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COMPLETE MONOTONICITY OF A FUNCTION INVOLVING THE DIVIDED DIFFERENCE OF PSI FUNCTIONS

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Abstract

In the paper, necessary and sufficient conditions are presented for a function involving the divided difference of the psi function to be completely monotonic and for a function involving the ratio of two gamma functions to be logarithmically completely monotonic. From these, some double inequalities are derived for bounding polygamma functions, divided differences of polygamma functions and the ratio of two gamma functions.

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1. Introduction

A function f is called completely monotonic on an interval $I \subseteq \mathbb{R}$ if f has derivatives of all orders on I and

$$(-1)^k f^{(k)}(x) \ge 0 \tag{1.1}$$

holds for all $k \ge 0$ on I. For our own convenience, in what follows, the class of completely monotonic functions on I is denoted by $\mathcal{C}[I]$. The class of completely monotonic functions may be characterized by [14, p. 161, Theorem 12b] which reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t), \tag{1.2}$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. This means that $f \in \mathcal{C}[(0, \infty)]$ if and only if f is a Laplace transform of the measure μ.

A function f is said to be logarithmically completely monotonic on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on I and its logarithm $\log f$ satisfies

$$(-1)^k (\log f(x))^{(k)} \ge 0 \tag{1.3}$$

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for $k \in \mathbb{N}$ on I. In what follows, the set of all logarithmically completely monotonic functions on I will be denoted by $\mathcal{L}[I]$. The logarithmically completely monotonic functions on $(0,\infty)$ are characterized in [1] as the infinitely divisible completely monotonic functions studied in [4].

The inclusive relationship $\mathcal{L}[I] \subset C[I]$ has been proved in several papers. For detailed information, please refer to [6, Section 1.5], [9, Section 1.3], [10, Section 1] and closely related references therein. Furthermore, it was discovered in [1] that every Stieltjes transform belongs to $\mathcal{L}[(0,\infty)]$, where a function f defined on $(0,\infty)$ is called a Stieltjes transform if it can be expressed in the form

$$f(x) = a + \int_0^\infty \frac{1}{s+x} d\mu(s) \tag{1.4}$$

for some nonnegative number a and some nonnegative measure μ on $[0,\infty)$ satisfying $\int_0^\infty \frac{1}{1+s} d\mu(s) < \infty$. For more information on this topic, please refer to [11].

The classical Euler's gamma function may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{1.5}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi function, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions. The special functions $\Gamma(x)$, $\psi(x)$ and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are fundamental and important and have many applications in mathematical sciences.

In [7, Thorem 1.3], the following necessary and sufficient conditions were established: The function

$$\psi(x) - \log x + \frac{\alpha}{x} \tag{1.6}$$

belongs to $C[(0,\infty)]$ if and only if $\alpha \ge 1$, and so is the negative of (1.6) if and only if $\alpha \le \frac{1}{2}$. For more information on equivalences of these necessary and sufficient conditions, please refer to [2, 5], [8, pp.1977–1978, Section 1.5], and the review articles [6, 9] and plenty of references cited therein.

In order to alternatively verify the monotonicity and convexity of the function

$$\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)} - x \tag{1.7}$$

for $x \in (-\alpha,\infty)$, where s and t are real numbers and $\alpha = \min\{s,t\}$, the following complete monotonicity of the divided difference of the psi functions was discovered in [5, 8]: For real numbers s and t and $\alpha = \min\{s,t\}$, the function

$$\delta_{s,t}(x) = \begin{cases} \dfrac{\psi(x+t)-\psi(x+s)}{t-s} - \dfrac{2x+s+t+1}{2(x+s)(x+t)}, & s \ne t \\ \psi'(x+s) - \dfrac{1}{x+s} - \dfrac{1}{2(x+s)^2}, & s = t \end{cases} \tag{1.8}$$

for $|t-s| < 1$ and its negative $-\delta_{s,t}(x)$ for $|t-s| > 1$ belong to $C[(-\alpha,\infty)]$. For the history, background, and recent developments of the study of the function (1.7), please refer to [3, 8], [6, Section 3.9, Section 3.20.1 and Section 6.1] and closely related references therein.

Now we generalize the function $\delta_{s,t}(x)$ in (1.8) by introducing a parameter λ as follows. For real numbers s and t, define

$$\delta_{s,t;\lambda}(x) = \begin{cases} \dfrac{\psi(x+t)-\psi(x+s)}{t-s} - \dfrac{2x+s+t+2\lambda}{2(x+s)(x+t)}, & s \neq t \\ \psi'(x+s) - \dfrac{1}{x+s} - \dfrac{\lambda}{(x+s)^2}, & s = t \end{cases} \tag{1.9}$$

on $(-\alpha, \infty)$, where $\lambda \in \mathbb{R}$ and $\alpha = \min\{s, t\}$. It is clear from (1.8) and (1.9) that $\delta_{s,t;1/2}(x) = \delta_{s,t}(x)$.

Motivating both by the necessary and sufficient conditions for the function (1.6) to belong to $C[(0, \infty)]$ and by the complete monotonicity of the function (1.8), we naturally pose a question: What are the necessary and sufficient conditions such that the function (1.9) belongs to $C[(-\alpha, \infty)]$? This question is answered by our Theorem 1.1 below.

THEOREM 1.1. *Let s and t be real numbers and let $\alpha = \min\{s, t\}$.*

1. *For $|t-s| < 1$,*
   (a) $\delta_{s,t;\lambda}(x) \in C[(-\alpha, \infty)]$ *if and only if* $\lambda \le \frac{1}{2}$;
   (b) $-\delta_{s,t;\lambda}(x) \in C[(-\alpha, \infty)]$ *if and only if* $\lambda \ge 1 - \frac{|t-s|}{2}$;
2. *For $|t-s| > 1$,*
   (a) $\delta_{s,t;\lambda}(x) \in C[(-\alpha, \infty)]$ *if and only if* $\lambda \le 1 - \frac{|t-s|}{2}$;
   (b) $-\delta_{s,t;\lambda}(x) \in C[(-\alpha, \infty)]$ *if and only if* $\lambda \ge \frac{1}{2}$;
3. *For $|t-s| = 1$,*
   (a) $\delta_{s,t;\lambda}(x) \in C[(-\alpha, \infty)]$ *if and only if* $\lambda < \frac{1}{2}$;
   (b) $-\delta_{s,t;\lambda}(x) \in C[(-\alpha, \infty)]$ *if and only if* $\lambda > \frac{1}{2}$;
   (c) *the function* $\delta_{s,t;\lambda}(x)$ *is identically zero if and only if* $\lambda = \frac{1}{2}$.

As a direct application of Theorem 1.1, the logarithmically complete monotonicity results of a function involving the ratio of two gamma functions can be deduced as follows.

THEOREM 1.2. *For real numbers s and t, define*

$$H_{s,t;\lambda}(x) = \begin{cases} \dfrac{(x+t)^{\lambda/(t-s)-1/2}}{(x+s)^{\lambda/(t-s)+1/2}} \left(\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)}, & s \neq t \\ \dfrac{1}{x+t} \exp\left(\psi(x+t) + \dfrac{\lambda}{x+t}\right), & s = t \end{cases} \tag{1.10}$$

on $(-\alpha, \infty)$, where $\lambda \in \mathbb{R}$ and $\alpha = \min\{s, t\}$. The following conclusions are valid:

1. *For $|t-s| < 1$,*
   (a) $H_{s,t;\lambda}(x) \in \mathcal{L}[(-\alpha, \infty)]$ *if and only if* $\lambda \ge 1 - \frac{|t-s|}{2}$;
   (b) $(H_{s,t;\lambda}(x))^{-1} \in \mathcal{L}[(-\alpha, \infty)]$ *if and only if* $\lambda \le \frac{1}{2}$;

2. *For* $|t-s|>1$,
   (a) $H_{s,t;\lambda}(x)\in\mathcal{L}[(-\alpha,\infty)]$ *if and only if* $\lambda\ge\frac{1}{2}$;
   (b) $(H_{s,t;\lambda}(x))^{-1}\in\mathcal{L}[(-\alpha,\infty)]$ *if and only if* $\lambda\le 1-\frac{|t-s|}{2}$;
3. *For* $|t-s|=1$,
   (a) $H_{s,t;\lambda}(x)\in\mathcal{L}[(-\alpha,\infty)]$ *if and only if* $\lambda>\frac{1}{2}$;
   (b) $(H_{s,t;\lambda}(x))^{-1}\in\mathcal{L}[(-\alpha,\infty)]$ *if and only if* $\lambda<\frac{1}{2}$;
   (c) *the function* $H_{s,t;\lambda}(x)$ *identically equals* 1 *on* $(-\alpha,\infty)$ *if and only if* $\lambda=\frac{1}{2}$.

As consequences of Theorem 1.1 and Theorem 1.2, the following double inequalities are immediately derived for the polygamma functions, the divided differences of polygamma functions, and the ratio of two gamma functions.

Theorem 1.3. *The following statements are true:*

1. *For* $x>0$, *the double inequality*

$$\beta_1\frac{k!}{x^{k+1}}<(-1)^{k+1}\psi^{(k)}(x)-\frac{(k-1)!}{x^k}<\gamma_1\frac{k!}{x^{k+1}} \tag{1.11}$$

   holds if and only if $\beta_1\le\frac{1}{2}$ *and* $\gamma_1\ge 1$.
2. *Let* a *and* b *be positive numbers and* $k\in\mathbb{N}$.
   (a) *For* $0<|b-a|<1$, *the double inequality*

$$\begin{aligned}\frac{(k-1)!}{2}\left(\frac{1}{a^k}+\frac{1}{b^k}\right)+\beta_2\frac{(k-1)!}{b-a}\left(\frac{1}{a^k}-\frac{1}{b^k}\right)&<\frac{(-1)^{k-1}(\psi^{(k-1)}(b)-\psi^{(k-1)}(a))}{b-a}\\&<\frac{(k-1)!}{2}\left(\frac{1}{a^k}+\frac{1}{b^k}\right)+\gamma_2\frac{(k-1)!}{b-a}\left(\frac{1}{a^k}-\frac{1}{b^k}\right)\end{aligned} \tag{1.12}$$

   holds if and only if $\beta_2\le\frac{1}{2}$ *and* $\gamma_2\ge 1-\frac{|b-a|}{2}$;
   (b) *For* $|b-a|>1$, *the double inequality* (1.12) *is reversed if and only if* $\beta_2\le 1-\frac{|b-a|}{2}$ *and* $\gamma_2\ge\frac{1}{2}$.
3. *Let* a *and* b *be positive numbers and* $k\in\mathbb{N}$.
   (a) *For* $0<|b-a|<1$, *the double inequality*

$$\frac{a^{\beta_3/(b-a)+1/2}}{b^{\beta_3/(b-a)-1/2}}<\left(\frac{\Gamma(b)}{\Gamma(a)}\right)^{1/(b-a)}<\frac{a^{\gamma_3/(b-a)+1/2}}{b^{\gamma_3/(b-a)-1/2}} \tag{1.13}$$

   holds if and only if $\beta_3\ge 1-\frac{|b-a|}{2}$ *and* $\gamma_3\le\frac{1}{2}$;
   (b) *For* $|b-a|>1$, *the double inequality* (1.13) *is reversed if and only if* $\beta_3\le 1-\frac{|b-a|}{2}$ *and* $\gamma_3\ge\frac{1}{2}$.
4. *Let* s *and* t *be real numbers,* $\alpha=\min\{s,t\}$, *and* $x\in(\rho,\infty)\subset(-\alpha,\infty)$.

(a) *For* $0<|t-s|<1$, *the double inequality*

$$\frac{(\rho+t)^{\beta_4/(t-s)-1/2}}{(\rho+s)^{\beta_4/(t-s)+1/2}}\left(\frac{\Gamma(\rho+t)}{\Gamma(\rho+s)}\right)^{1/(t-s)}\frac{(x+s)^{\beta_4/(t-s)+1/2}}{(x+t)^{\beta_4/(t-s)-1/2}}<\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)}$$
$$<\frac{(\rho+t)^{\gamma_4/(t-s)-1/2}}{(\rho+s)^{\gamma_4/(t-s)+1/2}}\left(\frac{\Gamma(\rho+t)}{\Gamma(\rho+s)}\right)^{1/(t-s)}\frac{(x+s)^{\gamma_4/(t-s)+1/2}}{(x+t)^{\gamma_4/(t-s)-1/2}} \quad (1.14)$$

holds if and only if $\beta_4\le\frac{1}{2}$ *and* $\gamma_4\ge1-\frac{|t-s|}{2}$;

(b) *For* $|t-s|>1$, *the inequality* (1.14) *is reversed if and only if* $\beta_4\ge\frac{1}{2}$ *and* $\gamma_4\le1-\frac{|t-s|}{2}$.

Remark 1.4. *We remark that taking* $a=x+\frac{1}{2}$ *and* $b=x+1$ *in the right-hand side of* (1.13) *yields*

$$\left(\frac{\Gamma(x+1)}{\Gamma(x+1/2)}\right)^2<\frac{(x+1/2)^{3/2}}{(x+1)^{1/2}}=\left(1-\frac{1}{2(x+1)}\right)^{1/2}\left(x+\frac{1}{2}\right) \quad (1.15)$$

on $(-\frac{1}{2},\infty)$, *which is obviously better than the inequality*

$$\left(\frac{\Gamma(x+1)}{\Gamma(x+1/2)}\right)^2<x+\frac{1}{2} \quad (1.16)$$

on $(-\frac{1}{2},\infty)$. *The inequality* (1.16) *is a long standing upper bound obtained in [12]. For more information about the inequality* (1.16), *see [6, pp. 21–22, Section 3.1].*

Remark 1.5. *There have been some similar but different results to our Theorems 1.2 and 1.3. For details, see the newly published paper [10] or related contents in [6, 9].*

2. Proofs of theorems

Now we are in a position to prove our theorems.

Proof of Theorem 1.1. For $|s-t|=1$, then from (1.9), it is equivalent to discuss the complete monotonicity of the function

$$\delta_{s,s+1;\lambda}(x)=\frac{1-2\lambda}{2(x+s)(x+s+1)},$$

which follows obviously from the fact that the product of finitely many completely monotonic functions is still completely monotonic.

For $s=t$, it is equivalent to discuss the complete monotonicity of the function

$$\psi'(x)-\frac{1}{x}-\frac{\lambda}{x^2}$$

on $(0,\infty)$, which can be derived directly from [2, Theorem 2] and [7, Thorem 1.3] mentioned above.

For $s \neq t$ and $s - t \neq \pm 1$, the function $\delta_{s,t;\lambda}(x)$ can be rewritten as

$$\delta_{s,t;\lambda}(x) = \frac{1}{t-s}\int_s^t \psi'(x+u)du - \frac{1}{2}\left(\left(1 - \frac{2\lambda}{t-s}\right)\frac{1}{x+t} + \left(1 + \frac{2\lambda}{t-s}\right)\frac{1}{x+s}\right).$$

Differentiating consecutively and employing the well-known formulas

$$\psi^{(k)}(x) = (-1)^{k+1}\int_0^\infty \frac{t^k}{1-e^{-t}} e^{-xt} dt \tag{2.1}$$

and

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-xt}dt \tag{2.2}$$

for $k \in \mathbb{N}$ and positive numbers $x > 0$ and $r > 0$ yield

$$\begin{aligned}
\delta^{(k)}_{s,t;\lambda}(x) &= \frac{1}{t-s}\int_s^t \psi^{(k+1)}(x+u)du \\
&\quad - \frac{(-1)^k k!}{2}\left(\left(1 - \frac{2\lambda}{t-s}\right)\frac{1}{(x+t)^{k+1}} + \left(1 + \frac{2\lambda}{t-s}\right)\frac{1}{(x+s)^{k+1}}\right) \\
&= \frac{(-1)^k}{t-s}\int_s^t\int_0^\infty \frac{v^{k+1}}{1-e^{-v}}e^{-(x+u)v}dvdu \\
&\quad - \frac{(-1)^k}{2}\left(\left(1 - \frac{2\lambda}{t-s}\right)\int_0^\infty v^k e^{-(x+t)v}dv + \left(1 + \frac{2\lambda}{t-s}\right)\int_0^\infty v^k e^{-(x+s)v}dv\right) \\
&= (-1)^k \int_0^\infty \left(\frac{1}{t-s}\int_s^t \frac{v}{1-e^{-v}}e^{-uv}du \right.\\
&\quad \left. - \frac{1}{2}\left(\left(1 - \frac{2\lambda}{t-s}\right)e^{-tv} + \left(1 + \frac{2\lambda}{t-s}\right)e^{-sv}\right)\right)v^k e^{-xv}dv \\
&= (-1)^k \int_0^\infty \left(\frac{e^{-sv} - e^{-tv}}{(t-s)(1-e^{-v})} - \frac{e^{-sv} + e^{-tv}}{2} - \frac{\lambda(e^{-sv} - e^{-tv})}{t-s}\right)v^k e^{-xv}dv \\
&= (-1)^k \int_0^\infty \left(\left(\frac{1}{1-e^{-v}} - \lambda\right)\frac{e^{-sv} - e^{-tv}}{t-s} - \frac{e^{-sv} + e^{-tv}}{2}\right)v^k e^{-xv}dv \\
&= (-1)^k \int_0^\infty \left(\frac{1}{1-e^{-v}} - \lambda - \frac{(t-s)(e^{-sv} + e^{-tv})}{2(e^{-sv} - e^{-tv})}\right)\frac{e^{-sv} - e^{-tv}}{t-s}v^k e^{-xv}dv \\
&= (-1)^k \int_0^\infty \left(\frac{1}{1-e^{-v}} - \frac{t-s}{2\tanh((t-s)v/2)} - \lambda\right)\frac{e^{-sv} - e^{-tv}}{t-s}v^k e^{-xv}dv
\end{aligned}$$

for $k \in \{0\} \cup \mathbb{N}$. Therefore, if

$$\begin{aligned}
\lambda &\le \frac{1}{1-e^{-v}} - \frac{t-s}{2\tanh((t-s)v/2)} \\
&= \frac{2e^v \tanh((t-s)v/2) - (t-s)(e^v - 1)}{2(e^v - 1)\tanh((t-s)v/2)} \triangleq \lambda(v, t-s)
\end{aligned} \tag{2.3}$$

for all $v \in (0,\infty)$, then $(-1)^k\delta^{(k)}_{s,t;\lambda}(x) \ge 0$ and $\delta_{s,t;\lambda}(x) \in C[(-\alpha,\infty)]$; if the inequality (2.3) reverses, then $(-1)^k\delta^{(k)}_{s,t;\lambda}(x) \le 0$ and $-\delta_{s,t;\lambda}(x) \in C[(-\alpha,\infty)]$.

Straightforward computation gives

$$\frac{\partial\lambda(v,r)}{\partial v}=\frac{1}{4}r^2\operatorname{csch}^2\Big(\frac{rv}{2}\Big)-\frac{e^v}{(e^v-1)^2}=\frac{1}{v^2}\bigg(\Big(\frac{rv}{2}\Big)^2\operatorname{csch}^2\Big(\frac{rv}{2}\Big)-\Big(\frac{v}{2}\Big)^2\operatorname{csch}^2\Big(\frac{v}{2}\Big)\bigg).$$

Since the function $x\operatorname{csch}x$ is strictly positive and decreasing on $(0,\infty)$, it follows that

$$\frac{\partial\lambda(v,r)}{\partial v}\begin{cases}<0, & \text{if } r>1,\\ >0, & \text{if } 0<r<1.\end{cases}$$

Accordingly, the function $\lambda(v,r)$ is increasing for $0<r<1$ and decreasing for $r>1$ on $(0,\infty)$. Using L'Hôspital's rule yields

$$\lim_{v\to0^+}\lambda(v,r)=\lim_{v\to0^+}\frac{e^v(2\tanh(rv/2)+r(\operatorname{sech}^2(rv/2)-1))}{2e^v\tanh(rv/2)+r(e^v-1)\operatorname{sech}^2(rv/2)}=\frac{1}{2}.$$

It is easy to see that

$$\lim_{v\to\infty}\lambda(v,r)=1-\frac{|t-s|}{2}.$$

Since the function $\lambda(v,r)$ is even with respect to the variable $r\in\mathbb{R}$ with $r\neq0$, for $0<|t-s|<1$, we have

$$\frac{1}{2}<\lambda(v,s-t)<1-\frac{|t-s|}{2}. \tag{2.4}$$

The inequality (2.4) is reversed for $|t-s|>1$. Consequently,

1. the function $\delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha,\infty)$ if either $\lambda\le\frac{1}{2}$ and $0<|t-s|<1$ or $\lambda<1-\frac{|t-s|}{2}$ and $|t-s|>1$;
2. the function $-\delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha,\infty)$ if either $\lambda\ge1-\frac{|t-s|}{2}$ and $0<|t-s|<1$ or $\lambda\ge\frac{1}{2}$ and $|t-s|>1$.

Conversely, if the function $\delta_{s,t;\lambda}(x)$ is completely monotonic, then $\delta_{s,t;\lambda}(x)\ge0$ on $(-\alpha,\infty)$, which can be rearranged as

$$\lambda\le(x+s)(x+t)\bigg(\frac{1}{t-s}\int_s^t\psi'(x+u)du-\frac{1}{2}\Big(\frac{1}{x+s}+\frac{1}{x+t}\Big)\bigg)\triangleq\lambda_{s,t}(x).$$

In the proof of [8, p. 1981, Lemma 2.4], it was cited that

$$\frac{1}{2x^2}-\frac{1}{6x^3}<\frac{1}{x}-\psi'(x+1)<\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5} \tag{2.5}$$

for $x>0$. From the left-hand side inequality in (2.5) it follows that

$$\begin{aligned}\lambda_{s,t}(x)&<(x+s)(x+t)\\&\quad\times\bigg(\frac{1}{t-s}\int_s^t\Big(\frac{1}{x+u}+\frac{1}{2(x+u)^2}+\frac{1}{6(x+u)^3}\Big)du-\frac{1}{2}\Big(\frac{1}{x+s}+\frac{1}{x+t}\Big)\bigg)\\&=\frac{(x+s)(x+t)}{t-s}\ln\frac{x+s}{x+t}-x+\frac{1}{2}-\frac{s+t}{2}+\frac{(x+t)^2-(x+s)^2}{12(t-s)(x+s)(x+t)}\to\frac{1}{2}\end{aligned}$$

as $x \to \infty$. Similarly, using the right-hand side inequality in (2.5) yields

$$\lambda_{s,t}(x) > (x+s)(x+t)\biggl(\frac{1}{t-s}\int_s^t\biggl(\frac{1}{x+u}+\frac{1}{2(x+u)^2}+\frac{1}{6(x+u)^3}-\frac{1}{30(x+u)^5}\biggr)du-\frac{1}{2}\biggl(\frac{1}{x+s}+\frac{1}{x+t}\biggr)\biggr)\to\frac{1}{2}$$

as $x \to \infty$. As a result, we have the limit

$$\lim_{x\to\infty}\lambda_{s,t}(x)=\frac{1}{2}.$$

Since $x\psi(x)=x\psi(x+1)-1$, then

$$\lim_{x\to0^+}(x\psi(x))=-1.$$

From this it follows using a standard argument that

$$\lim_{x\to(-\alpha)^+}\lambda_{s,t}(x)=1-\frac{|s-t|}{2}.$$

Therefore, if $|t-s|<1$, then

$$\lambda\le\lim_{x\to\infty}\lambda_{s,t}(x)=\frac{1}{2}<1-\frac{|s-t|}{2}=\lim_{x\to(-\alpha)^+}\lambda_{s,t}(x)$$

and if $|t-s|>1$, then

$$\lambda\le\lim_{x\to(-\alpha)^+}\lambda_{s,t}(x)=1-\frac{|s-t|}{2}<\lim_{x\to\infty}\lambda_{s,t}(x)=\frac{1}{2}.$$

The necessity for the function $-\delta_{s,t;\lambda}(x)$ to be completely monotonic can be similarly reasoned by repeating the above procedure. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. This follows from the fact that $(\log H_{s,t;\lambda}(x))'=\delta_{s,t;\lambda}(x)$ as defined in Theorem 1.1 and the definition of the logarithmically completely monotonic functions. □

Proof of Theorem 1.3. The double inequalities in (1.11) may be deduced readily from the complete monotonicity of the function $\delta_{0,0;\lambda}(x)$ turned out in Theorem 1.1.

With the help of the complete monotonicity of the function $\delta_{s,t;\lambda}(x)$ for $s\ne t$, it easily follows that the double inequality

$$\beta_2\frac{(k-1)!}{t-s}\biggl(\frac{1}{(x+s)^k}-\frac{1}{(x+t)^k}\biggr)<\frac{(-1)^{k-1}(\psi^{(k-1)}(x+t)-\psi^{(k-1)}(x+s))}{t-s}-\frac{(k-1)!}{2}\biggl(\frac{1}{(x+s)^k}+\frac{1}{(x+t)^k}\biggr)<\gamma_2\frac{(k-1)!}{t-s}\biggl(\frac{1}{(x+s)^k}-\frac{1}{(x+t)^k}\biggr)\qquad(2.6)$$

holds on $(-\alpha,\infty)$ if and only if $\beta_2 \le \frac12$ and $\gamma_2 \ge 1-\frac{|s-t|}{2}$ for $0<|t-s|<1$, and that the double inequality (2.6) is reversed if and only if $\beta_2 \le 1-\frac{|s-t|}{2}$ and $\gamma_2 \ge \frac12$ for $|t-s|>1$. Setting $x+s$ as a and $x+t$ as b in (2.6) produces (1.12).

As early as in 1948, by applying Hölder's integral inequality to the definition (1.5) of the gamma function $\Gamma(x)$ and using the well-known formula $\Gamma(x+1)=x\Gamma(x)$, it was established in [13] that the classical asymptotic relation

$$\lim_{x\to\infty}\frac{\Gamma(x+s)}{x^s\Gamma(x)}=1 \tag{2.7}$$

holds for real s and x. This implies that

$$H_{s,t;\lambda}(x)=\frac{(x+t)^{\lambda/(t-s)}}{(x+s)^{\lambda/(t-s)}}\cdot\frac{x}{\sqrt{(x+s)(x+t)}}\left(x^{s-t}\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)}\to 1 \tag{2.8}$$

as $x\to\infty$ for $s\ne t$. By virtue of Theorem 1.2, when $0<|t-s|<1$, the function $H_{s,t;\lambda}(x)$ is decreasing on $(-\alpha,\infty)$ if and only if $\lambda \ge 1-\frac{|t-s|}{2}$ and it is increasing on $(-\alpha,\infty)$ if and only if $\lambda\le\frac12$. Hence, employing the limit (2.8) implies that the inequality

$$\frac{(x+t)^{\lambda/(t-s)-1/2}}{(x+s)^{\lambda/(t-s)+1/2}}\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)}>1 \tag{2.9}$$

holds on $(-\alpha,\infty)$ if and only if $\lambda\ge 1-\frac{|t-s|}{2}$ and reverses on $(-\alpha,\infty)$ if and only if $\lambda\le\frac12$. Similarly, when $|t-s|>1$, the function $H_{s,t;\lambda}(x)$ is decreasing on $(-\alpha,\infty)$ if and only if $\lambda\ge\frac12$ and it is increasing on $(-\alpha,\infty)$ if and only if $\lambda\le 1-\frac{|t-s|}{2}$. This leads to the conclusion that the reversed version of the inequality (2.9) is valid. Finally, the double inequality

$$\frac{(x+s)^{\beta_3/(t-s)+1/2}}{(x+t)^{\beta_3/(t-s)-1/2}}<\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)}<\frac{(x+s)^{\gamma_3/(t-s)+1/2}}{(x+t)^{\gamma_3/(t-s)-1/2}} \tag{2.10}$$

holds on $(-\alpha,\infty)$ if and only if $\beta_3\ge 1-\frac{|t-s|}{2}$ and $\gamma_3\le\frac12$ for $0<|t-s|<1$; the double inequality (2.10) is reversed if and only if $\beta_3\le 1-\frac{|t-s|}{2}$ and $\gamma_3\ge\frac12$ for $|t-s|>1$. Furthermore, replacing $x+s$ by a and $x+t$ by b reproduces (1.13).

The double inequality (1.14) comes from the fact that the inequality $H_{s,t;\lambda}(\rho)>H_{s,t;\lambda}(x)$ holds on (ρ,∞) if and only if either $\lambda\ge 1-\frac{|t-s|}{2}$ for $0<|t-s|<1$ or $\lambda\ge\frac12$ and $|t-s|\ge 1$, and that it is revered if and only if either $\lambda\le\frac12$ for $0<|t-s|<1$ or $\lambda\le 1-\frac{|t-s|}{2}$ and $|t-s|\ge 1$. The proof of Theorem 1.3 is complete. □

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