Graph-Theoretic Determination of the Nonlinear Zeros at Infinity: Computational Results

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1 Introduction

Recently a new "linear algebraic" framework for the analysis of nonlinear systems has been introduced by Di Benedetto et al. [1]. This approach is centered around the study of a finite chain of ordinary vector spaces consisting of differentials of functions constructed from the output of a nonlinear system. Moreover, the work of Di Benedetto et al. is largely inspired by the differential algebraic approach of Fliess [2] which has enabled a fundamentally new understanding of system theory.

A basic concept in the new algebraic framework is the notation of the *zero structure at infinity* of the system. It is well-known that this concept is very useful, in the class of linear systems, for tackling such problems as noninteracting control [3], disturbances decoupling [4] and model matching [5]. The properties of the abstract algebraic definition of the infinite zeros of a nonlinear system are more consistent to the linear situation than the properties of an other definition of Nijmeijer and Schumacher [6] which is based on the geometric approach.

The nonlinear infinite zero structure can easily be obtained from the dimensions of the vector spaces of output differentials. Recently, it has been proved (see [7]) that the relevant parts of these vector spaces can be formed from the input-output paths of an associated weighted directed graph of the nonlinear system. A great number of formal computations which normally are necessary to construct these vector spaces can be avoid by using this property.

The aim of this paper is to give some computational results [8] obtained by two programs using the language MACSYMA for formal calculus. The first one computes the vector spaces in a conventional way by formal differentiations of the outputs. The second one makes use of the results in [7].

2 Notation and Preliminaries

Consider a nonlinear control system $\sum$ of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) = f(x) + g(x,u) \quad (1)$$
$$y(t) = h(x(t)) = Cx(t) \quad (2)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, and $f(\cdot)$ and the columns of $g(\cdot)$ are *meromorphic* functions of x; that is, they are elements of the fraction field $\mathcal{F}$ of the ring of functions of the variable x which are analytic on a domain $\mathcal{D} \subset \mathbb{R}^n$. In the following we suppress the time argument to simplify the notations.

Remark: The class (1,2) of nonlinear systems under consideration is not really more restricted as the class considered in [1] since every system with a nonlinear output equation $y = h(x)$ can be transformed to an augmented system of the form (1,2) (see [9] page 171) having the same infinite zero structure.

Following [1], we associate to $\sum$ a chain of vector spaces over the field $\mathcal{K}$ of meromorphic functions of x, u, ..., $u^{(n-1)}$ defined as follows. Let us recall, first, that denoting by $v = (v_1, ..., v_j)$ the components of $(x, u, ..., u^{(n-1)})$, the action of the partial derivative operator $\partial/\partial v_i$ on a meromorphic function

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$\eta(\boldsymbol{v}) = p(\boldsymbol{v})/q(\boldsymbol{v})$, where $p(\cdot)$ and $q(\cdot)$ are analytic, is defined, by the usual rule of calculus, as

$$\frac{\partial}{\partial v_i}\frac{p(\boldsymbol{v})}{q(\boldsymbol{v})} := \frac{q(\boldsymbol{v})\frac{\partial}{\partial v_i}p(\boldsymbol{v}) - p(\boldsymbol{v})\frac{\partial}{\partial v_i}q(\boldsymbol{v})}{q^2(\boldsymbol{v})}. \tag{3}$$

Then, the differential of η is given by

$$d\eta(\boldsymbol{v}) = \sum_{i=1}^{j} \frac{\partial \eta(\boldsymbol{v})}{\partial v_i} dv_i. \tag{4}$$

The components of the time derivatives of the output of $\sum$

$$\dot{\boldsymbol{y}} = \dot{\boldsymbol{y}}(\boldsymbol{x},\boldsymbol{u}) = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}] \tag{5}$$

$$\vdots$$

$$\boldsymbol{y}^{(k+1)} = \boldsymbol{y}^{(k+1)}(\boldsymbol{x},\boldsymbol{u},...,\boldsymbol{u}^{(k)}) \tag{6}$$

$$= \frac{\partial \boldsymbol{y}^{(k)}}{\partial \boldsymbol{x}}[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}] + \sum_{i=0}^{k-1} \frac{\partial \boldsymbol{y}^{(k)}}{\partial \boldsymbol{u}^{(i)}} \boldsymbol{u}^{(i+1)} \tag{7}$$

are meromorphic functions of $\boldsymbol{x},\boldsymbol{u},...,\boldsymbol{u}^{(k)}$, whose components are elements of $\mathcal{K}$.

Let $\mathcal{E}$ denote the vector space spanned over $\mathcal{K}$ by $\{d\boldsymbol{x}, d\boldsymbol{u}, ..., d\boldsymbol{u}^{(n-1)}\}$. One defines the subspaces $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_n$ of $\mathcal{E}$ by

$$\begin{aligned} \mathcal{E}_0 &:= \text{span } \{d\boldsymbol{x}\} \\ &\vdots \\ \mathcal{E}_n &:= \text{span } \{d\boldsymbol{x}, d\dot{\boldsymbol{y}}, ..., d\boldsymbol{y}^{(n)}\} \end{aligned} \tag{8}$$

and the associated list of dimensions $\rho_0 \le \cdots \le \rho_n$ by

$$\rho_k := \dim \mathcal{E}_k. \tag{9}$$

By using this list of dimensions the infinite zeros or structure at infinity can be defined as follows:

Definition 1 [10]
The number σ_k of zeros at infinity of order less than or equal to k, $k \ge 1$, is $\sigma_k = \rho_k - \rho_{k-1}$. The structure at infinity is given by the list $\{\sigma_1, \dots, \sigma_n\}$.

Remark: The number of zeros at infinity of order k is then $\sigma_k - \sigma_{k-1}$ and the total number of zeros at infinity σ_n corresponds precisely to the rank of $\sum$ (see [1]) as well as, in a suitable context, to the differential output rank of Fliess [2]. When specialized to the class of linear systems, this abstract algebraic definition agrees with the usual linear notion of the structure at infinity [11].

With the given nonlinear system (1,2) we associate a weighted directed graph (weighted digraph) $\mathcal{G}$ defined by a vertex-set and edge-set as follows (cf. [12]):

The vertex-set is given by m input vertices denoted by $u_1, u_2, ..., u_m$, by n state vertices denoted by $1, 2, ..., n$ and by p output vertices denoted by $y_1, y_2, ..., y_p$.

The edge-set results from the following rules:

- If the state variable x_j really occurs in $f_i(\boldsymbol{x}) + g_i(\boldsymbol{x},\boldsymbol{u})$, i.e. $\partial(f_i + g_i)/\partial x_j \ne 0$, then there exists an edge from vertex j to vertex i with the edge weight $\partial(f_i + g_i)/\partial x_j$.
- If the input variable u_k really occurs in $g_i(\boldsymbol{x},\boldsymbol{u})$, i.e. $\partial g_i/\partial u_k \ne 0$, then there exists an edge from input vertex u_k to state vertex i with the edge weight $\partial g_i/\partial u_k$.
- If the state variable x_i really occurs in $h_k(\boldsymbol{x})$, i.e. $\partial h_k/\partial x_i \ne 0$, then there exists an edge from state vertex i to output vertex y_k with the edge weight $\partial h_k/\partial x_i$.

A (directed) *path* is a sequence of edges $\{e_i, e_j, ...\}$ such that the initial vertex of the succeeding edge is the final vertex of the preceding edge. The edges occurring in the sequence $\{e_i, e_j, ...\}$ are not necessarily distinct. The number of edges contained in the sequence $\{e_i, e_j, ...\}$ is called the *length* of the path. Now the subspaces $\mathcal{E}_k$ can easily be obtained from the digraph of the nonlinear system by using the following theorem:

Theorem 1 [7]
Each path of length l between an input vertex u_k and an output vertex y_j in the digraph of the nonlinear system corresponds to a term $P_{i_l} du_k$ of the differential $dy^{(l-1)}$. The factor P_{i_l} is equal to the product of the edge weights of the path, i.e.

$$P_{i_l} = \frac{\partial y_j}{\partial x_{i_1}} \cdot \frac{\partial \dot{x}_{i_1}}{\partial x_{i_2}} \cdot \frac{\partial \dot{x}_{i_2}}{\partial x_{i_3}} \cdots \frac{\partial \dot{x}_{i_{l-2}}}{\partial x_{i_{l-1}}} \cdot \frac{\partial \dot{x}_{i_{l-1}}}{\partial u_k}. \tag{10}$$

In particular the set of input-output paths of length l contains the paths corresponding to these terms of $dy^{(l-1)}$ which increase the integer $\sigma_{l-1} = \dim \mathcal{E}_{l-1} - \dim \mathcal{E}_{l-2}$.

In [7] a well known example of Fliess [2] was used to explain the application of this theorem. Due to lack of space we refer to the paper of Svaricek [7].

3 Computational Results

In this section we will compare the performance of two programs developed by Wey [8] under the supervision

of the authors, using the symbolic programming language MACSYMA to compute the chain of subspaces $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_k$. The first program computes the complete output differentials by calculating the differentiations of the outputs. The second one needs formal differentiations only for the building of the system graph. Then the relevant parts of the vector spaces are formed from the input-output paths of the system graph. This is realized in the following way:

1. Computation of the Jacobian

$$D(x,u) = \frac{\partial \dot{x}}{\partial x} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \cdots & \frac{\partial \dot{x}_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \dot{x}_n}{\partial x_1} & \cdots & \frac{\partial \dot{x}_n}{\partial x_n} \end{bmatrix}. \tag{11}$$

2. Computation of a matrix

$$E_l = C D^{l-2}(x,u) g(x), \ \ l = 2, \ldots, n+1 . \tag{12}$$

   Then the elements e_{ji} of the matrix E_l are equal to the sum of the edge weights of all paths of length l between the input i and the output j.

Now we consider a nonlinear model of a primary controlled hydraulic rotary drive (cf. Figure 1), which consists of a pressure generating unit (primary unit), a line-network for the distribution of hydraulic energy and a secondary unit, the drive which has to be controlled in the presence of changing load. The behavior of this drive can be described by a nonlinear model (for details see [13]) of order 10:

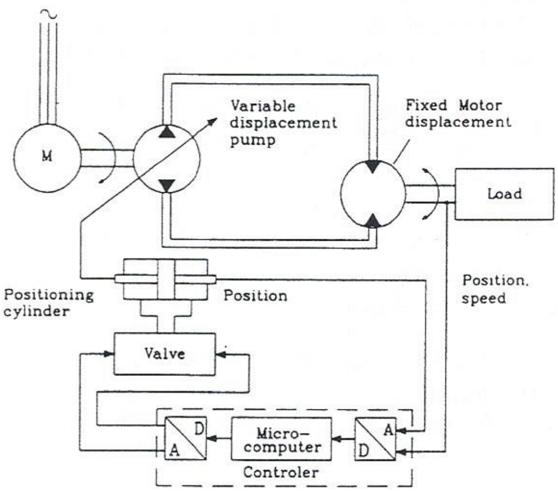

Fig. 1 Equipment plan of a hydraulic system

$$f(x) = \begin{bmatrix} x_2 \\ \frac{1}{\Theta}\left(\frac{V}{2\pi}(x_3 - x_4) - d_M \cdot x_2 \times \right. \\ \left. \times - M_R(x_2) - M_L\right) \\ \frac{E_{oil}(x_3)}{V_{AM}}\left(Q_{AM}(x_5) - K_{\dot{\phi}} \cdot x_2 \times \right. \\ \left. \times - K_{Li}(x_3 - x_4) - K_{Le} \cdot x_3\right) \\ \frac{E_{oil}(x_4)}{V_{BM}}\left(Q_{BM}(x_5) + K_{\dot{\phi}} \cdot x_2 \times \right. \\ \left. \times + K_{Li}(x_3 - x_4) + K_{Le} \cdot x_4\right) \\ x_6 \\ \frac{1}{m_K}\left(A_K(x_7 - x_8) - d_z \cdot x_6 \times \right. \\ \left. \times - F_R(x_6) - F_E\right) \\ \frac{E_{oil}(x_7)}{V_R}\left(Q_A(x_7, x_9) - A \cdot x_6 \times \right. \\ \left. \times - K_{Li}(x_7 - x_8) - K_{Le} \cdot x_7\right) \\ \frac{E_{oil}(x_8)}{V_R}\left(Q_B(x_8, x_9) + A \cdot x_6 \times \right. \\ \left. \times + K_{Li}(x_7 - x_8) + K_{Le} \cdot x_7\right) \\ x_{10} \\ \omega_v^2 \cdot x_9 - 2D_v\omega_v \cdot x_{10} \times \\ \times - F_R(x_{10}) \cdot \omega_v^2 \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \omega_v^2 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with one input (valve voltage) and three outputs (position and speed of the secondary drive, position of the stroke cylinder) and nonlinear functions $F_R, E_{oil}, Q_{A,B}, Q_{AM,BM}$ and M_R. Fig. 2 shows the associated system graph. For the sake of clearness the edge weights has been neglected.

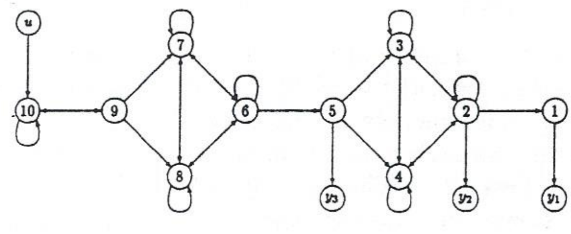

Fig. 2 The unweighted digraph of the hydraulic system

The nonlinear model of the hydraulic drive has one zero at infinity of order 5 so that the subspaces $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_5$ must be computed. On a HP 9000/400 computer the conventional program needs more than 154 seconds to generate the full subspaces. The program based on the described graph-theoretic approach computes the relevant parts of the subspaces in only 11 seconds. More than the half (6 sec.) of this computing time is necessary to calculate the Jacobian (11).

It is clear that the computing time depends on the supremum of the orders of the infinite zeros. In the case, that the considered nonlinear model of a hydraulic drive has only two outputs (position and speed of the secondary drive) the order of the infinite zero increases to 7. Hence, it is necessary to build two further subspaces $\mathcal{E}_6$ and $\mathcal{E}_7$. In this case the 16 MB memory of the computer is not large enough to compute and store the complete subspaces $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_7$. In contrast to this the relevant parts of all these spaces can be calculated with the graph-theoretic approach in less then 10 seconds.

This example shows that the performance of a symbolic computation of the dimensions of the subspaces $\mathcal{E}_i$ can evidently be increased if only the relevant parts of these spaces are considered. In addition to this only such a course of action will actually makes it possible to determine the subspace dimensions in much cases.

4 Conclusions

The infinite zero structure of a nonlinear system is directly related to the dimensions of a chain of ordinary vector spaces consisting of differentials of functions constructed from the output of the system. In this paper we have presented our experience with a recently introduced graph-theoretic approach for the determination of the relevant parts of these spaces. The results for a model of a technical plant show that a MACSYMA-program based on the graph-theoretic approach is not only much faster than a conventional program but makes it first possible to compute the dimensions of these vector spaces for larger nonlinear systems.

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