mathbb{T}^d : \pi > x_1 > x_2 > \cdots > x_d > 0, x_1 > 2^{-K+4}, x_d < \pi - 2^{-K+4}\}.$$

Let

$$S_{(i,j),k} := \begin{cases} x \in S : x_i - x_j > 2^{-k+2}, \pi \leq x_i - x_j \leq \pi, & \text{if } k < d; \\ x \in S : x_i - x_j > 2^{-k+2}, \pi \leq x_i - x_j \leq \pi, & \text{if } k = d \end{cases}$$

and

$$S_{(i,j),k,1} := \begin{cases} x \in S_{(i,j)}, k : x_j > 2^{-k+2}, x_{j-1} \leq \pi/2, & \text{if } k < d; \\ x \in S_{(i,j)}, k : x_{j-1} > 2^{-k+2}, x_{j-1} \leq \pi/2, & \text{if } k = d, \end{cases}$$

$$S_{(i,j),k,2} := \begin{cases} x \in S_{(i,j),k} : x_i \leq 2^{-k+2}, x_{i-1} \leq \pi/2, & \text{if } k < d; \\ x \in S_{(i,j),k} : x_{i-1} \leq 2^{-k+2}, x_{i-1} \leq \pi/2, & \text{if } k = d \end{cases}.$$
Then
\[
\int I \sup_{n \geq 1} \left| \int a(u)K_n(x-u)1_{S}(x-u) \, du \right|^p \, dx
\]
\[
\leq \sum_{(i,j) \in I} \sum_{k=1}^d \left( \prod_{l=1}^{d-1} (x_i - u_l - (x_j - u_j))^p \right) \, dx. \tag{9}
\]

We consider again the set \(S_{(i,j), k, 1}\), only. If \(1 \leq k \leq d - 1\) then (9) can be estimated by
\[
C_p2^{Kd} \sum_{(i,j) \in I} \sum_{k=1}^{d-1} \left( \prod_{l=1}^{d-2} (x_i - u_l - (x_j - u_j))^p \right) \, dx.
\]

Suppose that \(d \geq 3\). However, after some technical difficulties, the proof works for \(d = 2\) as well. Since \(x_{i_{d-1}} - u_{i_{d-1}} - (x_{j_{d-1}} - u_{j_{d-1}}) \leq x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l})\), (6) implies
\[
\int I K_n(x-u)1_{S_{(i,j), k, 1}}(x-u) \, dx \leq C \int I \prod_{l=1}^{k-1} (x_i - u_l - (x_j - u_j))^{-1} \beta
\]
\[
\times \prod_{l=k}^{d-2} (x_i - u_l - (x_j - u_j))^{-1} \beta + 1/(d-1) (x_{i_{d-1}} - u_{i_{d-1}} - (x_{j_{d-1}} - u_{j_{d-1}}))^{1/(d-1)}
\]
\[
\times (x_{j_{d-1}} - u_{j_{d-1}})^{\beta(d-2)-2} 1_{S_{(i,j), k, 1}}(x-u) \, du.
\]

In the first product we estimate the factors and in the second one we integrate. More exactly,
\[
x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l}) > x_{i_l} - x_{j_l} - 2^{-K}, \quad l = 1, \ldots, k-1,
\]
and
\[
x_{j_{d-1}} - u_{j_{d-1}} > x_{j_{d-1}} - 2^{-K-1}.
\]

For the integration first we choose the index \(i_{d-1} (=i’_{d-1})\) and then \(i_{d-2}\) if \(i_{d-2} \neq i_{d-1}\) or \(j_{d-2}\) if \(j_{d-2} \neq j_{d-1}\). Repeating this process we get an injective sequence \((i_l’, l = k, \ldots, d - 1)\).

We integrate the term \((x_{i_k} - u_{i_k} - (x_{j_k} - u_{j_k}))^{-1} \beta + 1/(d-k) \) in \(u_{i’_k}\), the term \((x_{i_k+1} - u_{i_k+1} - (x_{j_k+1} - u_{j_k+1}))^{-1} \beta + 1/(d-k) \) in \(u_{i’_{k+1}}\), \ldots, and finally the term \((x_{i_{d-1}} - u_{i_{d-1}} - (x_{j_{d-1}} - u_{j_{d-1}}))^{-1} \beta + 1/(d-k) \) in \(u_{i’_{d-1}}\). Since \(x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l}) \leq 2^{-K+2} (l = k, \ldots, d - 1)\) and we can choose \(\beta\) such that \(\beta < 1/(d-1)\), we have
\[
\int I (x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l}))^{-1} \beta + 1/(d-k) 1_{S_{(i,j), k, 1}}(x-u) \, du_{i’_l} \quad (du_{j_l})
\]
\[
\leq C2^{-K(1/(d-k) - \beta)}
\]
\((l = k, \ldots, d - 1)\). If \(x-u \in S_{(i,j), k, 1}\) then
\[
x_{i_l} - x_{j_l} > 2^{-K+2} + u_{i_l} - u_{j_l} > 2^{-K+2} - 2^{-K} > 2^{-K+1}, \quad l = 1, \ldots, k-1,
\]
and
\[ x_{jd-1} > 2^{-K+2} + u_{jd-1} > 2^{-K+2} - 2^{-K-1} > 2^{-K+1}. \]

Moreover,
\[ x_{ij} - x_{ji} \leq 2^{-K+2} + u_{ij} - u_{ji} < 2^{-K+3}, \quad l = k, \ldots, d - 1, \]
and
\[ x_{ij} - x_{ji} > u_{ij} - u_{ji} > -2^{-K}, \quad l = k, \ldots, d - 1. \]

Hence
\[
\int I K_n(i_l, j_l)(x - u) 1_{S(i_l, j_l), k, 1}(x - u) \, du \\
\leq C 2^{-K} k 2^{-K(1/(d-k) - \beta)(d-k-1) 2^{-K}(d-k)} \prod_{i=1}^{d-1} (x_{il} - x_{jl} - 2^{-K})^{-1 - \beta} 1_{[x_{il} - x_{jl} > 2^{-K+1}]} \\
\times \prod_{l=k}^{d-1} 1_{[-2^{-K} < x_{il} - x_{jl} < 2^{-K+3}] (x_{jd-1} - 2^{-K-1}) (d-2) - 2} 1_{[x_{jd-1} > 2^{-K+1}]} 
\]
and
\[
C_p 2^{Kd} \sum_{(i_l, j_l) \in I} \sum_{k=1}^{d-1} \int I K_n(i_l, j_l)(x - u) 1_{S(i_l, j_l), k, 1}(x - u) \, du \right)^p \, dx \\
\leq C_p 2^{Kd} 2^{-K} k p 2^{-K(1 - (d-k-1))} p \\
\times \sum_{(i_l, j_l) \in I} \sum_{k=1}^{d-1} \int I K_n(i_l, j_l)(x - u) 1_{S(i_l, j_l), k, 1}(x - u) \, du \right)^p \, dx \\
\times \prod_{l=k}^{d-1} 1_{[-2^{-K} < x_{il} - x_{jl} < 2^{-K+3}] (x_{jd-1} - 2^{-K-1}) (d-2) - 2} 1_{[x_{jd-1} > 2^{-K+1}]} \, dx \\
\leq C_p 2^{Kd} 2^{-K} k p 2^{-K(1 - (d-k-1))} p \\
\times \sum_{(i_l, j_l) \in I} \sum_{k=1}^{d-1} 2^{-K(1 - (1 + \beta)p)(k-1) 2^{-K(d-k)} 2^{-K(1 - (2 - \beta(d-2))p)} \\
\leq C_p,
\]
whenever \( 1 - (1 + \beta)p < 0, 1 - (2 - \beta(d-2))p < 0 \) and \( \beta < 1/(d-1) \). Since \( \beta \) can be arbitrarily near to \( 1/(d-1) \), we obtain \( p > \frac{d-1}{d} \).

For \( k = d \) we have \( x_{il} - x_{jl} > 2^{-K+1} \) for all \( l = 1, \ldots, d - 1 \) and \( x_{jd-1} > 2^{-K+1} \). We may suppose that the center of \( I \) is zero; in other words \( I := \prod_{j=1}^{d} (-v, v). \) Let
\[ A_1(u) := \int_{-v}^{u_1} a(t_1, u_2, \ldots, u_d) \, dt_1 \]
and
\[ A_k(u) := \int_{-v}^{u_k} A_{k-1}(u_1, \ldots, u_{k-1}, t_k, u_{k+1}, \ldots, u_d) \, dt_k, \quad (2 \leq k \leq d). \]
Observe that
\[ |A_k(u)| \leq C_p 2^{K(d/p - k)}. \]

Integrating by parts we can see that
\[
\int_{I_1} \int_{I_2} a(u)K_{n,(i_j,j)}(x-u)1_{S(j,j,d,1)}(x-u) \, du_1 \, du_2
= A_1(v, u_2, \ldots, u_d)(K_{n,(i_j,j)}1_{S(j,j,d,1)})(x_1 - v, x_2 - u_2, \ldots, x_d - u_d)
+ \int_{-v}^{v} A_1(u)\partial_1 K_{n,(i_j,j)}(x-u)1_{S(j,j,d,1)}(x-u) \, du_1,
\]
because \( A_1(-v, u_2, \ldots, u_d) = 0 \). Integrating the first term again by parts we obtain
\[
\int_{I_1} \int_{I_2} a(u)K_{n,(i_j,j)}(x-u)1_{S(j,j,d,1)}(x-u) \, du_1 \, du_2
= A_2(v, v_3, \ldots, u_d)(K_{n,(i_j,j)}1_{S(j,j,d,1)})(x_1 - v, x_2 - v, x_3 - u_3, \ldots, x_d - u_d)
+ \int_{-v}^{v} A_2(v, u_2, \ldots, u_d)(\partial_2 K_{n,(i_j,j)}1_{S(j,j,d,1)})(x_1 - v, x_2 - u_2, \ldots, x_d - u_d) \, du_2
+ \int_{I_1} \int_{I_2} A_1(u)(\partial_1 K_{n,(i_j,j)}1_{S(j,j,d,1)})(x-u) \, du_1 \, du_2.
\]

Since \( A_d(v, \ldots, v) = \int_I a = 0 \), repeating this process we get that
\[
\int_I a(u)K_{n,(i_j,j)}(x-u)1_{S(j,j,d,1)}(x-u) \, du = \sum_{k=1}^{d} \int_{I_k} \cdots \int_{I_d} A_k(v, \ldots, v, u_k, \ldots, u_d)
\times (\partial_k K_{n,(i_j,j)}1_{S(j,j,d,1)})(x_1 - v, \ldots, x_1 - v, x_k - u_k, \ldots, x_d - u_d) \, du_k \cdots du_d.
\]

Inequality (7) implies
\[
\left| \int_I a(u)K_{n,(i_j,j)}(x-u)1_{S(j,j,d,1)}(x-u) \, du \right| \leq C_p \sum_{k=1}^{d} 2^{K(d/p - k)}
\times \int_{I_1} \cdots \int_{I_d} \prod_{l=1}^{k-1} (x_{i_l} - v - (x_{j_l} - v))^{-1-\beta} \prod_{l=k}^{d-1} (x_{i_l} - u_{i_l} - (x_{j_l} - u_{j_l}))^{-1-\beta}
\times (x_{j_{d-1}} - u_{j_{d-1}})^{\beta(d-1)-2} 1_{S(j,j,d,1)}(x-u) \, du_k \cdots du_d
\leq C_p 2^{K(d/p - k)} 2^{-K(d-k+1)} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l} - 2^{-K})^{-1-\beta} 1_{\{x_{i_l} - x_{j_l} > 2^{-K+1}\}}
\times (x_{j_{d-1}} - 2^{-K-1})^{\beta(d-1)-2} 1_{\{x_{j_{d-1}} > 2^{-K+1}\}}
\]
and
\[
\sum_{(i,j) \in \mathcal{I}} \int_{\mathbb{T}^d} \sup_{n \geq 1} \left| \int_I a(u)K_{n,(i_j,j)}(x-u)1_{S(j,j,d,1)}(x-u) \, du \right|^p \, dx
\leq C_p 2^{Kd} 2^{-Kdp - Kp} \sum_{(i,j) \in \mathcal{I}} \int_{\mathbb{T}^d} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l} - 2^{-K})^{-(1+\beta)p} 1_{\{x_{i_l} - x_{j_l} > 2^{-K+1}\}}
\]


\[ \times (x_{jd-1} - 2^{-K-1})^{(\beta(d-1) - 2)p} 1_{\{x_{jd-1} > 2^{-K+1}\}} \, dx \]

\[ \leq C_p 2^{Kd} 2^{-Kp} \sum_{(i,j) \in I} 2^{-K(1-(1+\beta)p)(d-1)} 2^{-K(1-(2-\beta(d-1))p)} \]

\[ \leq C_p , \]

whenever \( 1 - (1 + \beta)p < 0, 1 - (2 - \beta(d - 1))p < 0 \) and \( \beta < 2/(d-1) \); in other words

\[ p > \frac{d}{d+1} , \]

which proves the theorem. \( \square \)

**Corollary 1.** If \( f \in L_1(\mathbb{I}^d) \) then

\[ \lim_{n \to \infty} \sigma_n f = f \quad a.e. \]

**Proof.** Since the trigonometric polynomials are dense in \( L_1(\mathbb{I}^d) \), the corollary follows from Theorem 3 and the usual density argument due to Marcinkiewicz and Zygmund [13]. \( \square \)

### 6. \( \theta \)-summation and conjugate functions

Now we introduce a general summability method, the so-called \( \theta \)-summability. In what follows the following conditions are always supposed.

\[
\begin{aligned}
&\text{The support of } \theta \text{ is } [-c, c)(0 < c \leq \infty), \theta \text{ is even and continuous, } \theta(0) = 1, \\
&\sum_{k=0}^{\infty} k^d \left| \Delta_1 \theta \left( \frac{k}{n} \right) \right| < \infty, \lim_{t \to \infty} t^d \theta(t) = 0, \\
&\theta'' \text{ is twice continuously differentiable on } (0, c), \\
&\theta'' \neq 0 \text{ except of finitely many points and finitely many intervals}, \\
&\lim_{t \to 0^+} t \theta'(t) \in \mathbb{R}, \lim_{t \to c^-} t \theta'(t) \in \mathbb{R}, \lim_{t \to \infty} t \theta'(t) = 0,
\end{aligned}
\]

(10)

where

\[ \Delta_1 \theta \left( \frac{k}{n} \right) := \theta \left( \frac{k}{n} \right) - \theta \left( \frac{k+1}{n} \right) \]

is the first difference. This implies by Abel rearrangement that

\[ \sum_{j \in \mathbb{Z}^d} \theta \left( \frac{|j|}{n} \right) \leq C \sum_{k=0}^{\infty} k^d \left| \theta \left( \frac{k}{n} \right) \right| \leq C \sum_{k=0}^{\infty} k^d \left| \Delta_1 \theta \left( \frac{k}{n} \right) \right| < \infty. \]

The \( \ell_1-\theta \)-means of \( f \in L_1(\mathbb{I}^d) \) are given by

\[ \sigma_n^\theta f(x) := \sum_{j \in \mathbb{Z}^d} \theta \left( \frac{|j|}{n} \right) \hat{f}(j)e^{ij \cdot x} = \int_{\mathbb{I}^d} f(x - u) K_n^\theta(u) \, du = \sum_{k=0}^{\infty} \Delta_1 \theta \left( \frac{k}{n} \right) s_k f(x), \]

where

\[ K_n^\theta(u) := \sum_{j \in \mathbb{Z}^d} \theta \left( \frac{|j|}{n} \right) e^{ij \cdot u} = \sum_{j \in \mathbb{Z}^d} \sum_{k \geq |j|} \Delta_1 \theta \left( \frac{k}{n} \right) e^{ij \cdot u} = \sum_{k=0}^{\infty} \Delta_1 \theta \left( \frac{k}{n} \right) D_k(u). \]
Again, $|K_n^\theta| \leq Cn^d$. If $\theta(t) = \max((1 - |t|), 0)$ then we get the Fejér means. Different types of $\theta$-summation have been considered in a great number of papers and books, such as [4,19,9,10, 20,17].

For a distribution

$$f \sim \sum_{n \in \mathbb{Z}^d} \hat{f}(n)e^{in \cdot x}$$

the conjugate distributions are defined by

$$\tilde{f}^{(i)} \sim \sum_{n \in \mathbb{Z}^d} -\frac{n_i}{\|n\|_2} \hat{f}(n)e^{in \cdot x} \quad (i = 1, \ldots, d).$$

As is well known, if $f \in L_1(\mathbb{T}^d)$ then the conjugate functions $\tilde{f}^{(i)}$ do exist almost everywhere, but they are not integrable in general (see [15] or [20]).

The conjugate $\theta$-means and conjugate maximal operator of a distribution $f$ are introduced by

$$\tilde{\sigma}_{n}^{(i);\theta} f(x) := \tilde{f}^{(i)} * K_n, \quad (i = 1, \ldots, d)$$

and

$$\tilde{\sigma}_{\ast}^{(i);\theta} f := \sup_{n \geq 1} |\tilde{\sigma}_{n}^{(i);\theta} f|.$$ 

We use the notation $\tilde{f}^{(0)} = f$, $\tilde{\sigma}_{n}^{(0);\theta} f = \sigma_n f$ and $\tilde{\sigma}_{\ast}^{(0);\theta} f = \sigma_\ast f$. We have proved in [21] that if an inequality (e.g. Theorems 1–3) holds for the Fejér means, then it holds for the $\theta$-means as well. These inequalities can be extended to the conjugate means $(i = 1, \ldots, d)$ with methods used in [20, pp. 220–221]. The details are left to the reader.

**Theorem 4.** If $i = 0, 1, \ldots, d$ then

$$\|\tilde{\sigma}_{\ast}^{(i);\theta} f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p(\mathbb{T}^d))$$

and

$$\|\tilde{\sigma}_{n}^{(i);\theta} f\|_{H_p} \leq C_p \|f\|_{H_p} \quad (n \geq 1, f \in H_p(\mathbb{T}^d))$$

for all $\frac{d}{d+1} < p < \infty$. In particular, if $f \in L_1(\mathbb{T}^d)$ then

$$\lambda(\tilde{\sigma}_{n}^{(i);\theta} f > \rho) \leq \frac{C}{\rho} \|f\|_1 \quad (\rho > 0).$$

**Corollary 2.** If $i = 0, 1, \ldots, d$ and $f \in L_1(\mathbb{T}^d)$ then

$$\lim_{n \to \infty} \tilde{\sigma}_{n}^{(i);\theta} f = \tilde{f}^{(i)} \quad a.e.$$

Moreover, if $f \in H_p(\mathbb{T}^d)$ with $d/(d + 1) < p < \infty$ then this convergence holds in $H_p(\mathbb{T}^d)$ norm.

7. Applications to various summability methods

In this section we consider some summability methods as special cases of the $\ell_1$–$\theta$-summation. It is easy to see that all the following examples satisfy (10).
Example 1 (Fejér Summation). Let
\[ \theta(t) := \begin{cases} 1 - |t| & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1. \end{cases} \]

Example 2 (de la Vallée Poussin Summation). Let
\[ \theta(t) = \begin{cases} 1 & \text{if } |t| \leq 1/2 \\ -2|t| + 2 & \text{if } 1/2 < |t| \leq 1 \\ 0 & \text{if } |t| > 1. \end{cases} \]

Example 3 (Jackson–de la Vallée Poussin Summation). Let
\[ \theta(t) = \begin{cases} 1 - 3t^2/2 + 3|t|^3/4 & \text{if } |t| \leq 1 \\ (2 - |t|)^3/4 & \text{if } 1 < |t| \leq 2 \\ 0 & \text{if } |t| > 2. \end{cases} \]

Example 4. Let \( 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m \) and \( \beta_0, \ldots, \beta_m \) (\( m \in \mathbb{N} \)) be real numbers, \( \beta_0 = 1, \beta_m = 0 \). Suppose that \( \theta \) is even, \( \theta(\alpha_j) = \beta_j \) (\( j = 0, 1, \ldots, m \)), \( \theta(t) = 0 \) for \( t \geq \alpha_m \), \( \theta \) is a polynomial on the interval \( [\alpha_{j-1}, \alpha_j] \) (\( j = 1, \ldots, m \)).

Example 5 (Rogosinski Summation). Let
\[ \theta(t) = \begin{cases} \cos \pi t/2 & \text{if } |t| \leq 1 + 2j \\ 0 & \text{if } |t| > 1 + 2j \end{cases} \quad (j \in \mathbb{N}). \]

Example 6 (Weierstrass Summation). Let \( \theta(t) = e^{-|t|^\gamma} \) for some \( 1 \leq \gamma < \infty \). Note that if \( \gamma = 1 \) then we obtain the Abel means.

Example 7. \( \theta(t) = e^{-(1+|t|^q)^\gamma} \) (\( t \in \mathbb{R}, 1 \leq q < \infty, 0 < \gamma < \infty \)).

Example 8 (Picard and Bessel Summations). \( \theta(t) = (1 + |t|^\gamma)^{-\alpha} \) (\( 0 < \alpha < \infty, 1 \leq \gamma < \infty, \alpha \gamma > d \)).

Example 9 (Riesz Summation). Let
\[ \theta(t) = \begin{cases} (1 - |t|^\gamma)^\beta & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases} \]
for some \( 1 \leq \beta, \gamma < \infty \).

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References

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