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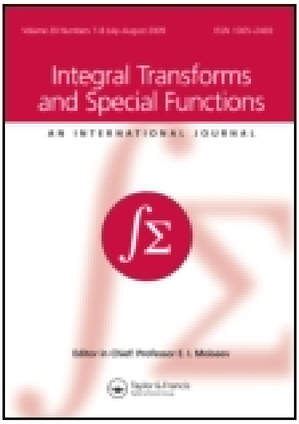

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A class of logarithmically completely monotonic functions related to the gamma function with applications

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A class of logarithmically completely monotonic functions related to the gamma function with applications

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In this article, a necessary and sufficient condition and a necessary condition are established for a class of functions involving the gamma function to be logarithmically completely monotonic on $(0, \infty)$. As applications of the necessary and sufficient condition, several two-sided bounding inequalities for the psi and polygamma functions and the ratio of two gamma functions are derived.

Keywords: necessary and sufficient condition; logarithmically completely monotonic function; inequalities; gamma function; psi (or digamma) function; polygamma function; zero-balanced hypergeometric ${}_pF_{p-1}$ series; applications involving two-sided bounding inequalities

2010 Mathematics Subject Classification: Primary: 26A48, 33B15; Secondary: 26D07, 26D20

1. Introduction

We begin by recalling from the earlier works [3,19] that a positive function f is said to be logarithmically completely monotonic on an open interval I if f has derivatives of all orders on I and

$$(-1)^n[\ln f(x)]^{(n)} \geqq 0 \qquad (n \in \mathbb{N} := \{1, 2, 3, \cdots\}). \tag{1}$$

This type of functions has very close relationships with the Laplace transforms, Stieltjes transforms and infinitely divisible completely monotonic functions. For more detailed information, the interested reader may refer to several earlier investigations (see, for example, [9,11,13,18–21]).

It is well known that the classical (Euler's) gamma function $\Gamma(z)$ is defined, for $\Re(z) > 0$, by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t \qquad \big(\Re(z) > 0\big). \tag{2}$$

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The logarithmic derivative of $\Gamma(z)$, denoted by

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)},$$

is called psi (or digamma) function. Moreover, its derivatives $\psi^{(k)}(z)$ $(k \in \mathbb{N})$ are called the polygamma functions.

For $\alpha \in \mathbb{R}$ and $\beta \geqq 0$, we define a function $f_{\alpha,\beta,\pm 1}(x)$ by

$$f_{\alpha,\beta,\pm 1}(x) = \left(\frac{e^x \Gamma(x+\beta)}{x^{x+\beta-\alpha}}\right)^{\pm 1} \quad \big(x \in (0,\infty)\big). \tag{3}$$

About four decades ago, Kečlić and Vasić [14, Theorem 1] showed that

$$\frac{b^{b-1}}{a^{a-1}} e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}} e^{a-b} \qquad (b > a > 1) \tag{4}$$

and thereby obtained the monotonic properties of the functions

$$\ln f_{\alpha,0,+1}(x) \qquad \text{and} \qquad \ln f_{\alpha,0,+1}(x)$$

on the semi-infinite interval $(1,\infty)$. More recently, Anderson *et al.* [2, Theorem 3.2] proved that the function $f_{1/2,0,+1}(x)$ is decreasing and logarithmically convex from $(0,\infty)$ onto $\big(\sqrt{2\pi},\infty\big)$ and that the function $f_{1,0,+1}(x)$ is increasing and logarithmically concave from $(0,\infty)$ onto $(1,\infty)$ (see also a closely-related earlier work on the subject of zero-balanced hypergeometric series ${}_pF_{p-1}(p \in \mathbb{N})$ by Saigo and Srivastava [23]). Alzer [1, Theorem 2], on the other hand, proved that the function $f_{\alpha,0,+1}(x)$ is decreasing on (c,∞) for $c \geqq 0$ if and only if $\alpha \leqq \frac{1}{2}$ and increasing on (c,∞) if and only if

$$\alpha \geqq \begin{cases} c[\ln c - \psi(c)] & (c > 0) \\ 1 & (c = 0). \end{cases} \tag{5}$$

The necessary and sufficient conditions for the functions

$$f_{\alpha,0,+1}(x) \qquad \text{and} \qquad f_{\alpha,0,-1}(x)$$

to be logarithmically completely monotonic on $(0,\infty)$ were given by Chen and Qi [4, Theorem 2]. Moreover, the function $f_{\alpha,\beta,+1}(x)$ was proved in [4, Theorem 1] to be logarithmically completely monotonic on $(0,\infty)$ if

$$2\alpha \leqq 1 \leqq \beta.$$

Using monotonic properties of the functions

$$f_{1/2,0,+1}(x) \qquad \text{and} \qquad f_{1,0,-1}(x),$$

the inequality (4) was extended in [4, Remark 1] from $b > a > 1$ to $b > a > 0$.

After proving the logarithmically completely monotonic property of the functions

$$f_{1/2,0,+1}(x) \qquad \text{and} \qquad f_{1,0,-1}(x),$$

by making use of Jensen's inequality for convex functions, the upper and lower bounds for the Gürland's ratio were established as follows by Wei *et al.* [24]:

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For positive numbers x and y, the following inequality holds true:

$$\frac{x^{x-1/2}y^{y-1/2}}{\left(\frac{x+y}{2}\right)^{x+y-1}} \leqq \frac{\Gamma(x)\Gamma(y)}{\left[\Gamma\left(\frac{x+y}{2}\right)\right]^2} \leqq \frac{x^{x-1}y^{y-1}}{\left(\frac{x+y}{2}\right)^{x+y-2}} \tag{6}$$

the second member in (6) *being called Gürland's ratio* [15].

Recently, the following new conclusions on logarithmically completely monotonic properties of the function $f_{\alpha,\beta,+1}(x)$ were drawn by Guo *et al.* [12, Theorem 1].

1. *If* $\beta \in (0,\infty)$ and $\alpha \leqq 0$, *then the function* $f_{\alpha,\beta,+1}(x)$ *is logarithmically completely monotonic on* $(0,\infty)$;
2. *If* $\beta \geqq 1$, *then the function* $f_{\alpha,\beta,+1}(x)$ *is logarithmically completely monotonic on* $(0,\infty)$ *if and only if* $\alpha \leqq \frac{1}{2}$.

As *direct* consequences of these results, one can immediately deduce the following assertions. It is deduced immediately that, if x and y are positive numbers with $x \neq y$, then

1. *The following inequality*:

$$I(x,y) > \left[\left(\frac{x}{y}\right)^{\alpha-\beta}\frac{\Gamma(x+\beta)}{\Gamma(y+\beta)}\right]^{1/(x-y)} \tag{7}$$
$$(x>0;\quad y>0;\quad x\neq y;\quad \beta \geqq 1)$$

holds true if and only if $\alpha \leqq \frac{1}{2}$, *where*

$$I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \qquad (a>0;\quad b>0;\quad a\neq b) \tag{8}$$

is the identric or exponential mean;

2. *The inequality* (7) *for* $\beta \in (0,\infty)$ *also holds true if* $\alpha \leqq 0$.

Guo and Srivastava [10] established, among other results, a necessary and sufficient condition for the function $f_{\alpha,\beta,+1}(x)$ to be logarithmically completely monotonic on $(0,\infty)$ for

$$\beta \in \{0\} \cup \left[\frac{1}{2}+\sqrt{\frac{3}{6}},\infty\right).$$

For more information on this topic, one may refer to a survey-cum-expository article by Qi [17], in which a large number of closely-related earlier works are also cited. Some of the most recent investigations on the subjects of this paper include the works by (for example) Chen *et al.* [5,6] (see also [16,22]).

In this paper, we consider the logarithmically completely monotonic property of the function $f_{\alpha,\beta,-1}(x)$ on $(0,\infty)$ and apply our result to derive several two-sided bounding inequalities for the psi and polygamma functions and the ratio of two gamma functions are derived.

2. The main results and their applications

In this section, we first state our main results as follows.

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THEOREM 1 *If the function $f_{\alpha,\beta,-1}(x)$ is logarithmically completely monotonic on $(0,\infty)$, then either*

$$\beta > 0 \qquad \text{and} \qquad \alpha \geqq \max\left\{\beta, \frac{1}{2}\right\}$$

or

$$\beta = 0 \qquad \text{and} \qquad \alpha \geqq 1.$$

THEOREM 2 *If $\beta \geqq \frac{1}{2}$, the necessary and sufficient condition for the function $f_{\alpha,\beta,-1}(x)$ to be logarithmically completely monotonic on $(0,\infty)$ is that $\alpha \geqq \beta$.*

As our first application, the following inequalities are derived by using logarithmically completely monotonic properties of the function $f_{\alpha,\beta,\pm1}(x)$ on $(0,\infty)$.

THEOREM 3

1. *For $k \in \mathbb{N}$, each of the following two-sided inequalities*:

$$\ln x - \frac{1}{x} \leqq \psi(x) \leqq \ln x - \frac{1}{2x} \tag{9}$$

and

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} \leqq (-1)^{k+1}\psi^{(k)}(x) \leqq \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \tag{10}$$

holds true on $(0,\infty)$.

2. *When $\beta > 0$, the following inequalities*:

$$\psi(x+\beta) \leqq \ln x + \frac{\beta}{x} \tag{11}$$

and

$$(-1)^k\psi^{(k-1)}(x+\beta) \geqq \frac{(k-2)!}{x^{k-1}} - \frac{\beta(k-1)!}{x^k} \tag{12}$$

hold true on $(0,\infty)$ for $k \geqq 2$.

3. *When $\beta \geqq \frac{1}{2}$, the following inequalities*:

$$\psi(x+\beta) \geqq \ln x \quad \text{and} \quad (-1)^k\psi^{(k-1)}(x+\beta) \leqq \frac{(k-2)!}{x^{k-1}} \tag{13}$$

hold true on $(0,\infty)$ for $k \geqq 2$.

4. *When $\beta \geqq 1$, the following inequalities*:

$$\psi(x+\beta) \leqq \ln x + \frac{\beta - \frac{1}{2}}{x} \tag{14}$$

and

$$(-1)^k\psi^{(k-1)}(x+\beta) \geqq \frac{(k-2)!}{x^{k-1}} - \frac{\left(\beta - \frac{1}{2}\right)(k-1)!}{x^k} \tag{15}$$

hold true on $(0,\infty)$ for $k \geqq 2$.

As our second application, the following inequalities are derived by using logarithmically convex properties of the function $f_{\alpha,\beta,\pm1}(x)$ on $(0,\infty)$.

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THEOREM 4 *Let $n \in \mathbb{N}$ and $x_k > 0$ $(1 \leqq k \leqq n)$. Suppose also that*

$$\sum_{k=1}^{n} p_k = 1 \qquad (p_k \geqq 0).$$

If either

$$\beta > 0 \qquad \text{and} \qquad \alpha \leqq 0$$

or

$$\beta \geqq 1 \qquad \text{and} \qquad \alpha \leqq \frac{1}{2},$$

then

$$\frac{\prod_{k=1}^{n} [\Gamma(x_k+\beta)]^{p_k}}{\Gamma\left(\sum_{k=1}^{n} p_k x_k + \beta\right)} \geqq \frac{\prod_{k=1}^{n} x_k^{p_k(x_k+\beta-\alpha)}}{\left(\sum_{k=1}^{n} p_k x_k\right)^{\sum_{k=1}^{n} p_k x_k + \beta - \alpha}}. \tag{16}$$

If $\alpha \geqq \beta \geqq \frac{1}{2}$, then the inequality (16) *is reversed.*

As our final application, the following inequality can be derived by using the decreasingly monotonic property of the function $f_{\alpha,\beta,-1}(x)$ on $(0, \infty)$.

THEOREM 5 *If $\alpha \geqq \beta \geqq \frac{1}{2}$, then*

$$I(x,y) < \left[\left(\frac{x}{y}\right)^{\alpha-\beta} \frac{\Gamma(x+\beta)}{\Gamma(y+\beta)}\right]^{1/(x-y)} \qquad (x > 0; \quad y > 0; \quad x \neq y), \tag{17}$$

where $I(x, y)$, defined by (8), *is the identric or exponential mean.*

3. Proofs of the main results

Now, we are in a position to prove our theorems stated in Section 2.

Proof of Theorem 1. Suppose that the function $f_{\alpha,\beta,-1}(x)$ is logarithmically completely monotonic on $(0, \infty)$. Then,

$$[\ln f_{\alpha,\beta,-1}(x)(x)]' = \ln x - \psi(x+\beta) + \frac{\beta-\alpha}{x} \leqq 0, \tag{18}$$

which readily yields

$$\beta - \alpha \leqq x[\psi(x+\beta) - \ln x] \qquad (0 < x < \infty). \tag{19}$$

If $\beta > 0$, then

$$\beta - \alpha \leqq \lim_{x \to 0+} [x\psi(x+\beta) - x\ln x] = 0,$$

that is,

$$\alpha \geqq \beta. \tag{20}$$

Using the following asymptotic formula [7, p. 47]:

$$\psi(x) = \ln x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right) \qquad (x \to \infty), \tag{21}$$

in (19) for $\beta > 0$, we obtain

$$\begin{aligned}\beta - \alpha &\leqq \lim_{x\to\infty} x\left[\ln(x+\beta) - \frac{1}{2(x+\beta)} + O\left(\frac{1}{x^2}\right) - \ln x\right] \\ &= \lim_{x\to\infty}\left[x\ln\left(1+\frac{\beta}{x}\right)\right] - \frac{1}{2} \\ &= \beta\lim_{x\to\infty}\left[\frac{x}{\beta}\ln\left(1+\frac{\beta}{x}\right)\right] - \frac{1}{2} \\ &= \beta - \frac{1}{2},\end{aligned}$$

from which we get

$$\alpha \geqq \frac{1}{2}. \tag{22}$$

By combining (20) and (22), we have

$$\alpha \geqq \max\left\{\beta, \frac{1}{2}\right\} \qquad (\beta > 0). \tag{23}$$

If $\beta = 0$, by considering the equation:

$$f_{\alpha,0,-1}(x) = f_{\alpha,1,-1}(x)$$

and (23), we find that

$$\alpha \geqq \max\left\{1, \frac{1}{2}\right\} = 1.$$

The proof of Theorem 1 is thus completed. ■

Proof of Theorem 2. By Theorem 1, the condition is necessary.

Now, if we differentiate (18) and make use of the following known results [8, p. 884]:

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}t \qquad \big(x \in (0,\infty)\big)$$

and

$$\frac{1}{x^n} = \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-xt}\,\mathrm{d}t \qquad \big(x \in (0,\infty)\big),$$

we find for $n \geqq 2$ that

$$\begin{aligned}&(-1)^n[\ln f_{\alpha,\beta,-1}(x)]^{(n)} \\ &= \frac{(n-2)!}{x^{n-1}} - (-1)^n\psi^{(n-1)}(x+\beta) - \frac{(\beta-\alpha)(n-1)!}{x^n} \\ &= \int_0^\infty t^{n-2}e^{-xt}\,\mathrm{d}t - \int_0^\infty \frac{t^{n-1}}{1-e^{-t}}e^{-(x+\beta)t}\,\mathrm{d}t - (\beta-\alpha)\int_0^\infty t^{n-1}e^{-xt}\,\mathrm{d}t \\ &= \int_0^\infty\left[\alpha - \beta - \frac{1}{t}\left(e^{(1-\beta)t}\frac{t}{e^t-1} - 1\right)\right]t^{n-1}e^{-xt}\,\mathrm{d}t.\end{aligned} \tag{24}$$

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It can be verified that

$$\frac{t}{e^t - 1} < \frac{1}{e^{\frac{1}{2}t}} \qquad \big(t \in (0, \infty)\big). \tag{25}$$

Substituting (25) into (24) leads to

$$(-1)^n[\ln f_{\alpha,\beta,-1}(x)]^{(n)} \geqq \int_0^\infty \left[\alpha - \beta - \frac{e^{(\frac{1}{2}-\beta)t} - 1}{t}\right] t^{n-1} e^{-xt} \mathrm{d}t \qquad (n \geqq 2).$$

Since the function

$$\frac{e^{(\frac{1}{2}-\beta)t} - 1}{t} \qquad \big(t \in (0, \infty)\big) \tag{26}$$

is increasing, if

$$\alpha \geqq \beta \geqq \frac{1}{2},$$

then we get

$$(-1)^n[\ln f_{\alpha,\beta,-1}(x)]^{(n)} \geqq (\alpha - \beta) \int_0^\infty t^{n-1} e^{-xt} \mathrm{d}t \geqq 0 \qquad (n \geqq 2). \tag{27}$$

By using (21), it follows that

$$[\ln f_{\alpha,\beta,-1}(x)(x)]' = \ln\left(1 + \frac{\beta}{x}\right) - \frac{1}{2(x+\beta)} + \frac{\alpha - \beta}{x} + O\left(\frac{1}{x^2}\right) \qquad (x \to \infty).$$

Thus, for all admissible values of α and β, we have

$$\lim_{x\to\infty} [\ln f_{\alpha,\beta,-1}(x)(x)]' = 0. \tag{28}$$

We see from (28) and (27) that, if

$$\alpha \geqq \beta \geqq \frac{1}{2},$$

then

$$[\ln f_{\alpha,\beta,-1}(x)]' \leqq 0. \tag{29}$$

We also observe from (29) and (27) that, if

$$\alpha \geqq \beta \qquad \text{and} \qquad n \in \mathbb{N},$$

then

$$(-1)^n[\ln f_{\alpha,\beta,-1}(x)]^{(n)} \geqq 0 \qquad \left(\beta \geqq \frac{1}{2}\right),$$

which implies that the condition is also sufficient. The proof of Theorem 2 is completed. ■

Proof of Theorem 3. If the function $f_{\alpha,\beta,-1}(x)$ is logarithmically completely monotonic on $(0, \infty)$, then

$$(-1)^k[\ln f_{\alpha,\beta,-1}(x)]^{(k)} \geqq 0 \qquad \big(x \in (0, \infty);\quad k \in \mathbb{N}\big),$$

which is equivalent to the following inequality:

$$\psi(x + \beta) \geqq \ln x + \frac{\beta - \alpha}{x}, \tag{30}$$

and, for $k \geqq 2$,

$$(-1)^k \psi^{(k-1)}(x + \beta) \leqq \frac{(k-2)!}{x^{k-1}} - \frac{(\beta - \alpha)(k-1)!}{x^k} \qquad \big(x \in (0, \infty);\quad k \in \mathbb{N} \setminus \{1\}\big). \tag{31}$$

Hence, Theorem 2 implies the inequalities in (13).

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If $\beta \geqq 1$, Theorem 1 of Guo *et al.* [12] states that the function $f_{\alpha,\beta,+1}(x)$ is logarithmically completely monotonic on $(0,\infty)$ if and only if $\alpha \leqq \frac{1}{2}$. This means that the inequalities in (30) and (31) are reversed, and so the inequalities in (14) and (15) are valid.

If $\beta > 0$ and $\alpha \leqq 0$, Theorem 1 of Guo *et al.* [12] also states that the function $f_{\alpha,\beta,+1}(x)$ is logarithmically completely monotonic on $(0,\infty)$. This means that the inequalities in (30) and (31) are also reversed, and so the inequalities (11) and (12) are valid.

When $\beta = 0$, the functions

$$f_{\alpha,0,+1}(x) \qquad \text{and} \qquad f_{\alpha,0,-1}(x)$$

are logarithmically completely monotonic on $(0,\infty)$ if and only if

$$\alpha \leqq \frac{1}{2} \qquad \text{and} \qquad \alpha \geqq 1,$$

respectively (see [4]), which (by the reasoning as above) imply the two-sided inequalities (9) and (10). The proof of Theorem 3 is thus completed. ■

Proof of Theorem 4. The first conclusion in the aforecited result of Guo *et al.* [12, Theorem 1] asserts that the function $f_{\alpha,\beta,+1}(x)$ is logarithmically convex for $\beta > 0$ and $\alpha \leqq 0$ on $(0,\infty)$. By combining this assertion with Jensen's inequality for convex functions, we get

$$\ln\left(\frac{\exp\left(\sum_{k=1}^{n} p_k x_k\right)\Gamma\left(\sum_{k=1}^{n} p_k x_k + \beta\right)}{\left(\sum_{k=1}^{n} p_k x_k\right)^{\sum_{k=1}^{n} p_k x_k + \beta - \alpha}}\right) \leqq \left(\sum_{k=1}^{n} p_k \ln \frac{\exp(x_k)\Gamma(x_k+\beta)}{x_k^{x_k+\beta-\alpha}}\right) \tag{32}$$

$$\big(n \in \mathbb{N};\quad x_k > 0 \ \ (1 \leqq k \leqq n);\quad \alpha \leqq 0;\quad \beta > 0\big),$$

where

$$\sum_{k=1}^{n} p_k = 1 \qquad (p_k \geqq 0).$$

Rearranging this last inequality (32) would lead to the inequality (16).

The final conclusion in the result of Guo *et al.* [12, Theorem 1] asserts that the function $f_{\alpha,\beta,+1}(x)$ is also logarithmically convex for

$$\beta \geqq 1 \qquad \text{and} \qquad \alpha \leqq \frac{1}{2}$$

on $(0,\infty)$. Hence, the inequality (32) is also valid for

$$\beta \geqq 1 \qquad \text{and} \qquad \alpha \leqq \frac{1}{2}$$

on $(0,\infty)$.

Theorem 2 above implies that the function $f_{\alpha,\beta,+1}(x)$ is logarithmically concave for

$$\alpha \geqq \beta \geqq \frac{1}{2}$$

on $(0,\infty)$. Therefore, the inequality (32) is reversed. Our demonstration of Theorem 4 is thus completed. ■

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Proof of Theorem 5. Theorem 2 implies that the function $f_{\alpha,\beta,-1}(x)$ is decreasing on $(0,\infty)$ if

$$\alpha \geqq \beta \geqq \frac{1}{2}.$$

Hence, we have

$$\frac{e^y\Gamma(y+\beta)}{y^{y+\beta-\alpha}} > \frac{e^x\Gamma(x+\beta)}{x^{x+\beta-\alpha}} \qquad (y > x > 0),$$

which can be rearranged as follows:

$$\frac{\Gamma(y+\beta)}{\Gamma(x+\beta)} > e^{x-y}\frac{y^{y+\beta-\alpha}}{x^{x+\beta-\alpha}} \quad (y > x > 0)$$

or

$$\left[\left(\frac{y}{x}\right)^{\alpha-\beta}\frac{\Gamma(y+\beta)}{\Gamma(x+\beta)}\right]^{1/(y-x)} > \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)} \qquad (y > x > 0). \tag{33}$$

This last inequality (33) is obviously equivalent to (17). For $x > y > 0$, the conclusion is the same. The proof of Theorem 5 is now completed. ■

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References

[1] H. Alzer, *On some inequalities for the gamma and psi functions*, Math. Comput. 66 (1997), pp. 373–389.
[2] G.D. Anderson, R.W. Barnard, K.C. Richards, M.K. Vamanamurthy, and M. Vuorinen, *Inequalities for zero-balanced hypergeometric functions*, Trans. Amer. Math. Soc. 347 (1995), pp. 1713–1723.
[3] R.D. Atanassov and U.V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. 41 (1988), pp. 21–23.
[4] C.-P. Chen and F. Qi, *Logarithmically completely monotonic functions relating to the gamma function*, J. Math. Anal. Appl. 321 (2006), pp. 405–411.
[5] C.-P. Chen and H.M. Srivastava, *Some inequalities and monotonicity properties associated with the gamma and psi functions and the Barnes G-function*, Integral Transforms Spec. Funct. 22 (2011), pp. 1–15.
[6] C.-P. Chen, H.M. Srivastava, L. Li, and S. Manyama, *Inequalities and monotonicity properties for the psi* (*or digamma*) *function and estimates for the Euler–Mascheroni constant*, Integral Transforms Spec. Funct. 22 (2011), pp. 681–693.
[7] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill Book Company, New York, Toronto and London, 1953.
[8] I.S. Gradshteyn and I.M. Ryzhik (eds.), *Tables of Integrals, Series, and Products* (Corrected and Enlarged edition prepared by A. Jeffrey), 6th ed., Academic Press, New York and London, 2000.
[9] S. Guo and F. Qi, *A class of logarithmically completely monotonic functions associated with the gamma function*, J. Comput. Appl. Math. 224 (2009), pp. 127–132.
[10] S. Guo and H.M. Srivastava, *A class of logarithmically completely monotonic functions*, Appl. Math. Lett. 21 (2008), pp. 1134–1141.
[11] S. Guo, F. Qi, and H.M. Srivastava, *Necessary and sufficient conditions for two classes of functions to be logarithmically completely monotonic*, Integral Transforms Spec. Funct. 18 (2007), pp. 819–826.
[12] S. Guo, F. Qi, and H.M. Srivastava, *Supplements to a class of logarithmically completely monotonic functions associated with the gamma function*, Appl. Math. Comput. 197 (2008), pp. 768–774.
[13] R.A. Horn, *On infinitely divisible matrices, kernels, and functions*, Z. Wahrscheinlichkeitstheorie Verw. Geb. 8 (1967), pp. 219–230.
[14] J.D. Kečlić and P.M. Vasić, *Some inequalities for the gamma function*, Publ. Inst. Math. (Beograd) (N.S.) 11 (1971), pp. 107–114.
[15] M. Merkle, *Gürland's ratio for the gamma function*, Comput. Math. Appl. 49 (2005), pp. 389–406.
[16] A. Prabhu and H.M. Srivastava, *Some limit formulas for the gamma and psi* (*or digamma*) *functions at their singularities*, Integral Transforms Spec. Funct. 22 (2011), pp. 587–592.

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[17] F. Qi, *Three classes of logarithmically completely monotonic functions involving gamma and psi functions*, Integral Transforms Spec. Funct. 18 (2007), pp. 503–509.
[18] F. Qi, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. 2010 (2010), pp. 1–84 , Article ID 493058.
[19] F. Qi and C.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. 296 (2004), pp. 603–607.
[20] F. Qi and B.-N. Guo, *Necessary and sufficient conditions for functions involving the tri- and tetra-gamma functions to be completely monotonic*, Adv. Appl. Math. 44 (2010), pp. 71–83.
[21] F. Qi, B.-N. Guo, and C.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Austral. Math. Soc. 80 (2006), pp. 81–88.
[22] Th.M. Rassias and H.M. Srivastava (eds.), *Analytic and Geometric Inequalities and Applications*, Series on Mathematics and its Applications, Vol. 478, Kluwer Academic Publishers, Dordrecht, Boston and London, 1999.
[23] M. Saigo and H.M. Srivastava, *The behavior of the zero-balanced hypergeometric series* $_pF_{p-1}$ *near the boundary of its convergence region*, Proc. Amer. Math. Soc. 110 (1990), pp. 71–76.
[24] Y.-J. Wei, S.-L. Zhang, and C.-P. Chen, *Logarithmically completely monotonic functions and Gürland's ratio for the gamma function*, Adv. Stud. Contemp. Math. 15 (2007), pp. 253–257.

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