U.P.B. Sci. Bull., Series A, Vol. 72, Iss. 3, 2010 ISSN 1223-7027

GENERALIZATIONS OF JORDAN'S INEQUALITY AND CONCERNED RELATIONS

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In the paper, two new Jordan type inequalities are established for bounding the Bessel function, some concerned relations among some recent results are discussed, and several simple applications are presented.

Keywords: Jordan's inequality, Bessel function, gamma function, Kober's inequality, sine integral, hyperbolic sine integral, generalization, extension, relation

MSC2000: 26D05, 26D15, 33B15, 33C10

1. Introduction

Jordan's inequality, an elementary and celebrated inequality related to the sine function, reads that

$$\frac{2}{\pi} \le \frac{\sin x}{x} < 1 \tag{1}$$

for $0 < |x| \le \frac{\pi}{2}$. The equality in (1) is valid if and only if $x = \frac{\pi}{2}$.

Jordan's inequality (1) has been refined, generalized and applied by many mathematicians. For detailed information, please refer to the expository and survey articles [7], especially the newest version [10], and related references therein.

Now let us recall some recent refinements and generalizations of the inequality (1), which will be discussed in this paper. In [7, 9], the following general refinement of Jordan's inequality (1) was presented: For $0 < x \le \frac{\pi}{2}$ and $n \in \mathbb{N}$, the inequality

$$\frac{2}{\pi} + \sum_{k=1}^{n} \alpha_k \left(\pi^2 - 4x^2\right)^k \le \frac{\sin x}{x} \le \frac{2}{\pi} + \sum_{k=1}^{n} \beta_k \left(\pi^2 - 4x^2\right)^k \tag{2}$$

holds with equalities if and only if $x = \frac{\pi}{2}$, where the constants

$$\alpha_k = \frac{(-1)^k}{(4\pi)^k k!} \sum_{i=1}^{k+1} \left(\frac{2}{\pi}\right)^i a_{i-1,k} \sin\left(\frac{k+i}{2}\pi\right) \tag{3}$$

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and

$$\beta_k = \begin{cases} \dfrac{1 - \frac{2}{\pi} - \sum_{i=1}^{n-1} \alpha_i \pi^{2i}}{\pi^{2n}}, & k = n \\ \alpha_k, & 1 \le k < n \end{cases} \tag{4}$$

with

$$a_{i,k} = \begin{cases} (i + k - 1)a_{i-1,k-1} + a_{i,k-1}, & 0 < i \le k \\ 1, & i = 0 \\ 0, & i > k \end{cases} \tag{5}$$

are the best possible.

In [6], the identity

$$\frac{\sin x}{x} = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{(-1)^k R_k}{k!\pi^{2k}} \left(\pi^2 - 4x^2\right)^k, \quad x > 0 \tag{6}$$

was established, where

$$R_k = \sum_{n=k}^{\infty} \frac{(-1)^n n!}{(n-k)!(2n+1)!} \left(\frac{\pi}{2}\right)^{2n} \tag{7}$$

satisfying $(-1)^k R_k > 0$, from which one can obtain the inequality

$$\frac{\sin x}{x} \ge \frac{2}{\pi} + \sum_{k=1}^{n} \frac{(-1)^k R_k}{k!\pi^{2k}} \left(\pi^2 - 4x^2\right)^k \tag{8}$$

for $n \in \mathbb{N}$ and $x \in \left(0, \frac{\pi}{2}\right]$.

In [12], the inequality

$$a_{N+1}\left(\pi^2 - 4x^2\right)^{N+1} \le \frac{\sin x}{x} - P_{2N}(x) \le \frac{1 - \sum_{n=0}^{N} a_n \pi^{2n}}{\pi^{2(N+1)}} \left(\pi^2 - 4x^2\right)^{N+1} \tag{9}$$

was presented, where $x \in \left(0, \frac{\pi}{2}\right]$,

$$P_{2N}(x) = \sum_{n=0}^{N} a_n \left(\pi^2 - 4x^2\right)^n$$

for $N \in \mathbb{N}$, and

$$\begin{aligned} a_0 &= \frac{2}{\pi}, \\ a_1 &= \frac{1}{\pi^3}, \\ a_{n+1} &= \frac{2n+1}{2(n+1)\pi^2} a_n - \frac{1}{16n(n+1)\pi^2} a_{n-1}. \end{aligned} \tag{10}$$

In [11], the inequality (9) was generalized as

$$b_{N+1}\left(r^2 - x^2\right)^{N+1} \le \frac{\sin x}{x} - Q_{2N}(x) \le \frac{1 - \sum_{n=0}^{N} b_n r^{2n}}{r^{2(N+1)}} \left(r^2 - x^2\right)^{N+1} \tag{11}$$

for $x \in (0, r]$, where $r \le \frac{\pi}{2}$,

$$Q_{2N}(x) = \sum_{n=0}^{N} b_n \left(r^2 - x^2\right)^n$$

for $N \in \mathbb{N}$, and

$$\begin{aligned} b_0 &= \frac{\sin r}{r}, \\ b_1 &= \frac{\sin r - r\cos r}{2r^3}, \\ b_{n+1} &= \frac{2n+1}{2(n+1)r^2} b_n - \frac{1}{4n(n+1)r^2} b_{n-1}. \end{aligned}$$

The inequality (9) can follow from letting $r = \frac{\pi}{2}$ in (11).

In this paper, two new inequalities generalizing inequalities (2), (8), (9) and (11) will be established, concerned relations among inequalities (2), (8) and (9) will be studied. In passing, some basic properties of the coefficients α_k and β_k in (2) will be discussed.

In order to clearly and concisely state our main results in this paper, some basic notions about Bessel functions must be first introduced as follows. The function

$$v_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!\Gamma(p+n+(b+1)/2)} \left(\frac{x}{2}\right)^{2n+p}, \quad x \in \mathbb{R}, \tag{12}$$

which is a particular solution of the differential equation

$$x^2 y''(x) + bxy'(x) + [cx^2 - p^2 + p(1-b)]y(x) = 0,$$

is called the generalized Bessel function of first kind, where b, p, c are real numbers. When $b = c = 1$, it reduces to the classical Bessel function

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}, \quad x \in \mathbb{R}; \tag{13}$$

When $b = 1$ and $c = -1$, the function $v_p(x)$ becomes

$$I_p(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}, \quad x \in \mathbb{R}, \tag{14}$$

which is called the modified Bessel function of first kind.

The generalized and normalized Bessel function of first kind is defined as

$$u_p(x) = \frac{2^p}{x^{p/2}} \Gamma\left(p + \frac{b+1}{2}\right) v_p(\sqrt{x}\,), \quad x \in \mathbb{R}. \tag{15}$$

Using the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$$

for $a \neq 0$, we have

$$u_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{4^n (k)_n} \cdot \frac{x^n}{n!}, \tag{16}$$

where $x \in \mathbb{R}$ and $k = p + \frac{b+1}{2} \neq 0, -1, \dots$. Letting $\lambda_p(x) = u_p(x^2)$ gives

$$\lambda_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!(k)_n} \left(\frac{x}{2}\right)^{2n}, \quad x \in \mathbb{R}. \tag{17}$$

In particular, taking $c=\pm 1$ and $b=1$ in $\lambda_p(x)$ respectively lead to

$$\mathcal{J}_p(x)=\frac{2^p}{x^p}\Gamma(p+1)J_p(x)$$

and

$$\mathcal{I}_p(x)=\frac{2^p}{x^p}\Gamma(p+1)I_p(x),$$

with

$$\begin{aligned}\mathcal{J}_{-1/2}(x)&=\cos x, \quad \mathcal{I}_{-1/2}(x)=\cosh x,\\ \mathcal{J}_{1/2}(x)&=\frac{\sin x}{x}, \quad \mathcal{I}_{1/2}(x)=\frac{\sinh x}{x}.\end{aligned}\tag{18}$$

For more information about Bessel functions, please refer to [3, 4, 5] and related references therein.

Now the main results of this paper can be stated as follows.

Theorem 1. *If $k\ge\frac{1}{2}$ and $0\le c\le 1$, then the inequality*

$$\sum_{i=0}^{n}\gamma_i\big(\pi^2-4x^2\big)^i\le\lambda_p(x)\le\sum_{i=0}^{n}\eta_i\big(\pi^2-4x^2\big)^i\tag{19}$$

holds for $n\in\mathbb{N}$ and $x\in\big(0,\frac{\pi}{2}\big]$, where the constants

$$\gamma_i=\left(\frac{c}{16}\right)^i\frac{\lambda_{i+p}(\pi/2)}{i!(k)_i}, \quad 0\le i\le n\tag{20}$$

for $k=p+\frac{b+1}{2}$ and $b\in\mathbb{R}$ and

$$\eta_i=\begin{cases}\gamma_i, & 0\le i\le n-1\\ \dfrac{1-\sum_{\ell=0}^{n-1}\gamma_\ell\pi^{2\ell}}{\pi^{2n}}, & i=n\end{cases}\tag{21}$$

are the best possible. If $k>0$ and $c\le 0$, then the inequality (19) either holds for n being odd or reverses for n being even.

Theorem 2. *If $k\ge\frac{1}{2}$ and $0\le c\le 1$, then the inequality*

$$\sum_{i=0}^{n}\sigma_i\big(\theta^2-x^2\big)^i\le\lambda_p(x)\le\sum_{i=0}^{n}\nu_i\big(\theta^2-x^2\big)^i\tag{22}$$

is valid for $n\in\mathbb{N}$ and $0<x\le\theta\le\frac{\pi}{2}$, where the coefficients

$$\sigma_i=\left(\frac{c}{4}\right)^i\frac{\lambda_{i+p}(\theta)}{i!(k)_i}, \quad 0\le i\le n$$

for $k=p+\frac{b+1}{2}$ and $b\in\mathbb{R}$ and

$$\nu_i=\begin{cases}\sigma_i, & 0\le i\le n-1\\ \dfrac{1-\sum_{\ell=0}^{n-1}\sigma_\ell\theta^{2\ell}}{\theta^{2n}}, & i=n\end{cases}$$

are the best possible. If $k>0$ and $c\le 0$, then the inequality (22) either validates for n being odd and $0<x<\theta<\infty$ or reverses for n being even and $0<x<\theta<\infty$.

Proposition 1. *The inequality (19) generalizes the inequality (2).*

Proposition 2. *The inequality* (22) *generalizes the inequality* (11).

Proposition 3. *The inequality* (2) *is equivalent to the inequality* (9), *the inequality* (8) *is equivalent to the left-hand side of the inequality* (2).

Proposition 4. *Let n be a given natural number. Then the coefficients α_n and β_n in* (2) *satisfy*

$$\frac{1}{4^n(2n+1)!}\left[1-\frac{\pi^2}{2^3(2n+3)}\right]<\alpha_n<\frac{1}{4^n(2n+1)!} \tag{23}$$

and

$$0<\alpha_n\le\beta_n<\frac{1}{\pi^{2n}}\left(1-\frac{2}{\pi}\right). \tag{24}$$

2. Remarks and applications

Before proving the above theorems and propositions, we would like to present some remarks and several simple applications.

Remark 1. Setting $p=\frac{1}{2}$, $b=1$ and $c=-1$ in the inequality (19) yields

$$\sum_{i=0}^{2m+1}\gamma_i\big(\pi^2-4x^2\big)^i\le\frac{\sinh x}{x}\le\sum_{i=0}^{2m+1}\eta_i\big(\pi^2-4x^2\big)^i,\quad m\in\mathbb{N} \tag{25}$$

and

$$\sum_{i=0}^{2m}\eta_i\big(\pi^2-4x^2\big)^i\le\frac{\sinh x}{x}\le\sum_{i=0}^{2m}\gamma_i\big(\pi^2-4x^2\big)^i,\quad m\in\mathbb{N}. \tag{26}$$

Remark 2. If letting $\theta=\frac{\pi}{2}$, then the inequality (22) reduces to the inequality (19), which means that Theorem 2 extends Theorem 1.

Remark 3. When taking $p=\frac{1}{2}$, $b=1$ and $c=-1$, the inequality (22) becomes

$$\sum_{i=0}^{2m+1}\sigma_i\big(\theta^2-x^2\big)^i\le\frac{\sinh x}{x}\le\sum_{i=0}^{2m+1}\nu_i\big(\theta^2-x^2\big)^i,\quad m\in\mathbb{N} \tag{27}$$

and

$$\sum_{i=0}^{2m}\nu_i\big(\theta^2-x^2\big)^i\le\frac{\sinh x}{x}\le\sum_{i=0}^{2m}\sigma_i\big(\theta^2-x^2\big)^i,\quad m\in\mathbb{N}. \tag{28}$$

These two inequalities include inequalities (25) and (26).

Remark 4. The first three propositions above demonstrate concerned relations among inequalities (2), (8), (9), (11), (19) and (22).

Remark 5. From Proposition 3 we can see that inequalities (2) and (9) are essentially the same one.

Remark 6. Taking $p=\frac{1}{2}$ and $b=c=1$ in the inequality (19) can lead to the inequality (2), which implies that the inequality (19) generalizes the inequality (2). However, this conclusion is not very explicit and obvious, so we state it as Proposition 1 and prove it separately.

Remark 7. The inequalities (23) and (24) imply that

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \beta_n = 0.$$

Since $\alpha_k > 0$, it is easy to see that

$$\sum_{k=1}^{n} \alpha_k \bigl(\pi^2 - 4x^2\bigr)^k < \sum_{k=1}^{n+1} \alpha_k \bigl(\pi^2 - 4x^2\bigr)^k. \tag{29}$$

On the other hand, by the recurrent formula (4), we have

$$\alpha_n - \beta_n = -\pi^2 \beta_{n+1}.$$

From this, it follows that

$$\begin{aligned} &\sum_{k=1}^{n+1} \beta_k \bigl(\pi^2 - 4x^2\bigr)^k - \sum_{k=1}^{n} \beta_k \bigl(\pi^2 - 4x^2\bigr)^k \\ &= \beta_{n+1} \bigl(\pi^2 - 4x^2\bigr)^{n+1} + (\alpha_n - \beta_n) \bigl(\pi^2 - 4x^2\bigr)^n \\ &= \bigl(\pi^2 - 4x^2\bigr)^n \bigl[\beta_{n+1} \bigl(\pi^2 - 4x^2\bigr) + \alpha_n - \beta_n\bigr] \\ &= -4x^2 \beta_{n+1} \bigl(\pi^2 - 4x^2\bigr)^n \\ &< 0. \end{aligned} \tag{30}$$

Inequalities (29) and (30) show us that the inequality (2) becomes more and more accurate as n grows larger and larger.

Remark 8. We can also comprehend the effect of the polynomial

$$\frac{\pi}{2} + \sum_{k=1}^{n} \beta_k (\pi^2 - 4x^2)^k$$

approximating to the function $\frac{\sin x}{x}$ by considering the function

$$U(n,x) = \frac{\pi}{2} + \sum_{k=1}^{n} \beta_k \bigl(\pi^2 - 4x^2\bigr)^k - \frac{\sin x}{x},$$

the graphs of whose particular cases $U(2,x)$ and $U(3,x)$ can be depicted by Figure 1.

Remark 9. Inequalities (27) and (28) can be used to evaluate the hyperbolic sine integral

$$\operatorname{Shi} x = \int_0^x \frac{\sinh t}{t} \, \mathrm{d}\, t \tag{31}$$

as follows:

$$\begin{aligned} \sum_{i=0}^{2m+1} \sum_{k=0}^{i} \frac{(-1)^k \sigma_i \theta^{2i-2k} x^{2k+1}}{2k+1} \binom{i}{k} &\le \operatorname{Shi} x \\ &\le \sum_{i=0}^{2m+1} \sum_{k=0}^{i} \frac{(-1)^k \nu_i \theta^{2i-2k} x^{2k+1}}{2k+1} \binom{i}{k} \end{aligned} \tag{32}$$

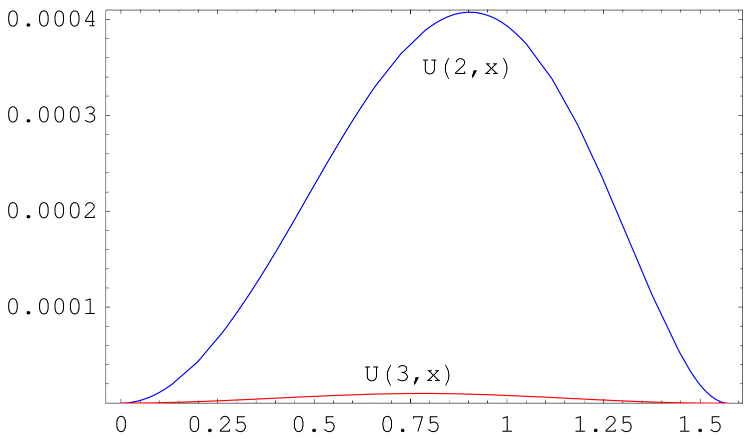

FIGURE 1.

and

$$\sum_{i=0}^{2m}\sum_{k=0}^{i}\frac{(-1)^k\nu_i\theta^{2i-2k}x^{2k+1}}{2k+1}\binom{i}{k}\le \operatorname{Shi}x\le \sum_{i=0}^{2m}\sum_{k=0}^{i}\frac{(-1)^k\sigma_i\theta^{2i-2k}x^{2k+1}}{2k+1}\binom{i}{k} \tag{33}$$

for $m\in\mathbb{N}$.

Remark 10. As direct consequences of the inequality (2), the following Kober type inequality and estimation of the sine integral

$$\operatorname{Si}x=\int_0^x\frac{\sin t}{t}\,\mathrm{d}\,t \tag{34}$$

may be obtained:

$$\left(\frac{\pi}{2}-x\right)\left[\sum_{k=1}^{n}\alpha_k(4x)^k(\pi-x)^k+\frac{2}{\pi}\right]\le\cos x \le\left(\frac{\pi}{2}-x\right)\left[\sum_{k=1}^{n}\beta_k(4x)^k(\pi-x)^k+\frac{2}{\pi}\right] \tag{35}$$

and

$$\frac{2}{\pi}x+\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-4)^i\alpha_k\pi^{2k-2i}}{2i+1}\binom{k}{i}x^{2i+1}\le\operatorname{Si}x \le\frac{2}{\pi}x+\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-4)^i\beta_k\pi^{2k-2i}}{2i+1}\binom{k}{i}x^{2i+1}, \tag{36}$$

where $0<x\le\frac{\pi}{2}$, k and n are natural numbers, and α_k and β_k for $k\ge0$ are defined by (3) and (4).

Remark 11. Combining the formula

$$\Gamma(1+z)\Gamma(1-z)=\frac{\pi z}{\sin\pi z} \tag{37}$$

with (2) yields

$$\frac{2}{\pi}+\sum_{k=1}^{n}\alpha_k\bigl(\pi^2-4x^2\bigr)^k\le\frac{1}{\Gamma(1+x/\pi)\Gamma(1-x/\pi)}\le\frac{2}{\pi}+\sum_{k=1}^{n}\beta_k\bigl(\pi^2-4x^2\bigr)^k \tag{38}$$

for $0<x<\frac{\pi}{2}$ and $n\in\mathbb{N}$.

Remark 12. By the way, the inequality (2) has been generalized in [7, 8] along another direction different from that in [6, 9, 11, 12, 13].

3. **Lemmas**

In order to prove the above theorems and propositions, the following two lemmas are needed.

Lemma 1 ([1, 2]). *Let f and g be continuous on $[a,b]$ and differentiable in (a,b) such that $g'(x)\ne 0$ in (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing respectively) in (a,b), then the functions $\frac{f(x)-f(b)}{g(x)-g(b)}$ and $\frac{f(x)-f(a)}{g(x)-g(a)}$ are also increasing (or decreasing respectively) in (a,b).*

Lemma 2. *The coefficients α_k for $1\le k\le n$ in (2) can be expressed as*

$$\alpha_k=\frac{(-1)^k}{\pi^{2k}}\sum_{n=k}^{\infty}\binom{n}{k}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n}. \tag{39}$$

Proof. For $x>0$ and $k\in\mathbb{N}$, let $u_0(x)=\frac{\sin x}{x}$ and $u_k(x)=\frac{u'_{k-1}(x)}{x}$. In [9, pp.162], it was obtained that $\alpha_k=\frac{u_k(\frac{\pi}{2})}{(-8)^k k!}$. On the other hand, let

$$d_k(x)=\sum_{n=k}^{\infty}\frac{(-1)^n n!}{(2n+1)!(n-k)!}x^{2n}.$$

It is clear that $d_k(\frac{\pi}{2})=R_k$, where R_k is defined in (7). It was also proved in [6] that

$$d_1(x)=\frac{x}{2}\left(\frac{\sin x}{x}\right)',\quad d_2(x)=\frac{x^3}{2^2}\left(\frac{1}{x}\left(\frac{\sin x}{x}\right)'\right)',\quad \cdots,$$

that is,

$$d_k(x)=\frac{x^{2k}}{2^k}u_k(x).$$

Hence,

$$\alpha_k=\frac{u_k(\pi/2)}{(-8)^k k!}=\frac{2^k}{(\pi/2)^{2k}}\cdot\frac{R_k}{(-8)^k k!}=\frac{(-1)^k}{\pi^{2k}}\sum_{n=k}^{\infty}\binom{n}{k}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n}. \tag{40}$$

The proof of Lemma 2 is finished. □

4. **Proofs of theorems and propositions**

Now we are in a position to prove the above theorems and propositions.

Proof of Theorem 1. By (16), we have

$$\begin{aligned}u_p'(x) &= \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(k)_n} \cdot \frac{x^{n-1}}{(n-1)!}\\ &= \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(k+1)_{n-1}k} \cdot \frac{x^{n-1}}{(n-1)!}\\ &= \left(-\frac{c}{4k}\right) u_{p+1}(x).\end{aligned}$$

Differentiating $\lambda_p(x)$ gives

$$\lambda_p'(x) = 2xu_p'(x^2) = -\frac{c}{2k} x u_{p+1}(x^2) \triangleq K_1 x \lambda_{p+1}(x)$$

and

$$\lambda_{p+i}'(x) = -\frac{c}{2(k+i)} x u_{p+i+1}(x^2) \triangleq K_i x \lambda_{p+i+1}(x), \tag{41}$$

where

$$K_i = -\frac{c}{2(k+i-1)}, \quad k \geq 1.$$

For $n \geq 1$, let

$$f_0(x) = \lambda_p(x) - \sum_{i=0}^{n-1} \gamma_i (\pi^2 - 4x^2)^i, \quad g_0(x) = (\pi^2 - 4x^2)^n,$$

$$f_m(x) = \prod_{i=1}^{m} K_i \lambda_{p+m}(x) - (-8)^m \sum_{i=m}^{n-1} \gamma_i m! \binom{i}{m} (\pi^2 - 4x^2)^{i-m}$$

and

$$g_m(x) = (-8)^m m! \binom{n}{m} (\pi^2 - 4x^2)^{n-m} \tag{42}$$

for $0 \leq m \leq n-1$. Direct differentiation and utilization of (41) yield

$$\begin{aligned}\frac{f_m'(x)}{g_m'(x)} &= \frac{1}{(-8)^{m+1}(m+1)!\binom{n}{m+1}(\pi^2 - 4x^2)^{n-m-1}} \Bigg[\lambda_{p+m+1}(x) \prod_{i=1}^{m+1} K_i\\ &\qquad - (-8)^{m+1}(m+1)! \sum_{i=m+1}^{n-1} \gamma_i \binom{i}{m+1} (\pi^2 - 4x^2)^{i-m-1}\Bigg] \\ &= \frac{f_{m+1}(x)}{g_{m+1}(x)}\end{aligned} \tag{43}$$

and

$$\frac{f_{n-1}'(x)}{g_{n-1}'(x)} = \frac{\prod_{i=1}^{n} K_i \lambda_{n+p}(x)}{n!(-8)^n}. \tag{44}$$

It is clear that

$$f_{m+1}\left(\frac{\pi}{2}\right) = g_{m+1}\left(\frac{\pi}{2}\right) = 0, \quad 0 \leq m \leq n-1.$$

Therefore, it is easy to see that

$$\frac{f_m'(x)}{g_m'(x)} = \frac{f_{m+1}(x) - f_{m+1}(\pi/2)}{g_{m+1}(x) - g_{m+1}(\pi/2)}, \quad 0 \le m \le n-1. \tag{45}$$

When $c \in [0,1]$ and $k \ge \frac{1}{2}$, it was proved in [4] that the function $\lambda_p(x)$ is decreasing in $(0, \frac{\pi}{2}]$. From (45) and Lemma 1, it readily follows that the functions $\frac{f_m'(x)}{g_m'(x)}$ and $\frac{f_{m-1}'(x)}{g_{m-1}'(x)}$ have the same monotonicity, which leads to the same monotonicity of $\frac{f_{n-1}'(x)}{g_{n-1}'(x)}$ and $\frac{f_0(x)}{g_0(x)}$ in $(0, \frac{\pi}{2}]$. So we have

$$\gamma_n \triangleq \lim_{x\to(\pi/2)^-} \frac{f_0(x)}{g_0(x)} \le \frac{f_0(x)}{g_0(x)} \le \lim_{x\to 0^+} \frac{f_0(x)}{g_0(x)} \triangleq \eta_n,$$

which implies (19) and the best possibility of γ_n and η_n, where the constant γ_n can obtained by using L'Hôspital's Rule as follows

$$\gamma_n = \lim_{x\to\frac{\pi}{2}} \frac{f_0(x)}{g_0(x)} = \lim_{x\to\frac{\pi}{2}} \frac{f_{n-1}'(x)}{g_{n-1}'(x)} = \frac{\prod_{i=1}^n K_i \lambda_{n+p}(\pi/2)}{n!(-8)^n} = \left(\frac{c}{16}\right)^n \frac{\lambda_{n+p}(\pi/2)}{n!(k)_n}.$$

When $c \le 0$ and $k > 0$, it was showed in [4] that the function $\lambda_p(x)$ increases in $(0,\infty)$. By (45) and Lemma 1, it is deduced that $\frac{f_{n-1}'(x)}{g_{n-1}'(x)}$ and $\frac{f_0(x)}{g_0(x)}$ have the same monotonicity. Since $K_i \ge 0$, the function $\frac{f_0(x)}{g_0(x)}$ decreases when n is odd, this implies (19). When n is even, the function $\frac{f_0(x)}{g_0(x)}$ increases, and so the inequality (19) reverses. The proof of Theorem 1 is complete. □

Proof of Theorem 2. For $n \ge 1$, let

$$f_0(x) = \lambda_p(x) - \sum_{i=0}^{n-1} \sigma_i \left(\theta^2 - x^2\right)^i, \quad g_0(x) = \left(\theta^2 - x^2\right)^n,$$

$$f_m(x) = \prod_{i=1}^m K_i \lambda_{p+m}(x) - (-2)^m \sum_{i=m}^{n-1} \sigma_i m! \binom{i}{m} \left(\theta^2 - x^2\right)^{i-m},$$

$$g_m(x) = (-2)^m m! \binom{n}{m} \left(\theta^2 - x^2\right)^{n-m}$$

for $0 \le m \le n-1$. Then applying (41) gives

$$\begin{aligned} \frac{f_m'(x)}{g_m'(x)} &= \frac{1}{(-2)^{m+1}(m+1)!\binom{n}{m+1}\left(\theta^2 - x^2\right)^{n-m-1}} \Bigg[\lambda_{p+m+1}(x) \prod_{i=1}^{m+1} K_i \\ &\quad - (-2)^{m+1}(m+1)! \sum_{i=m+1}^{n-1} \sigma_i \binom{i}{m+1} \left(\theta^2 - x^2\right)^{i-m-1}\Bigg] \\ &= \frac{f_{m+1}(x)}{g_{m+1}(x)} \end{aligned} \tag{46}$$

and

$$\frac{f_{n-1}'(x)}{g_{n-1}'(x)} = \frac{\prod_{i=1}^n K_i \lambda_{n+p}(x)}{n!(-2)^n}. \tag{47}$$

Since

$$f_{m+1}(\theta) = g_{m+1}(\theta) = 0$$

for $0 \le m \le n-1$, we have

$$\frac{f'_m(x)}{g'_m(x)} = \frac{f_{m+1}(x) - f_{m+1}(\theta)}{g_{m+1}(x) - g_{m+1}(\theta)} \tag{48}$$

for $0 \le m \le n-1$, when $c \in [0,1]$ and $k \ge \frac{1}{2}$, $\lambda_p(x)$ is proved decreasing in [4] for $0 < x < \theta \le \frac{\pi}{2}$ and thus $\lambda_{n+p}(x)$ decreases in $(0, \frac{\pi}{2}]$. By (48) and Lemma 1, $\frac{f'_m(x)}{g'_m(x)}$ and $\frac{f'_{m-1}(x)}{g'_{m-1}(x)}$ have the same monotonicity, then $\frac{f'_0(x)}{g'_0(x)}$ and $\frac{f'_{n-1}(x)}{g'_{n-1}(x)}$ are both decreasing. So we have

$$\sigma_n \triangleq \lim_{x\to\theta^-} \frac{f_0(x)}{g_0(x)} \le \frac{f_0(x)}{g_0(x)} \le \lim_{x\to 0^+} \frac{f_0(x)}{g_0(x)} \triangleq \nu_n,$$

which implies (22) and the best possibility of σ_n and ν_n, where the constant σ_n is obtained by using L'Hôspital's Rule

$$\sigma_n = \lim_{x\to\theta^-} \frac{f_0(x)}{g_0(x)} = \lim_{x\to\theta^-} \frac{f'_{n-1}(x)}{g'_{n-1}(x)} = \frac{\prod_{i=1}^n K_i \lambda_{n+p}(\theta)}{n!(-2)^n} = \left(\frac{c}{4}\right)^n \frac{\lambda_{n+p}(\theta)}{n!(k)_n}.$$

When $c \le 0$ and $k > 0$, it was proved in [4] that $\lambda_p(x)$ increases in $(0, \infty)$. By (48) and Lemma 1, the functions $\frac{f'_{n-1}(x)}{g'_{n-1}(x)}$ and $\frac{f_0(x)}{g_0(x)}$ have the same monotonicity. Since $K_i \ge 0$, then $\frac{f_0(x)}{g_0(x)}$ decreases when n is odd, which implies (22). When n is even, the function $\frac{f_0(x)}{g_0(x)}$ increases, and so the inequality (22) reverses. The proof is complete. □

Proof of Proposition 1. Using the Pochhammer symbol gives

$$(k)_i (k+i)_n = (k)_{n+i}.$$

From (17) and (20), it follows that

$$\begin{aligned}
\gamma_i &= \left(\frac{c}{16}\right)^i \frac{1}{i!(k)_i} \lambda_{i+p}\left(\frac{\pi}{2}\right) \\
&= \frac{c^i}{i! 2^{4i} (k)_i} \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!(k+i)_n} \left(\frac{\pi}{4}\right)^{2n} \\
&= \frac{c^i}{i! 2^{4i} (k)_i} \sum_{n=0}^{\infty} \frac{(-1)^n c^n (k)_i}{n!(k)_{n+i}} \left(\frac{\pi}{4}\right)^{2n} \\
&= \frac{c^i}{i! 2^{4i}} \sum_{n=i}^{\infty} \frac{(-1)^{n-i} c^{n-i}}{(n-i)!(k)_n} \left(\frac{\pi}{4}\right)^{2(n-i)} \\
&= \frac{(-1)^i}{\pi^{2i}} \sum_{n=i}^{\infty} \binom{n}{i} \frac{(-1)^n c^n}{n!(k)_n 2^{2n}} \left(\frac{\pi}{2}\right)^{2n},
\end{aligned}$$

where $k = p + \frac{b+1}{2}$. In particular, setting $b = c = 1$ and $p = \frac{1}{2}$ immediately yields

$$\gamma_i = \frac{(-1)^i}{\pi^{2i}} \sum_{n=i}^{\infty} \binom{n}{i} \frac{(-1)^n}{n!(3/2)_n 2^{2n}} \left(\frac{\pi}{2}\right)^{2n}$$

$$
\begin{aligned}
&= \frac{(-1)^i}{\pi^{2i}} \sum_{n=i}^{\infty} \binom{n}{i} \frac{(-1)^n}{(2n+1)!!n!2^n} \left(\frac{\pi}{2}\right)^{2n} \\
&= \frac{(-1)^i}{\pi^{2i}} \sum_{n=i}^{\infty} \binom{n}{i} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{2}\right)^{2n} \\
&= \alpha_i,
\end{aligned}
$$

where the last equality follows from Lemma 2. Meanwhile, it is not difficult to see that $\lambda_{1/2}(x) = \frac{\sin x}{x}$. Hence, the inequality (19) generalizes the inequality (2). □

Proof of Proposition 2. In [11], the coefficients of the inequality (11) are defined by

$$
b_{n+1} = \frac{2n+1}{2(n+1)r^2} b_n - \frac{1}{4n(n+1)r^2} b_{n-1},
$$

where

$$
b_n = \frac{1}{n!(2\theta)^n} \sqrt{\frac{\pi}{2\theta}}\, J_{n+1/2}(\theta), \quad \theta \in \left(0, \frac{\pi}{2}\right].
$$

When $c = b = 1$ and $p = \frac{1}{2}$, the coefficient σ_i in (22) can be rewritten as

$$
\begin{aligned}
\sigma_i &= \left(\frac{1}{4}\right)^i \frac{1}{i!(3/2)_i} \lambda_{i+1/2}(\theta) \\
&= \left(\frac{1}{4}\right)^i \frac{1}{i!(3/2)_i} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(i+3/2)_n} \left(\frac{\theta}{2}\right)^{2n} \\
&= \left(\frac{1}{4}\right)^i \frac{\Gamma(3/2)}{i!\Gamma(3/2+i)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(3/2+i)}{n!\Gamma(n+3/2+i)} \left(\frac{\theta}{2}\right)^{2n} \\
&= \left(\frac{1}{4}\right)^i \frac{\Gamma(3/2)}{i!} \left(\frac{2}{\theta}\right)^{1/2+i} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+3/2+i)} \left(\frac{\theta}{2}\right)^{2n+1/2+i} \\
&= \frac{1}{i!(2\theta)^i} \sqrt{\frac{\pi}{2\theta}}\, J_{i+1/2}(\theta) \\
&= b_i.
\end{aligned}
$$

This shows that the inequality (22) generalizes the inequality (11). □

Proof of Proposition 3. The polynomial $P_{2N}(x)$ involved in the inequality (9) is determined by the coefficients a_n which can be calculated by the recursing formula in (10). In fact, an alternative representation

$$
a_n = \frac{1}{n!(4\pi)^n} j_n\left(\frac{\pi}{2}\right) = \frac{1}{n!(4\pi)^n} \sqrt{\frac{\pi}{2} \cdot \frac{2}{\pi}}\, J_{n+1/2}\left(\frac{\pi}{2}\right) \tag{49}
$$

was given in [12, p. 2501]. Then taking $c = b = 1$ and $p = \frac{1}{2}$ in (20) produces

$$
\begin{aligned}
\gamma_i &= \left(\frac{1}{16}\right)^i \frac{1}{i!(3/2)_i} \lambda_{i+1/2}\left(\frac{\pi}{2}\right) \\
&= \left(\frac{1}{16}\right)^i \frac{1}{i!(3/2)_i} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(i+3/2)_n} \left(\frac{\pi}{4}\right)^{2n}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{16}\right)^i \frac{\Gamma(3/2)}{i!\Gamma(3/2+i)} \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(3/2+i)}{n!\Gamma(n+3/2+i)} \left(\frac{\pi}{4}\right)^{2n} \\
&= \frac{\Gamma(3/2)}{i!4^{2i}} \left(\frac{4}{\pi}\right)^{1/2+i} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+3/2+i)} \left(\frac{\pi}{4}\right)^{2n+1/2+i} \\
&= \frac{1}{i!(4\pi)^i} J_{i+1/2}\left(\frac{\pi}{2}\right) \\
&= a_i.
\end{aligned}
$$

In virtue of Proposition 1, we see that when $c = b = 1$ and $p = \frac{1}{2}$ in (20) we have $\gamma_i = \alpha_i$, where α_i is determined by (3), so we obtain $a_i = \alpha_i$ which means that the inequality (2) is equivalent to the inequality (9).

In view of Lemma 2, it immediately follows that $\alpha_k = \frac{(-1)^k R_k}{k!\pi^{2k}}$ which implies that the inequality (8) is equivalent to the left-hand side of the inequality (2). □

Proof of Proposition 4. For a converging series $S = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$ with $a_n > 0$, it is known that $a_1 - a_2 < S < a_1$. From this and (39) in Lemma 2, the inequality (23) follows.

The inequality (2) can be rearranged as

$$
0 \le \frac{\sin x}{x} - \frac{2}{\pi} - \sum_{k=1}^{n} \alpha_k \left(\pi^2 - 4x^2\right)^k \le (\beta_n - \alpha_n)\left(\pi^2 - 4x^2\right)^n, \tag{50}
$$

which implies $\beta_n \ge \alpha_n > 0$.

In view of (4), we have

$$
\beta_n = \frac{1 - 2/\pi - \sum_{i=1}^{n-1} \alpha_i \pi^{2i}}{\pi^{2n}} < \frac{1 - 2/\pi}{\pi^{2n}}.
$$

The proof of Proposition 4 is complete. □

Acknowledgements

The third author was supported in part by the China Scholarship Council and the Science Foundation of Tianjin Polytechnic University.

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