

VIEWING SOME NONLINEAR ODES AND THEIR SOLUTIONS FROM THE ANGLE OF DERIVATIVE POLYNOMIALS

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ABSTRACT. In the paper, the authors view some nonlinear ODEs and their solutions from the angle of (the modified) derivative polynomials and simplify some known identities for the Bernoulli numbers and polynomials, the Frobenius-Euler polynomials, the Euler numbers and polynomials, in terms of the Stirling numbers of the first and second kinds.

1. INTRODUCTION

We recall some results obtained in two different and independent directions. We also recall below a concept in combinatorics about higher derivatives of functions.

1.1. The first direction. In [12, p. 1127 and 1131], it was obtained that

$$\frac{1}{(1 - e^{-t})^2} = 1 + \frac{1}{e^t - 1} - \left(\frac{1}{e^t - 1} \right)'$$

and

$$\frac{1}{(1 - e^{-t})^3} = 1 + \frac{1}{e^t - 1} - \frac{3}{2} \left(\frac{1}{e^t - 1} \right)' + \frac{1}{2} \left(\frac{1}{e^t - 1} \right)''.$$

In the preprint [15] and its formally published version [5], the above two identities were generalized by the following eight identities:

$$\begin{aligned} \left(\frac{1}{e^t - 1} \right)^{(k)} &= \sum_{m=1}^{k+1} \lambda_{k,m} \left(\frac{1}{e^t - 1} \right)^m, \\ \left(\frac{1}{1 - e^{-t}} \right)^{(k)} &= \sum_{m=1}^{k+1} \mu_{k,m} \left(\frac{1}{1 - e^{-t}} \right)^m, \\ \left(\frac{1}{1 - e^{-t}} \right)^{(k)} &= \sum_{m=1}^{k+1} \lambda_{k,m} \left(\frac{1}{e^t - 1} \right)^m, \\ \left(\frac{1}{e^t - 1} \right)^{(k)} &= \sum_{m=1}^{k+1} \mu_{k,m} \left(\frac{1}{1 - e^{-t}} \right)^m, \\ \left(\frac{1}{1 - e^{-t}} \right)^k &= \sum_{m=1}^k a_{k,m-1} \left(\frac{1}{1 - e^{-t}} \right)^{(m-1)}, \\ \left(\frac{1}{e^t - 1} \right)^k &= \sum_{m=1}^k b_{k,m-1} \left(\frac{1}{e^t - 1} \right)^{(m-1)}, \end{aligned}$$

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$$\begin{aligned} \left(\frac{1}{1-e^{-t}}\right)^k &= 1 + \sum_{m=1}^k a_{k,m-1} \left(\frac{1}{e^t-1}\right)^{(m-1)}, \\ \left(\frac{1}{e^t-1}\right)^k &= 1 + \sum_{m=1}^k b_{k,m-1} \left(\frac{1}{1-e^{-t}}\right)^{(m-1)}, \end{aligned}$$

where $1 \leq m \leq k$, the quantities

$$S(k, m) = \frac{1}{m!} \sum_{\ell=1}^m (-1)^{m-\ell} \binom{m}{\ell} \ell^k$$

are the Stirling numbers of the second kind,

$$\begin{aligned} \lambda_{k,m} &= (-1)^k (m-1)! S(k+1, m), \quad \mu_{k,m} = (-1)^{m-1} (m-1)! S(k+1, m), \\ a_{k,m-1} &= (-1)^{m^2+1} M_{k-m+1}(k, m), \quad b_{k,m-1} = (-1)^{k-m} a_{k,m-1}, \end{aligned}$$

and

$$M_j(k, i) = \begin{vmatrix} \frac{1}{(i-1)!} \binom{k}{i} & S(i+1, i) & S(i+2, i) & \cdots & S(i+j-1, i) \\ \frac{1}{i!} \binom{k}{i+1} & S(i+1, i+1) & S(i+2, i+1) & \cdots & S(i+j-1, i+1) \\ \frac{1}{(i+1)!} \binom{k}{i+2} & 0 & S(i+2, i+2) & \cdots & S(i+j-1, i+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(i+j-2)!} \binom{k}{i+j-1} & 0 & 0 & \cdots & S(i+j-1, i+j-1) \end{vmatrix}$$

for $j \in \mathbb{N}$. In [5, Remark 5.1], it was pointed out that the four functions

$$\frac{1}{e^t-1}, \quad \frac{1}{1-e^{-t}}, \quad \frac{1}{1-e^{-t}}, \quad \text{and} \quad \frac{1}{e^t-1}$$

are respectively the unique solutions to the linear ordinary equations

$$\sum_{i=1}^k a_{k,i-1} y^{(i-1)} = F_n(t) \quad \text{and} \quad \sum_{i=1}^k (-1)^{i-1} a_{k,i-1} y^{(i-1)} = G_n(t)$$

for all $k \in \mathbb{N}$ and $n = 1, 2$, where

$$F_n(t) = \begin{cases} \frac{1}{(1-e^{-t})^k} - 1, & n = 1; \\ \frac{1}{(1-e^{-t})^k}, & n = 2 \end{cases} \quad \text{and} \quad G_n(t) = \begin{cases} \frac{1}{(e^t-1)^k} - 1, & n = 1; \\ \frac{1}{(e^t-1)^k}, & n = 2. \end{cases}$$

In [5, Remark 5.2], it was concluded that

$$\begin{pmatrix} \frac{1}{e^t-1} \\ \left(\frac{1}{e^t-1}\right)' \\ \left(\frac{1}{e^t-1}\right)'' \\ \vdots \\ \left(\frac{1}{e^t-1}\right)^{(i-1)} \\ \left(\frac{1}{e^t-1}\right)^{(i)} \end{pmatrix} = \begin{pmatrix} \lambda_{0,1} & 0 & 0 & \cdots & 0 & 0 \\ \lambda_{1,1} & \lambda_{1,2} & 0 & \cdots & 0 & 0 \\ \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{k-2,1} & \lambda_{k-2,2} & \lambda_{k-2,3} & \cdots & \lambda_{k-2,k-1} & 0 \\ \lambda_{k-1,1} & \lambda_{k-1,2} & \lambda_{k-1,3} & \cdots & \lambda_{k-1,k-1} & \lambda_{k-1,k} \end{pmatrix} \begin{pmatrix} \frac{1}{e^t-1} \\ \frac{1}{(e^t-1)^2} \\ \frac{1}{(e^t-1)^3} \\ \vdots \\ \frac{1}{(e^t-1)^i} \\ \frac{1}{(e^t-1)^{i+1}} \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{1}{e^t-1} \\ \frac{1}{(e^t-1)^2} \\ \frac{1}{(e^t-1)^3} \\ \vdots \\ \frac{1}{(e^t-1)^k} \end{pmatrix} = (-1)^{(k-1)k/2} \begin{pmatrix} a_{1,0} & 0 & 0 & \cdots & 0 \\ a_{2,0} & a_{2,1} & 0 & \cdots & 0 \\ a_{3,0} & a_{3,1} & a_{3,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k,0} & a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} \end{pmatrix} \begin{pmatrix} \frac{1}{e^t-1} \\ \left(\frac{1}{e^t-1}\right)' \\ \left(\frac{1}{e^t-1}\right)'' \\ \vdots \\ \left(\frac{1}{e^t-1}\right)^{(k-1)} \end{pmatrix}.$$

All of the above results in [5, 15] were established by induction.

In [22, Theorems 3.1 and 3.2], by the Faà di Bruno formula and combinatorial techniques, the above identities in [5, 15] were generalized and unified as follows. For real numbers $\alpha, \lambda \in \mathbb{R}$,

(1) if $n \in \mathbb{N}$, then

$$\left(\frac{1}{1-\lambda e^{\alpha t}}\right)^{(n)} = (-1)^n \alpha^n \sum_{k=1}^{n+1} (-1)^{k-1} (k-1)! S(n+1, k) \left(\frac{1}{1-\lambda e^{\alpha t}}\right)^k;$$

(2) if $n \in \mathbb{N}$, then

$$\left(\frac{1}{1-\lambda e^{\alpha t}}\right)^n = \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^n \frac{(-1)^{k-1}}{\alpha^{k-1}} s(n, k) \left(\frac{1}{1-\lambda e^{\alpha t}}\right)^{(k-1)};$$

where $s(n, k)$ for $n \geq k \geq 1$ denote the Stirling numbers of the first kind.

In [3, Theorem 2.1], the above identities in [5, 15, 22] were inductively proved once again and were rewritten as follows. For real constants $\lambda \neq 0$ and $\alpha \neq 0$ and for $k \in \mathbb{N}$, when $\lambda > 0$ and $t \neq -\frac{\ln \lambda}{\alpha}$ or when $\lambda < 0$ and $t \in \mathbb{R}$, we have

$$\frac{d^k}{dt^k} \left(\frac{1}{\lambda e^{\alpha t} - 1}\right) = (-1)^k \alpha^k \sum_{m=1}^{k+1} (m-1)! S(k+1, m) \left(\frac{1}{\lambda e^{\alpha t} - 1}\right)^m \quad (1.1)$$

and

$$\left(\frac{1}{\lambda e^{\alpha t} - 1}\right)^k = \frac{1}{(k-1)!} \sum_{m=1}^k \frac{(-1)^{m-1}}{\alpha^{m-1}} s(k, m) \frac{d^{m-1}}{dt^{m-1}} \left(\frac{1}{\lambda e^{\alpha t} - 1}\right). \quad (1.2)$$

1.2. The second direction. In [8, 9], it was obtained that

(1) the function $F(t) = \frac{1}{u-e^t}$ is a solution of the nonlinear differential equation

$$F^n(t) = \frac{n}{u^{n-1}} \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \sum_{\ell_1+\ell_2+\cdots+\ell_{k+1}=n} \frac{1}{\ell_1 \ell_2 \cdots \ell_{k+1}} F^{(k)}(t), \quad (1.3)$$

where $u \in \mathbb{C} \setminus \{1\}$ and $n \in \mathbb{N}$;

(2) for $m \in \mathbb{N}$ and $n \geq 0$, we have

$$H_n^{(m)}(u) = m \left(\frac{u-1}{u}\right)^{m-1} \sum_{k=0}^{m-1} \frac{1}{(k+1)!} \sum_{\ell_1+\ell_2+\cdots+\ell_{k+1}=m} \frac{H_{n+k}(u)}{\ell_1 \ell_2 \cdots \ell_{k+1}},$$

where $H_n(u)$ are called the n th Frobenius-Euler numbers which can be generated by

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \quad u \neq 1;$$

(3) for $m \in \mathbb{N}$ and $n \geq 0$, we have

$$\sum_{\ell_1+\ell_2+\cdots+\ell_m=n} \binom{n}{\ell_1, \ell_2, \dots, \ell_m} H_{\ell_1}(u) H_{\ell_2}(u) \cdots H_{\ell_m}(u)$$

$$= m \left(\frac{u-1}{u} \right)^{m-1} \sum_{k=0}^{m-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \ell_2 + \dots + \ell_{k+1} = m} \frac{H_{n+k}(u)}{\ell_1 \ell_2 \dots \ell_{k+1}};$$

(4) for $m \in \mathbb{N}$ and $n \geq 0$, we have

$$\begin{aligned} H_n^{(m)}(x|u) &= m \left(\frac{u-1}{u} \right)^{m-1} \sum_{k=0}^{m-1} \frac{1}{(k+1)!} \\ &\quad \times \sum_{\ell_1 + \ell_2 + \dots + \ell_{k+1} = m} \frac{1}{\ell_1 \ell_2 \dots \ell_{k+1}} \sum_{q=0}^n \binom{n}{q} H_{q+k}(u) x^{n-q}, \end{aligned}$$

where $H_n(x|u)$ are called the Frobenius-Euler polynomials which can be generated by

$$\frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x|u) \frac{t^n}{n!}, \quad u \neq 1.$$

Influenced by the papers [8, 9], the authors in [6] derived several formulas for the Bernoulli polynomials of the r th order

$$\left(\frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}$$

in terms of the Bernoulli numbers $B_n = B_n^{(1)}(0)$.

In [19, Theorem 2.1], it was stated that the function $F(t) = \frac{1}{e^t + 1}$ is a solution of the nonlinear differential equation

$$F^n(t) = n \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \sum_{\ell_1 + \ell_2 + \dots + \ell_{k+1} = n} \frac{1}{\ell_1 \ell_2 \dots \ell_{k+1}} F^{(k)}(t), \quad n \in \mathbb{N}. \quad (1.4)$$

By virtue of this, as did in [6], the authors of [19] presented several formulas for the Euler numbers and polynomials of r th order $E_n^{(r)}(0)$ and $E_n^{(r)}(x)$ defined by

$$\left(\frac{2}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}$$

in terms of the Euler numbers $E_n = E_n^{(1)}(0)$ which are different from the classical and traditional Euler numbers $E_n = 2^n E(\frac{1}{2})$.

In [7], it was acquired that the function $F(t) = \frac{1}{\ln(1+t)}$ is a solution of the nonlinear differential equation

$$F^{(n)}(t) = \frac{(-1)^n (n-1)!}{(1+t)^n} \sum_{j=2}^{n+1} (j-1)! H_{n-1, j-2} F^j(t) \quad (1.5)$$

for $n \in \mathbb{N}$, where

$$H_{n,0} = 1, \quad H_{n,1} = \sum_{k=1}^n \frac{1}{k}, \quad H_{0, j-1} = 0, \quad H_{n,j} = \sum_{k=1}^n \frac{H_{k-1, j-1}}{k}$$

for $2 \leq j \leq n$. Thereafter, the authors in [7] established an identity involving the higher order Bernoulli numbers of the second kind $b_n^{(k)}$ defined by

$$\left[\frac{t}{\ln(1+t)} \right]^k = \sum_{n=0}^{\infty} b_n^{(k)} \frac{t^n}{n!}.$$

In [11], it was procured that

- (1) the function $F(t) = \frac{1}{(1+\lambda t)^{1/\lambda+1}}$ is a solution of the nonlinear differential equation

$$F^{(n)}(t) = \frac{(-1)^n}{(1+\lambda t)^n} \sum_{i=1}^{n+1} (-1)^{i-1} a_i(n, \lambda) F^i(t) \quad (1.6)$$

for $n \in \mathbb{N}$, where

$$a_i(n, \lambda) = (i-1)! \lambda^{n-i+1} \sum_{m_{i-1}=0}^{n-i+1} \sum_{m_{i-2}=0}^{n-m_{i-1}-i+1} \cdots \sum_{m_1=0}^{n-m_{i-1}-\cdots-m_2-i+1} \binom{n-m_{i-1}}{n-m_{i-1}-m_{i-2}-1+\frac{i-1}{\lambda}} \cdots \binom{n-m_{i-1}}{-\cdots-m_1-i+2+\frac{2}{\lambda}} \binom{1}{\lambda}_{n-m_{i-1}-m_{i-2}-\cdots-m_1-i+1},$$

- (2) the function $F(t) = \frac{1}{(1+\lambda t)^{1/\lambda-1}}$ is a solution of the nonlinear differential equation

$$F^{(n)}(t) = \frac{(-1)^n}{(1+\lambda t)^n} \sum_{i=1}^{n+1} a_i(n, \lambda) F^i(t) \quad (1.7)$$

for $n \in \mathbb{N}$.

Hereafter, the authors in [11] gave some new identities involving degenerate Euler numbers and polynomials.

In [10], it was demonstrated that

- (1) the function $F(t) = \frac{1}{t+c}$, $t \neq -c$, is a solution of the nonlinear differential equation $F^{(n)}(t) = (-1)^n n! F^{n+1}(t)$ for $n \in \mathbb{N}$,
 (2) the function $G(t) = \frac{1}{e^t+1}$ is a solution of the nonlinear differential equation

$$G^{(n)}(t) = (-1)^n \sum_{k=1}^{n+1} (-1)^{k-1} a_{n,k} G^k(t), \quad (1.8)$$

where

$$a_{n,k} = (k-1)! \sum_{m_1=0}^{n-k+2} \sum_{m_2=0}^{n-k+2-m_1} \cdots \sum_{m_{k-2}=0}^{n-k+2-m_1-\cdots-m_{k-3}} k^{m_1} (k-1)^{m_2} \cdots 3^{m_{k-2}} \times (2^{n-m_1-m_2-\cdots-m_{k-2}-k+3} - 1). \quad (1.9)$$

Hereafter, the authors gave two identities for the Changhee polynomials of r th order $\text{Ch}_n^{(r)}(x)$ defined by

$$\left(\frac{2}{t+2}\right)^r (1+t)^x = \sum_{n=0}^{\infty} \text{Ch}_n^{(r)}(x) \frac{t^n}{n!}.$$

1.3. Derivative polynomials. Suppose f is a function whose derivative is a polynomial in f , that is, $f'(x) = P(f(x))$ for some polynomial P . Then all the higher order derivatives of f are also polynomials in f , so we have a sequence of polynomials P_n defined by $f^{(n)}(x) = P_n(f(x))$ for $n \geq 0$. As usual, we call $P_n(u)$ the derivative polynomials of f . See [2, 21] and closely related references therein.

If there exists a sequence of polynomials $P_n(x, u) = h_n(x)P_n(u)$ for $n \geq 0$ such that $f^{(n)}(x) = h_n(x)P_n(f(x))$ for $n \geq 0$, then we call $P_n(x, u)$ the modified derivative polynomials of f .

2. ALTERNATIVE VIEWPOINTS AND DERIVATIVE POLYNOMIALS

Now we are in a position to discuss the above conclusions from alternative viewpoints and (the modified) derivative polynomials.

2.1. All the above linear or nonlinear ordinary differential equations and their solutions can be alternatively regarded as problems of (the modified) derivative polynomials.

2.2. Taking $\lambda = -1$ and $\alpha = 1$ in (1.1) and simplifying yield

$$\frac{d^k}{dt^k} \left(\frac{1}{e^t + 1} \right) = (-1)^k \sum_{m=1}^{k+1} (-1)^{m-1} (m-1)! S(k+1, m) \left(\frac{1}{e^t + 1} \right)^m.$$

Comparing this with (1.8) reveals that the coefficients $a_{n,k}$ in (1.9) is just equal to $(k-1)! S(n+1, k)$. In other words,

$$\begin{aligned} S(n+1, k) &= \sum_{m_1=0}^{n-k+2} \sum_{m_2=0}^{n-k+2-m_1} \cdots \sum_{m_{k-2}=0}^{n-k+2-m_1-\cdots-m_{k-3}} k^{m_1} (k-1)^{m_2} \cdots 3^{m_{k-2}} \\ &\quad \times (2^{n-m_1-m_2-\cdots-m_{k-2}-k+3} - 1). \end{aligned}$$

We can also say that the derivative polynomials of the function $\frac{1}{e^t+1}$ is

$$P_k(x) = (-1)^k \sum_{m=1}^{k+1} (-1)^{m-1} (m-1)! S(k+1, m) x^m.$$

It is much straightforward and simple to see that the derivative polynomials of the function $\frac{1}{t+c}$ is $P_n(x) = (-1)^n n! x^{n+1}$. Consequently, we find an alternative viewpoint to examine the results in [10].

2.3. In combinatorial analysis, the Faà di Bruno formula plays an important role and can be described in terms of the Bell polynomials of the second kind

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!} \right)^{\ell_i}$$

for $n \geq k \geq 0$, see [1, p. 134, Theorem A], by

$$\frac{d^n}{dt^n} [f \circ h(t)] = \sum_{k=0}^n f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)) \quad (2.1)$$

for $n \geq 0$, see [1, p. 139, Theorem C]. Replacing $f(u)$ and $u = h(t)$ in (2.1) respectively by $\frac{1}{u \pm 1}$ and $(1 + \lambda t)^{1/\lambda}$ yields

$$\begin{aligned} f^{(k)}(u) &= \frac{(-1)^k k!}{(u \pm 1)^{k+1}}, \\ h^{(k)}(t) &= \left[\prod_{\ell=0}^{k-1} (1 - \ell\lambda) \right] (1 + \lambda t)^{1/\lambda - k}, \end{aligned}$$

and

$$\begin{aligned} \left[\frac{1}{(1 + \lambda t)^{1/\lambda \pm 1}} \right]^{(n)} &= \sum_{k=0}^n \frac{(-1)^k k!}{[(1 + \lambda t)^{1/\lambda \pm 1}]^{k+1}} B_{n,k} \left((1 + \lambda t)^{1/\lambda - 1}, \right. \\ &\quad \left. (1 - \lambda)(1 + \lambda t)^{1/\lambda - k}, \dots, \left[\prod_{\ell=0}^{n-k} (1 - \ell\lambda) \right] (1 + \lambda t)^{1/\lambda - n + k - 1} \right) \end{aligned}$$

$$= \frac{1}{(1 + \lambda t)^n} \sum_{k=0}^n \frac{(-1)^k k! (1 + \lambda t)^{k/\lambda}}{[(1 + \lambda t)^{1/\lambda} \pm 1]^{k+1}} B_{n,k} \left(1, 1 - \lambda, \dots, \prod_{\ell=0}^{n-k} (1 - \ell\lambda) \right),$$

where we used above the formula

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$$

listed in [1, p. 135].

In [1, p. 133], it was listed that

$$\frac{1}{k!} \left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!}$$

for $k \geq 0$. Therefore, we obtain that

$$\begin{aligned} \sum_{n=k}^{\infty} B_{n,k} \left(1, 1 - \lambda, \dots, \prod_{\ell=0}^{n-k} (1 - \ell\lambda) \right) \frac{t^n}{n!} &= \frac{1}{k!} \left(\sum_{m=1}^{\infty} \prod_{\ell=0}^{m-1} (1 - \ell\lambda) \frac{t^m}{m!} \right)^k \\ &= \frac{1}{k!} [(\lambda t + 1)^{1/\lambda} - 1]^k = \frac{1}{k!} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} (\lambda t + 1)^{\ell/\lambda}. \end{aligned}$$

Accordingly, it follows that

$$\begin{aligned} B_{n,k} \left(1, 1 - \lambda, \dots, \prod_{\ell=0}^{n-k} (1 - \ell\lambda) \right) &= \frac{1}{k!} \lim_{t \rightarrow 0} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} [(\lambda t + 1)^{\ell/\lambda}]^{(n)} \\ &= \frac{1}{k!} \lim_{t \rightarrow 0} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda) (1 + \lambda t)^{\ell/\lambda - n} \\ &= \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda). \end{aligned}$$

In a word, we derive that

$$\begin{aligned} \left[\frac{1}{(1 + \lambda t)^{1/\lambda} \pm 1} \right]^{(n)} &= \frac{1}{(1 + \lambda t)^n} \sum_{k=0}^n \left[\sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda) \right] \frac{(1 + \lambda t)^{k/\lambda}}{[(1 + \lambda t)^{1/\lambda} \pm 1]^{k+1}}. \quad (2.2) \end{aligned}$$

The coefficients in the equation (2.2) are more nice, more explicit, easier computable than $a_i(n, \lambda)$ in (1.6) and (1.7).

By the way we note that the form of the equation (2.2) is different from the one in (1.6) and (1.7).

2.4. In [13, 16], it was obtained that

$$\left[\frac{1}{\ln(1+t)} \right]^{(m)} = \frac{1}{(1+t)^m} \sum_{i=0}^m (-1)^i i! \frac{s(m, i)}{[\ln(1+t)]^{i+1}}, \quad m \geq 0$$

and

$$\left(\frac{1}{\ln x} \right)^{(n)} = \frac{(-1)^n}{x^n} \sum_{i=2}^{n+1} \frac{a_{n,i}}{(\ln x)^i}, \quad n \in \mathbb{N},$$

where $a_{n,2} = (n-1)!$ and

$$a_{n,i} = (i-1)!(n-1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{i-3}=1}^{\ell_{i-4}-1} \frac{1}{\ell_{i-3}} \sum_{\ell_{i-2}=1}^{\ell_{i-3}-1} \frac{1}{\ell_{i-2}}$$

for $n + 1 \geq i \geq 3$. Comparing this with (1.5) and rearranging lead to

$$H_{n-1, i-1} = \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{i-3}=1}^{\ell_{i-4}-1} \frac{1}{\ell_{i-3}} \sum_{\ell_{i-2}=1}^{\ell_{i-3}-1} \frac{1}{\ell_{i-2}} = (-1)^{n+i} \frac{s(n, i)}{(n-1)!}$$

for $3 \leq i \leq n$. This connects (1.5) with the Stirling numbers of the first kind $s(n, i)$ and the modified derivative polynomials: the modified derivative polynomials of $\frac{1}{\ln(1+t)}$ is

$$P_m(t, u) = \frac{1}{(1+t)^m} \sum_{i=0}^m (-1)^i i! s(m, i) \frac{1}{u^{i+1}}, \quad m \geq 0.$$

Hence, we are viewing the paper [7] from a different angle.

2.5. Letting $\lambda = -1$ and $\alpha = 1$ in (1.2) results in

$$\left(\frac{1}{e^t + 1} \right)^k = \frac{1}{(k-1)!} \sum_{m=1}^k (-1)^{m+k} s(k, m) \frac{d^{m-1}}{dt^{m-1}} \left(\frac{1}{e^t + 1} \right). \quad (2.3)$$

Comparing this with (1.4) gives

$$s(n, k+1) = (-1)^{n+k+1} \frac{n!}{(k+1)!} \sum_{\ell_1+\ell_2+\cdots+\ell_{k+1}=n} \frac{1}{\ell_1 \ell_2 \cdots \ell_{k+1}}, \quad n \in \mathbb{N}. \quad (2.4)$$

As a result, Theorem 2.1 in [19] can be simplified as (2.3) and has something to do with the Stirling numbers of the first kind $s(n, k)$.

2.6. Setting $\lambda = \frac{1}{u}$ and $\alpha = 1$ in (1.2) produces

$$\left(\frac{1}{u - e^t} \right)^k = (-1)^k \frac{1}{(k-1)!} \frac{1}{u^{k-1}} \sum_{m=0}^{k-1} (-1)^{m+1} s(k, m+1) \frac{d^m}{dt^m} \left(\frac{1}{u - e^t} \right)$$

for $k \in \mathbb{N}$. Comparing this with (1.3) recovers the formula (2.4) which is an alternative expression for the Stirling numbers of the first kind $s(n, k)$. Therefore, all the above-mentioned results in [8, 9] can be restated simply in terms of the Stirling numbers of the first kind $s(n, k)$.

2.7. In [3, 5, 22], some formulas for computing the Bernoulli numbers, the Euler polynomials, the Apostol-Bernoulli numbers, the Eulerian polynomials, and the Fubini numbers in terms of the Stirling numbers of the second kind were established. These formulas are more concise, simpler, more meaningful than those in [6, 7, 8, 9, 10, 19], as showed above. Due to limitation on the length of the paper, we will not elaborate in further detail.

2.8. In [4, 14, 17, 18, 20, 21], there are more infirmation and new conclusions about (the modified) derivative polynomials, explicit formulas for the Bernoulli numbers and polynomials, the Euler numbers and polynomials, higher derivatives of some elementary functions, properties of the functions $\frac{\pm 1}{e^{\pm t} - 1}$, and the like. Due to limitation on the length of the paper, we will not elaborate in further detail yet.

2.9. It should be common knowledge that mathematicians should try to represent, explain, or interpret a new mathematical quantities in terms of some known and popular quantities in mathematics. Once a new mathematical quantity were established connections or relations with some famous or important quantities, it would be more meaningful in mathematics.

Remark 1. By the way, the formula (3.5) in [16, p. 325, Corollary 3.1] should be slightly corrected as

$$\left[\frac{x}{\ln(1+x)} \right]^{(i)} = \frac{1}{(1+x)^i} \sum_{k=1}^i \frac{(-1)^k k! [xs(i, k) + i(1+x)s(i-1, k)]}{[\ln(1+x)]^{k+1}}, \quad i \geq 2$$

or

$$\left[\frac{x}{\ln(1+x)} \right]^{(i)} = \frac{1}{(1+x)^i} \sum_{k=0}^i \frac{(-1)^k k! [xs(i, k) + i(1+x)s(i-1, k)]}{[\ln(1+x)]^{k+1}}, \quad i \geq 1,$$

where $s(n, k)$ for $n, k \geq 0$ denote the Stirling numbers of the first kind.

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