A Class of Completely Monotonic Functions Related to the Remainder of Binet’s Formula with Applications *

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Abstract

In the note, the complete monotonicity of difference between remainders of Binet’s formula and the star-shaped and subadditive properties of the remainder of Binet’s formula are proved.

Keywords and Phrases: Completely monotonic function; Star-shaped function; Subadditive function; Remainder; Binet’s formula; Gamma function.

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1. Introduction

It is well-known [1, 4, 6] that a function \( f \) is said to be completely monotonic on an interval \( I \) if \( f \) has derivatives of all orders on \( I \) and \((-1)^n f^{(n)}(x) \geq 0\) for \( x \in I \) and \( n \geq 0 \), a function \( f(x) \) is said to be star-shaped on \((0, \infty)\) if \( f(\alpha x) \leq \alpha f(x) \) for \( x \in (0, \infty) \) and all \( 0 < \alpha < 1 \), a function \( f \) is said to be superadditive on \((0, \infty)\) if \( f(x + y) \geq f(x) + f(y) \) for all \( x, y > 0 \), and a function \( f \) is said to be subadditive if \(-f\) is superadditive. In [4, p. 453], it was presented that a star-shaped function must be superadditive.

The noted Binet’s formula [3, p. 11] states that
\[
\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \theta(x) \tag{1.1}
\]
for \( x > 0 \), where \( \Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt \) stands for Euler’s gamma function and
\[
\theta(x) = \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \frac{e^{-xt}}{t} \, dt \tag{1.2}
\]
is called the remainder of Binet’s formula (1.1).

For real numbers \( p > 0, q \in \mathbb{R} \) and \( r \neq 0 \), define
\[
f_{p,q,r}(x) = r[\theta(px) - q\theta(x)], \quad x \in (0, \infty). \tag{1.3}
\]

The aims of this note are to establish the complete monotonicity of \( f_{p,q,r}(x) \) and the star-shaped and subadditive properties of \( \theta(x) \).

The main results are the following theorems.

**Theorem 1.** If \( f_{p,q,r}(x) \) is completely monotonic in \((0, \infty)\), then either \( r > 0 \) and \((p, q) \in D_1\) or \( r < 0 \) and \((p, q) \in D_2\), where
\[
D_1 = (0, 1) \times (-\infty, 1] \cup (1, \infty) \times (-\infty, 1) \tag{1.4}
\]
and
\[
D_2 = (0, 1) \times (1, \infty) \cup [1, \infty) \times [1, \infty). \tag{1.5}
\]

**Theorem 2.** If either \( r > 0 \) and \( q \leq \min\{1, 1/p\} \) or \( r < 0 \) and \( q \geq \max\{1, 1/p\} \), then \( f_{p,q,r}(x) \) is completely monotonic in \((0, \infty)\).

**Theorem 3.**
1. The function \(-\theta(x)\) is star-shaped in \((0, \infty)\).
2. The function \(\theta(x)\) is subadditive in \((0, \infty)\).
2. A lemma

In order to prove above theorems, the lemma below is necessary.

**Lemma 1.** 1. The function

\[ \delta(t) = \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \]  

for \( t > 0 \) is strictly increasing onto \((0, 1/2)\); the derivative of \( \delta(t) \) is strictly decreasing onto \((0, 1/12)\).

2. If \( 0 < \alpha < 1 \), then

\[ \alpha \delta(t) < \delta(\alpha t) < 1 \cdot \delta(t) \]  

for \( t > 0 \). The constants \( \alpha \) and 1 in (2.2) are the best possible.

3. If \( \alpha > 1 \), inequality (2.2) is reversed and the constants \( \alpha \) and 1 in (2.2) are also the best possible.

**Proof.** The decreasing monotonicity of \( \delta'(t) \) has been verified in [5]. From this, the increasing monotonicity of \( \delta(t) \) can be deduced readily.

For \( 0 < \alpha < 1 \), it is clear that \( \delta(\alpha t) < \delta(t) \) for \( t > 0 \). The right hand side inequality in (2.2) follows.

For \( 0 < \alpha < 1 \), let \( g(t) = \delta(\alpha t) - \alpha \delta(t) \) for \( t > 0 \). Since \( \lim_{t \to 0^+} \delta(t) = 0 \), then \( \lim_{t \to 0^+} g(t) = 0 \). Since \( \delta'(t) \) is decreasing, then \( g'(t) = \alpha [\delta'(\alpha t) - \delta'(t)] > 0 \), and \( g(t) \) is strictly increasing, then \( g(t) = \delta(\alpha t) - \alpha \delta(t) > 0 \). The left hand side inequality in (2.2) follows.

Since \( \lim_{t \to \infty} \delta(t) = 1/2 \), then \( \lim_{t \to \infty} [\delta(\alpha t)/\delta(t)] = 1 \). Since \( \lim_{t \to 0^+} \delta'(t) = 1/12 \), then \( \lim_{t \to 0^+} [\delta'(t)/\delta(t)] = \lim_{t \to 0^+} \alpha \delta'(t)/\delta'(t) \) = \( \alpha \) by L'Hôpital’s rule. Therefore, the constants \( \alpha \) and 1 in (2.2) are the best possible.

By a similar argument, the reversed inequality (2.2) for \( \alpha > 1 \) can be proved. The proof of Lemma 1 is complete.

\[ \square \]

3. Proofs of theorems

Now we are in a position to prove our theorems.
Proof of Theorem 1. Utilization of (1.1) and straightforward computation gives
\[-f'_{p,q,r}(x) = r \left[ q\psi(x) - p\psi(px) + \frac{q-1}{2x} + (p-q)\ln x + p\ln p \right],
\quad (3.1)
\]
where \( \psi(x) = \Gamma'(x)/\Gamma(x) \). In [2, p. 893], the following formula is given for \( x > 0 \):
\[\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(n+x)},
\quad (3.2)\]
where \( \gamma \) is Euler-Mascheroni’s constant. Substituting (3.2) into (3.1) leads to
\[-f'_{p,q,r}(x) = r \left[ p\ln p + (p-q)\gamma + \frac{1-q}{2x} + (p-q)\ln x + \omega(x) \right],
\quad (3.3)\]
where
\[\omega(x) = qx \sum_{n=1}^{\infty} \frac{1}{n(n+x)} - p^{2}x \sum_{n=1}^{\infty} \frac{1}{n(n+px)}
\quad (3.4)\]
with \( \lim_{x \to 0^+} \omega(x) = 0 \). If \( f_{p,q,r}(x) \) is completely monotonic, then \( -f'_{p,q,r}(x) \) is nonnegative in \( (0, \infty) \). Hence
\[0 \leq \lim_{x \to 0^+} [-f'_{p,q,r}(x)] = r[p\ln p + (p-q)\gamma] + \lim_{x \to 0^+} \frac{1}{x} \left[ r(1-q) \frac{1}{2} + r(p-q)x\ln x \right].
\]
From this, it is concluded that either \( r(1-q) > 0 \) or \( r(1-q) = 0 \) and \( r(1-p) \geq 0 \). The proof of of Theorem 1 is complete.

Proof of Theorem 2. Direct calculation yields
\[f_{p,q,r}(x) = r \left[ \int_{0}^{\infty} \frac{\delta(u)}{u} e^{-pxu} \, du - q \int_{0}^{\infty} \frac{\delta(t)}{t} e^{-xt} \, dt \right]
= r \left[ \int_{0}^{\infty} \frac{\delta(t/p)}{t} e^{-xt} \, dt - q \int_{0}^{\infty} \frac{\delta(t)}{t} e^{-xt} \, dt \right]
\Delta \int_{0}^{\infty} k(p, q, r; t) e^{-xt} \, dt.
\]
By (2.2), it is reasoned that \( k(p, q, r; t) > 0 \) if \( p, q \) and \( r \) satisfy one of the following conditions:
1. \( r > 0, \ 0 < p < 1 \ \text{and} \ q \leq 1; \)
2. \( r > 0, \ p > 1 \ \text{and} \ q \leq 1/p; \)
3. \( r < 0, \ 0 < p < 1 \ \text{and} \ q \geq 1/p; \)
4. \( r < 0, \ p > 1 \ \text{and} \ q \geq 1. \)

For \( p = 1, \) then \( k(p,q,r;\,t) = r(1-q)\delta(t) \geq 0 \) if either \( r > 0, \ p = 1 \ \text{and} \ q \leq 1 \) or \( r < 0, \ p = 1 \ \text{and} \ q \geq 1. \)

In conclusion, if \( r > 0 \ \text{and} \ q \leq \min\{1, 1/p\} \ \text{or} \ r < 0 \ \text{and} \ q \geq \max\{1, 1/p\}, \) then \( k(p,q,r;\,t) \geq 0, \) which implies that \( f_{p,q,r}(x) \) is completely monotonic.

**Proof.**[Proof of Theorem 3] Applying \( r = 1 \ \text{and} \ 0 < p = q < 1 \) in Theorem 2 yields \( f_{p,q,r}(x) = \theta(px) - p\theta x \geq 0. \) Hence \( -\theta(x) \) is star-shaped in \((0, \infty), \) which implies that \( -\theta(x) \) is superadditive. Consequently, \( \theta(x) \) is subadditive. □

**References**


