A GENERALIZATION OF JORDAN’S INEQUALITY AND AN APPLICATION

ZHEN-HONG HUO, DA-WEI NIU, JIAN CAO, AND FENG QI

Abstract. In this article, a new generalization of Jordan’s inequality
\[ \sum_{k=1}^{n} \mu_k (\theta^t - x^t)^k \leq \frac{\sin x}{x} - \frac{\sin \theta}{\theta} \leq \sum_{k=1}^{n} \omega_k (\theta^t - x^t)^k \]
for \( t \geq 2 \), \( n \in \mathbb{N} \) and \( \theta \in (0, \pi] \) is established, where the coefficients \( \mu_k \) and \( \omega_k \) defined by recursion formulas (11) and (12) are the best possible. As an application, Yang’s inequality is refined.

1. Introduction

The well-known Jordan’s inequality (see [2, 5], [3, p. 143], [7, p. 269] and [10, p. 33]) states that
\[ \frac{2}{\pi} \leq \frac{\sin x}{x} < 1 \]
for \( 0 < |x| \leq \frac{\pi}{2} \). The equality in (11) is valid if and only if \( x = \frac{\pi}{2} \).

Jordan’s inequality and its refinements have important applications in several mathematical areas such as calculus, trigonometry, where specially the theory of limits are involved in [25]. These are important tools in approximating Riemann zeta function \( \zeta(x) \) in [8], in improving Yang’s inequality in [29] and its generalization which play an important role in the theory of distribution of values of functions. Therefore, many mathematicians have struggled to refine, generalize and apply it. For more detailed information, please refer to [2, pp. 274–275] and [11, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 28, 30, 33, 34, 35], especially [11, 20], and related references therein.

In [11, 15, 16, 17, 18, 19], among other things, Jordan’s inequality had been refined as
\[ \frac{1}{\pi^3} x (\pi^2 - 4x^2) \leq \sin x - \frac{2}{\pi} x \leq \frac{\pi - 2}{\pi^3} x (\pi^2 - 4x^2). \]
(2)

In [35], a stronger sharp double inequality for \( x \in (0, \frac{\pi}{2}] \) was obtained:
\[ \frac{12 - \pi^2}{16\pi^5} (\pi^2 - 4x^2)^2 \leq \frac{\sin x}{x} - \frac{2}{\pi} - \frac{1}{\pi^3} (\pi^2 - 4x^2)^2 \leq \frac{\pi - 3}{\pi^5} (\pi^2 - 4x^2)^2. \]
(3)

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Recently, the following general refinement of Jordan’s inequality was showed in [13]:

\[
\frac{2}{\pi} + \sum_{k=1}^{n} \alpha_k \left( \pi^2 - 4x^2 \right)^k \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \sum_{k=1}^{n} \beta_k \left( \pi^2 - 4x^2 \right)^k, 
\]

where the constants

\[
\alpha_k = \frac{(-1)^k}{(4\pi)^{k+1}} \sum_{i=1}^{k} \left( \frac{2}{\pi} \right)^i c_{i-1} \sin \left( \frac{k+i}{2\pi} \right)
\]

and

\[
\beta_k = \begin{cases} 
1 \quad & k = n \\
\alpha_k \quad & 1 \leq k < n 
\end{cases}
\]

with

\[
c_i^k = \begin{cases} 
(i+k-1)c_{i-1}^{k-1} + c_i^{k-1} \quad & 0 < i \leq k \\
1 \quad & i = 0 \\
0 \quad & i > k 
\end{cases}
\]

are the best possible.

In [28], as a generalization of Jordan’s inequality [1], the following sharp inequality

\[
\frac{1}{2\tau^2} \left[ (1 + \lambda) \left( \frac{\sin \theta}{\theta} - \cos \theta \right) - \theta \sin \theta \right] \left( 1 - \frac{x^\tau}{\theta^\tau} \right)^2 
\]

\[
\leq \frac{\sin x}{x} - \frac{\sin \theta}{\theta} - \frac{1}{\lambda} \left( \frac{\sin \theta}{\theta} - \cos \theta \right) \left( 1 - \frac{x^\lambda}{\theta^\lambda} \right) 
\]

\[
\leq \left[ 1 - \frac{\sin \theta}{\theta} - \frac{1}{\lambda} \left( \frac{\sin \theta}{\theta} - \cos \theta \right) \right] \left( 1 - \frac{x^\tau}{\theta^\tau} \right)^2
\]

was obtained for \( 0 < x \leq \theta \in (0, \pi/2] \), \( \tau \geq 2 \) and \( \tau \leq \lambda \leq 2\tau \). The equalities in (8) hold if and only if \( x = \theta \). The coefficients of the term \( (1 - \frac{x^\tau}{\theta^\tau})^2 \) are the best possible. If \( 1 \leq \tau \leq \frac{\pi}{2} \) and either \( \lambda \neq 0 \) or \( \lambda \geq 2\tau \) then the inequality (8) is reversed. In particular, when \( \theta = \pi/2 \), the inequality (8) becomes

\[
\frac{4\lambda + 4 - \pi^2}{4\tau^2\pi^2\tau + 1} \left( \pi^\tau - 2^\tau x^\tau \right)^2 \leq \frac{\sin x}{x} - \frac{2}{\pi} - \frac{2}{\lambda\pi\lambda+1} \left( \pi^\lambda - 2^\lambda x^\lambda \right) 
\]

\[
\leq \frac{\lambda\pi - 2\lambda - 2}{\lambda\pi^2\tau + 1} \left( \pi^\tau - 2^\tau x^\tau \right)^2
\]

for \( 0 < x \leq \pi/2 \), \( \tau \geq 2 \) and \( \lambda \leq 2\tau \). If \( 1 \leq \tau \leq \frac{\pi}{2} \) and either \( \lambda \neq 0 \) or \( \lambda \geq 2\tau \) then the inequality (9) is reversed. If taking \( (\tau, \lambda) = (2, 2) \) in (9), then the inequality (3) can be deduced.

For recent developments of refinements, generalizations and applications of Jordan’s inequality, please refer to the survey paper [20] and related references therein.

The first aim of this paper is to generalize inequalities (4) and (8) as the following

\textbf{Theorem 1}.
Theorem 1. For $0 < x \leq \theta < \pi$, $n \in \mathbb{N}$ and $t \geq 2$, the inequality
\begin{equation}
\sum_{k=1}^{n} \mu_k (\theta^t - x^t)^k \leq \frac{\sin x}{x} - \frac{\sin \theta}{\theta} \leq \sum_{k=1}^{n} \omega_k (\theta^t - x^t)^k
\end{equation}
holds with the equalities if and only if $x = \theta$, where the constants
\begin{equation}
\mu_k = \frac{(-1)^k}{k! t^k} \sum_{i=1}^{n-i-1} a_{i-1}^k \theta^{k-i-1} \sin \left( \theta + \frac{k+i-1}{2} \pi \right)
\end{equation}
and
\begin{equation}
\omega_k = \begin{cases} 
1 - \frac{\sin \theta/\theta - \sum_{i=1}^{n-1} \mu_i \theta^i}{\theta^n}, & k = n \\
\mu_k, & 1 \leq k < n 
\end{cases}
\end{equation}
with
\begin{equation}
a_{i}^{k} = \begin{cases} 
a_{i-1}^{k-1} + [i + (k-1)(t-1)]a_{i-1}^{k-1}, & 0 \leq i \leq k \\
1, & i = 0 \\
0, & i > k
\end{cases}
\end{equation}
are the best possible.

Remark 1. Taking $t = 2$ in (10) yields inequality (4). Letting $n = 2$ in (10) leads to (8) for $\lambda = \tau = 2$.

The second aim of this paper is to apply Theorem 1 to refine Yang’s inequality [29] as follows.

Theorem 2. Let $0 \leq \lambda \leq 1$, $0 < x \leq \theta < \pi$, $t \geq 2$ and $A_i > 0$ with $\sum_{i=1}^{n} A_i \leq \pi$ for $n \in \mathbb{N}$. If $m \in \mathbb{N}$ and $n \geq 2$, then
\begin{equation}
\frac{\lambda}{2} \pi^2 \left[ \frac{\sin \theta}{\theta} + \sum_{k=1}^{m} 2^{-kt} \mu_k (2^t \theta^t - \lambda \pi^t)^k \right]^2 \cos^2 \left( \frac{\lambda}{2} \pi \right),
\end{equation}

where
\begin{equation}
\frac{\lambda}{2} \pi^2 \left[ \frac{\sin \theta}{\theta} + \sum_{k=1}^{m} 2^{-kt} \mu_k (2^t \theta^t - \lambda \pi^t)^k \right]^2 \cos^2 \left( \frac{\lambda}{2} \pi \right),
\end{equation}

and $\mu_k$ and $\omega_k$ are defined by (11).

2. LEMMAS

To prove our main results, the following lemmas are necessary.

Lemma 1. For $x > 0$, let $u_0(x) = \frac{\sin x}{x}$ and $u_k(x) = \frac{u_{k-1}(x)}{x^r}$ for $k \in \mathbb{N}$ and $r \geq 1$. Then
\begin{equation}
u_k(x) = \sum_{i=1}^{k+1} a_{i-1}^{k} \sin \left( x + (i + k - 1)\pi/2 \right),
\end{equation}
where $a_i^k$ is defined by (13).
Proof. It is apparent that
\[ u_1(x) = x^{-r} \left( \sin x \right)' = x^{-1-r} \cos x - x^{-2-r} \sin x, \]
which tells us that the formula (18) is valid for \( k = 1 \).

Now assume the formula (18) holds for some given \( k > 1 \). Direct computation and utilization of (13) give
\[
u_{k+1} = \sum_{i=1}^{k+1} a_i x_i^{k-i+1} \sin \left( x + \frac{k+i-1}{2} \right)
= a_0 x^{k+1} \cos \left( x + \frac{k+1}{2} \pi \right)
- \frac{1}{x^{k+1}} \sin \left( x + \frac{k+i}{2} \right) \]
\[
= a_0 x^{k+1} \cos \left( x + \frac{k+1}{2} \pi \right)
- \frac{1}{x^{k+1}} \sin \left( x + \frac{k+i}{2} \right) \]
\[
= a_0 x^{k+1} \cos \left( x + \frac{k+1}{2} \pi \right)
- \frac{1}{x^{k+1}} \sin \left( x + \frac{k+i}{2} \right) \]
\[
= \sum_{i=1}^{k+1} a_i x_i^{k-i+1} \sin \left( x + \frac{k+i}{2} \pi \right).
\]

By mathematical induction, Lemma 1 is proved. □

Lemma 2. For \( x > 0 \) and \( k \in \mathbb{N} \), let
\[
v_1(x) = \sum_{i=1}^{k+1} a_i x_i^{k-i+1} \sin \left( x + \frac{k+i-1}{2} \right)
\]
and
\[
v_{j+1}(x) = \frac{1}{x} v_j'(x) \quad \text{for } j \in \mathbb{N}.
\]
Then
\[
v_j(x) = \sum_{i=0}^{k-j+1} b_i x_i^{k-i-j+1} \sin \left( x + \frac{k+i+j-1}{2} \right)
\]
is valid for \( j \in \mathbb{N} \), where \( b_0^1 = a_0^k \), \( b_0^j = 1 \) and
\[
b_i^j = b_i^{j-1} - (k-i-j+3) b_{i-1}^{j-1}, \quad 0 < i \leq k-j+1, \quad j > 1.
\]

Proof. When \( j = 1 \), the formula (19) is valid clearly.

By induction, suppose that the formula (19) holds for some \( j > 1 \). Since \( k-j+1 > k-(j+1)+1 \), it deduced from (20) that \( b_{k-j+1}^j = b_{k-j+1}^j - b_{k-j}^j = 0 \) Thus,
\[
v_{j+1}(x) = \frac{1}{x} \left( \sum_{i=0}^{k-j+1} b_i^j \left( (k-i-j+1) x_i^{k-i-j+1} \sin \left( x + \frac{k+i+j-1}{2} \right) \right. \right.
+ x_i^{k-i-j+1} \cos \left( x + \frac{k+i+j-1}{2} \right) \]
holds with the equalities if and only if
\[
\sin \left( \frac{\theta}{1+t} - \frac{\sin \theta}{\theta} x^t \right) \leq \frac{\sin x}{x} \leq \frac{\sin \theta}{\theta} \left( \frac{1}{\theta} - \frac{\sin \theta}{\theta^t} \right) (\theta^t - x^t) \tag{21}
\]
holds with the equalities if and only if \( x = \theta \), where the constants
\[
\frac{1}{t} \left( \sin \frac{\theta}{\theta^t} - \cos \frac{\theta}{\theta^t} \right) \text{ and } \left( \frac{1}{\theta^t} - \frac{\sin \theta}{\theta^t} \right)
\]
are the best possible.

Proof. Let
\[
f(x) = \sin \frac{x}{x}, \quad g(x) = \theta^t - x^t, \quad f_1(x) = x \cos x - \sin x, \quad g_1(x) = -tx^{1+t}.
\tag{22}
\]
Then
\[
\frac{f(x)}{g(x)} = f(x) - f(0) \quad \frac{f'(x)}{g'(x)} = \frac{f_1(x) - f_1(0)}{g_1(x) - g_1(0)}, \quad \frac{f_1'(x)}{g_1'(x)} = \frac{\sin x}{t(1 + t)x^t}.
\]
Since \( \frac{\sin x}{x} \) is decreasing in \((0, \pi]\), then \( \frac{f_1'(x)}{g_1'(x)} \) is decreasing, and so, in virtue of Lemma 3, the function \( \frac{f(x)}{g(x)} \) is decreasing, and the function \( \frac{f'(x)}{g'(x)} \) is decreasing in \((0, \pi]\), thus,
\[
\frac{1}{t} \left( \sin \frac{\theta}{\theta^t} - \cos \frac{\theta}{\theta^t} \right) \leq \lim_{x \to \theta^+} \frac{f(x)}{g(x)} \leq \lim_{x \to 0^+} \frac{f(x)}{g(x)} = \frac{1}{\theta^t} \left( 1 - \frac{\sin \theta}{\theta} \right)
\]
and the two constants are proved to be the best possible. \( \square \)
3. Proofs of theorems

3.1. Proof of Theorem 1. If $n = 1$, the inequality (10) becomes (21).

For $n \geq 2$, let $t = r + 1$ and

$$
\varphi(x) = \frac{\sin x}{x} - \frac{\sin \theta}{\theta} - \sum_{k=1}^{n-1} \mu_k (\theta^{r+1} - x^{r+1})^k, \quad \psi(x) = (\theta^{r+1} - x^{r+1})^n,
$$

$$
\varphi_i(x) = \frac{\varphi(x)}{x^r}, \quad \varphi_{i+1}(x) = \frac{\varphi_i'(x)}{x^r}, \quad \psi_1(x) = \frac{\psi'(x)}{x^r}, \quad \psi_{i+1}(x) = \frac{\psi_i'(x)}{x^r},
$$

where $2 \leq i \leq n$. Then for $1 \leq k \leq n - 2$,

$$
\varphi_k(x) = u_k(x) - (r+1)! k! \mu_k - \sum_{i=1}^{n-k-1} \frac{(i+k)!}{i!} \mu_{i+k} (\theta^{1+r} - x^{1+r})^i,
$$

$$
\varphi_{n-1}(x) = u_{n-1}(x) - (n-1)! (r+1)! \mu_{n-1},
$$

and $\varphi_n(x) = u_n(x)$, where $u_k(x)$ for $1 \leq k \leq n$ is defined by (18).

In view of (15), it is deduced that

$$
[-(r+1)]^k k! \mu_k = u_k(\theta)
$$

for $1 \leq k \leq n - 1$, hence $\varphi_i(\theta) = 0$ for $1 \leq i \leq n - 1$. A simple calculation gives

$$
\psi_i(x) = [-1(r+1)]^i \prod_{\ell=0}^{i-1} (n-\ell)(\theta^{r+1} - x^{r+1})^{n-i}
$$

for $1 \leq i \leq n$, consequently $\psi_i(\theta) = 0$ for $1 \leq i \leq n - 1$. As a result, for $1 \leq i \leq n - 1$,

$$
\frac{\varphi_i(x)}{\psi(x)} = \frac{\varphi_i(x) - \varphi_i(\theta)}{\psi(x) - \psi_i(\theta)}, \quad \frac{\varphi_i'(x)}{\psi'(x)} = \frac{\varphi_i'(x) - \varphi_i'(\theta)}{\psi'(x) - \psi'(\theta)},
$$

$$
\frac{\varphi_{i+1}(x)}{\psi_{i+1}(x)} = \frac{\varphi_{i+1}(x) - \varphi_{i+1}(\theta)}{\psi_{i+1}(x) - \psi_{i+1}(\theta)}, \quad \frac{\varphi_{i+1}'(x)}{\psi_{i+1}'(x)} = \frac{\varphi_{i+1}'(x) - \varphi_{i+1}'(\theta)}{\psi_{i+1}'(x) - \psi_{i+1}'(\theta)}.
$$

Let $h_1(x) = x^{nr+r+1}$ and $h_{i+1}(x) = \frac{1}{r} h'_i(x)$ for $1 \leq i \leq n$ and $n \in \mathbb{N}$. Then it is easy to see that

$$
h_{i+1}(x) = \prod_{\ell=1}^{i} (nr + n - 2\ell + 3)x^{nr+n-2i+1}
$$

for $1 \leq i \leq n$. Utilization of Lemma 1 and Lemma 2 leads to

$$
\frac{\varphi_{n-1}'(x)}{\psi_{n-1}'(x)} = \sum_{i=1}^{n+1} \frac{a_{n-1}^\prime x^{n-i+1} \sin \left(x + \frac{n+i-1}{2}\right)}{n![-(r+1)]^n x^{n-n+1}} = \frac{v_1(x)}{n![-(r+1)]^n h_1(x)}
$$

and, since $v_i(0) = h_i(0) = 0$ for $1 \leq i \leq n + 1$,

$$
\frac{v_1(x)}{h_1(x)} = \frac{v_1(x) - v_1(0)}{h_1(x) - h_1(0)}, \quad \frac{v_j'(x)}{h_j'(x)} = \frac{v_j'(x) - v_j'(0)}{h_j'(x) - h_j'(0)}, \quad \frac{v_{i+1}'(x)}{h_{i+1}'(x)} = \frac{v_{i+1}'(x) - v_{i+1}'(0)}{h_{i+1}'(x) - h_{i+1}'(0)} = \prod_{\ell=1}^{i} (nr + n - 2\ell + 3)x^{nr-n+1}
$$

for $1 \leq j \leq n - 1$. Since $\frac{\sin x}{x^{n-r-1}}$ and $x^{-n(r-1)}$ is decreasing on $(0, \pi)$, then the function $\frac{\sin x}{x^{n-r-1}}$ is decreasing and $\frac{(-1)^n v_i'(x)}{h_n'(x)}$ is decreasing. Accordingly, from Lemma 3 it follows that the functions $\frac{(-1)^n v_i'(x)}{h_i(x)}$ and $\frac{(-1)^i v_{i+1}'(x)}{h_{i+1}(x)}$ for $2 \leq i \leq n$ are decreasing.
Thus, the functions \( \frac{(-1)^n \psi_1(x)}{n! \psi(x)} \) and \( \frac{(-1)^n \psi_2(x)}{n! \psi(x)} \) are decreasing, and so \( \frac{\psi'(x)}{\psi(x)} \) is decreasing in \((0, \pi)\). Utilizing Lemma 3 again reveals that the functions \( \frac{\psi'(x)}{\psi(x)} \) and \( \frac{\psi''(x)}{\psi'(x)} \) for \( 2 \leq j \leq n - 1 \) are decreasing, which implies the decreasingly monotonicity of \( \frac{\varphi(x)}{\psi(x)} \) in \((0, \pi)\). By L’Hôpital’s rule, it is easy to deduce that

\[
\lim_{x \to \theta^-} \frac{\varphi(x)}{\psi(x)} = \lim_{x \to \theta^-} \frac{\varphi'(x)}{\psi'(x)} = \lim_{x \to \theta^-} \frac{\varphi''(x)}{\psi''(x)} = \frac{u_n(\theta)}{n!(1 + r)^n} = \mu_n
\]

for \( 1 \leq i \leq n - 1 \) and \( \lim_{x \to 0^+} \frac{\varphi(x)}{\psi(x)} = \omega_n \), which implies \( \mu_n \leq \frac{\varphi(x)}{\psi(x)} \leq \omega_n \) and the constants \( \mu_k \) and \( \omega_k \) are the best possible.

By the mathematical induction, the inequality (10) is proved. The proof of Theorem 4 is complete.

3.2. Proof of Theorem 2

It was proved in [31] and [32, (2.13)] that

\[
\sin^2(\lambda \pi) \leq \cos^2(\lambda A_i) + \cos^2(\lambda A_j) - 2 \cos(\lambda A_i) \cos(\lambda A_j) \cos(\lambda \pi)
\]

\[
\triangleq H_{ij} \leq 4 \sin^2\left(\frac{\lambda \pi}{2}\right).
\] (23)

Summing up (23) for \( 1 \leq i < j \leq n \) yields

\[
\left(\frac{n}{2}\right) \sin^2(\lambda \pi) \leq \sum_{1 \leq i < j \leq n} H_{ij} = H(n, \lambda) \leq 4 \left(\frac{n}{2}\right) \sin^2\left(\frac{\lambda \pi}{2}\right).
\] (24)

By virtue of inequality (10) in Theorem 4

\[
4 \sin^2\left(\frac{\lambda \pi}{2}\right) \leq \lambda^2 \pi^2 \left[\frac{\sin \theta}{\theta} + \sum_{k=1}^{m} 2^{-k} \mu_k \left(2^k \theta^k - \lambda^k \pi^k\right)^k\right]^2,
\] (25)

\[
\sin^2(\lambda \pi) = 4 \cos^2\left(\frac{\lambda \pi}{2}\right) \sin^2\left(\frac{\lambda \pi}{2}\right)
\]

\[
\geq \lambda^2 \pi^2 \left[\frac{\sin \theta}{\theta} + \sum_{k=1}^{m} 2^{-k} \mu_k \left(2^k \theta^k - \lambda^k \pi^k\right)^k\right]^2 \cos^2\left(\frac{\lambda \pi}{2}\right).
\] (26)

Substituting (25) and (26) into (24) leads to (14). The proof of Theorem 2 is complete.

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