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Research Paper

An integral representation of the Catalan numbers

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Abstract

In the paper, the authors establish an integral representation of the Catalan numbers, connect the Catalan numbers with the (logarithmically) complete monotonicity, and pose an open problem on the logarithmically complete monotonicity of a function involving ratio of gamma functions.

Keywords: *Catalan number; integral representation; complete monotonicity; logarithmically complete monotonicity; open problem*

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1. Introduction

It is known [22] that, in combinatorics, the Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems of the type, "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" The solution is the Catalan number C_{n-2}. They are named after the Belgian mathematician Eugène Charles Catalan. The first few Catalan numbers C_n for $0 \le n \le 11$ are

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786.

Explicit formulas of C_n for $n \ge 0$ include

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \frac{2^n(2n-1)!!}{(n+1)!} = (-1)^n 2^{2n+1}\binom{\frac12}{n+1} = \frac{1}{n}\binom{2n}{n-1} = {}_2F_1(1-n,-n;2;1)$$

and

$$C_n = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \tag{1}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t, \quad \Re(z) > 0$$

is the classical Euler gamma function and

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{n=0}^\infty \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}$$

is the generalized hypergeometric series defined for complex numbers $a_i \in \mathbb{C}$ and $b_i \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, for positive integers $p, q \in \mathbb{N}$, and in terms of the rising factorials

$$(x)_n = \begin{cases} x(x+1)(x+2)\cdots(x+n-1), & n \ge 1, \\ 1, & n = 0. \end{cases}$$

The asymptotic form for the Catalan numbers is

$$C_x \sim \frac{4^x}{\sqrt{\pi}}\left(x^{-3/2} - \frac{9}{8}x^{-5/2} + \frac{145}{128}x^{-7/2} + \cdots\right).$$

For more information on the Catalan numbers C_n, please also refer to the monographs [1, 2] and the website `https://en.wikipedia.org/wiki/Catalan_number` and references therein.

In this paper, motivated by the explicit expression (1) and by virtue of an integral representation of the gamma function $\Gamma(x)$, we establish an integral representation of the Catalan numbers C_x for $x \ge 0$.

Our main result can be stated as the following theorem.

Theorem 1. *For $x \ge 0$, we have*

$$C_x = \frac{e^{3/2} 4^x (x+1/2)^x}{\sqrt{\pi}\,(x+2)^{x+3/2}} \exp\left[\int_0^\infty \beta(t)\big(e^{-t/2} - e^{-2t}\big)e^{-xt}\,\mathrm{d}\,t\right], \tag{2}$$

where

$$\beta(t) = \frac{1}{t}\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right).$$

2. A remark and an open problem

Before proving Theorem 1, we give a remark on the formula (2) and pose an open problem as follows.

Recall from [4, Chapter XIII], [20, Chapter 1], and [24, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies

$$0 \le (-1)^k f^{(k)}(x) < \infty$$

on I for all $k \ge 0$. Recall from [8, 9] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if

$$0 \le (-1)^k [\ln f(x)]^{(k)} < \infty$$

hold on I for all $k \in \mathbb{N}$. For more information on logarithmically completely monotonic functions, please refer to [10, 11, 14, 19].

The formula (2) can be rearranged as

$$\ln\left[\frac{\sqrt{\pi}\,(x+2)^{x+3/2}}{e^{3/2}4^x(x+1/2)^x}C_x\right] = \int_0^\infty \beta(t)\big(e^{-t/2} - e^{-2t}\big)e^{-xt}\,\mathrm{d}\,t. \tag{3}$$

Since the function $\beta(t)$ is positive on $(0, \infty)$, see [3, 15, 25] and references therein, the right hand side of (3) is a completely monotonic function on $(0, \infty)$. This means that the function

$$\frac{(x+2)^{x+3/2}}{4^x(x+1/2)^x}C_x, \quad x > 0 \tag{4}$$

is logarithmically completely monotonic on $(0, \infty)$. Because any logarithmically completely monotonic function must be completely monotonic, see [11] and references therein, the function (4) is also completely monotonic on $(0, \infty)$.

The function (4) can be rewritten as

$$\frac{(x+2)^{x+3/2}\Gamma(x+1/2)}{(x+1/2)^x\Gamma(x+2)}, \quad x > 0. \tag{5}$$

Hence, the logarithmically complete monotonicity of (4) implies the logarithmically complete monotonicity of (5). The function (5) is a special case of the function

$$\frac{\Gamma(x+a)}{(x+a)^x}\frac{(x+b)^{x+b-a}}{\Gamma(x+b)} \tag{6}$$

for $a, b \in \mathbb{R}$, $a \ne b$, and $x \in (-\min\{a,b\}, \infty)$. It seems that the function (6) does not appear in the expository and survey articles [6, 7, 11, 12, 13] and plenty of references therein. Therefore, we naturally pose an open problem below.

Open Problem 1. *What are the necessary and sufficient conditions on $a, b \in \mathbb{R}$ such that the function (6) is (logarithmically) completely monotonic in $x \in (-\min\{a,b\}, \infty)$?*

3. Proof of Theorem 1

Now we are in a position to give a proof of Theorem 1.

Let

$$h(x) = (2\ln 2)x - \ln\sqrt{\pi} + \ln\Gamma\left(x + \frac{1}{2}\right) - \ln\Gamma(x+2), \quad x > 0. \tag{7}$$

Employing the formula [23, (3.22)]

$$\ln\Gamma(z) = \ln\left(\sqrt{2\pi}\, z^{z-1/2} e^{-z}\right) + \int_0^\infty \beta(t) e^{-zt} \,\mathrm{d}\, t$$

gives

$$\begin{aligned} h(x) &= (2\ln 2)x - \ln\sqrt{\pi} + \ln\left[\sqrt{2\pi}\left(x + \frac{1}{2}\right)^x e^{-(x+1/2)}\right] + \int_0^\infty \beta(t) e^{-(x+1/2)t} \,\mathrm{d}\, t \\ &\quad - \ln\left[\sqrt{2\pi}\,(x+2)^{x+3/2} e^{-(x+2)}\right] - \int_0^\infty \beta(t) e^{-(x+2)t} \,\mathrm{d}\, t \\ &= (2\ln 2)x - \ln\sqrt{\pi} + \frac{3}{2} + \ln\frac{(x+1/2)^x}{(x+2)^{x+3/2}} + \int_0^\infty \beta(t)\left(e^{-t/2} - e^{-2t}\right) e^{-xt} \,\mathrm{d}\, t. \end{aligned}$$

As a result, we acquire

$$\begin{aligned} C_x &= \exp\left\{(2\ln 2)x - \ln\sqrt{\pi} + \frac{3}{2} + \ln\frac{(x+1/2)^x}{(x+2)^{x+3/2}} + \int_0^\infty \beta(t)\left(e^{-t/2} - e^{-2t}\right) e^{-xt} \,\mathrm{d}\, t\right\} \\ &= \frac{e^{3/2} 4^x (x+1/2)^x}{\sqrt{\pi}\,(x+2)^{x+3/2}} \exp\left[\int_0^\infty \beta(t)\left(e^{-t/2} - e^{-2t}\right) e^{-xt} \,\mathrm{d}\, t\right]. \end{aligned}$$

The proof of Theorem 1 is complete.

Remark 1. This paper is a companion of the articles [5, 16, 17, 18] and a slightly revised version of the preprint [21].

References

[1] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974.

[2] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics—A Foundation for Computer Science*, 2nd ed., Addison-Wesley Publishing Company, Reading, MA, 1994.

[3] A.-Q. Liu, G.-F. Li, B.-N. Guo, and F. Qi, *Monotonicity and logarithmic concavity of two functions involving exponential function*, Internat. J. Math. Ed. Sci. Tech. **39** (2008), no. 5, 686–691; Available online at http://dx.doi.org/10.1080/00207390801986841.

[4] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993; Available online at http://dx.doi.org/10.1007/978-94-017-1043-5.

[5] F. Qi, *A logarithmically completely monotonic function involving the gamma function and originating from the Catalan numbers*, ResearchGate Research, available online at http://dx.doi.org/10.13140/RG.2.1.1401.2009.

[6] F. Qi, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. **2010** (2010), Article ID 493058, 84 pages; Available online at http://dx.doi.org/10.1155/2010/493058.

[7] F. Qi, *Bounds for the ratio of two gamma functions: from Gautschi's and Kershaw's inequalities to complete monotonicity*, Turkish J. Anal. Number Theory **2** (2014), no. 5, 152–164; Available online at http://dx.doi.org/10.12691/tjant-2-5-1.

[8] F. Qi and C.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), 603–607; Available online at http://dx.doi.org/10.1016/j.jmaa.2004.04.026.

[9] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 8, 63–72; Available online at http://rgmia.org/v7n1.php.

[10] F. Qi, S. Guo, and B.-N. Guo, *Complete monotonicity of some functions involving polygamma functions*, J. Comput. Appl. Math. **233** (2010), no. 9, 2149–2160; Available online at http://dx.doi.org/10.1016/j.cam.2009.09.044.

[11] F. Qi and W.-H. Li, *A logarithmically completely monotonic function involving the ratio of gamma functions*, J. Appl. Anal. Comput. **5** (2015), no. 4, 626–634; Available online at http://dx.doi.org/10.11948/2015049.

[12] F. Qi and Q.-M. Luo, *Bounds for the ratio of two gamma functions—From Wendel's and related inequalities to logarithmically completely monotonic functions*, Banach J. Math. Anal. **6** (2012), no. 2, 132–158.

[13] F. Qi and Q.-M. Luo, *Bounds for the ratio of two gamma functions: from Wendel's asymptotic relation to Elezović-Giordano-Pečarić's theorem*, J. Inequal. Appl. 2013, **2013**:542, 20 pages; Available online at http://dx.doi.org/10.1186/1029-242X-2013-542.

[14] F. Qi, Q.-M. Luo, and B.-N. Guo, *Complete monotonicity of a function involving the divided difference of digamma functions*, Sci. China Math. **56** (2013), no. 11, 2315–2325; Available online at http://dx.doi.org/10.1007/s11425-012-4562-0.

[15] F. Qi, Q.-M. Luo, and B.-N. Guo, *The function $(b^x - a^x)/x$: Ratio's properties*, In: *Analytic Number Theory, Approximation Theory, and Special Functions*, G. V. Milovanović and M. Th. Rassias (Eds), Springer, 2014, pp. 485–494; Available online at http://dx.doi.org/10.1007/978-1-4939-0258-3_16.

[16] F. Qi, X.-T. Shi, and F.-F. Liu, *An exponential representation for a function involving the gamma function and originating from the Catalan numbers*, ResearchGate Research, available online at http://dx.doi.org/10.13140/RG.2.1.1086.4486.

[17] F. Qi, X.-T. Shi, and F.-F. Liu, *An integral representation, complete monotonicity, and inequalities of the Catalan numbers*, ResearchGate Technical Report, available online at http://dx.doi.org/10.13140/RG.2.1.3754.4806.

[18] F. Qi, X.-T. Shi, and F.-F. Liu, *Several formulas for special values of the Bell polynomials of the second kind and applications*, ResearchGate Technical Report, available online at http://dx.doi.org/10.13140/RG.2.1.3230.1927.

[19] F. Qi, C.-F. Wei, and B.-N. Guo, *Complete monotonicity of a function involving the ratio of gamma functions and applications*, Banach J. Math. Anal. **6** (2012), no. 1, 35–44; Available online at http://dx.doi.org/10.15352/bjma/1337014663.

[20] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions—Theory and Applications*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; Available online at http://dx.doi.org/10.1515/9783110269338.

[21] X.-T. Shi, F.-F. Liu, and F. Qi, *An integral representation of the Catalan numbers*, ResearchGate Research, available online at http://dx.doi.org/10.13140/RG.2.1.2273.6485.

[22] R. Stanley and E. W. Weisstein, *Catalan Number*, From MathWorld–A Wolfram Web Resource; Available online at http://mathworld.wolfram.com/CatalanNumber.html.

[23] N. M. Temme, *Special Functions: An Introduction to Classical Functions of Mathematical Physics*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996; Available online at http://dx.doi.org/10.1002/9781118032572.

[24] D. V. Widder, *The Laplace Transform*, Princeton Mathematical Series, Volume **6**, Princeton University Press, Princeton, N. J., 1941.

[25] S.-Q. Zhang, B.-N. Guo, and F. Qi, *A concise proof for properties of three functions involving the exponential function*, Appl. Math. E-Notes **9** (2009), 177–183.