A MODEL FOR HIGH REYNOLDS NUMBER VORTEX RINGS

F. Kaplanski*[a)], Y.Fukumoto**& Y. Rudi*
*Laboratory of Multiphase Media Physics, Tallinn University of Technology, Tallinn, 12618 Estonia
**Graduate School of Mathematics and Mathematical Research Centre for Industrial Technology, Kyushu University, Fukuoka, 812-8581 Japan

Summary The vortex ring solution of the Stokes equations is modified to allow it to represent some aspects of high-Reynolds-number vortex rings. Based on this modification, new expressions for the time dependent translation velocity, energy, circulation and streamfunction are derived. The model introduces two nondimensional parameters that govern an elliptical shape of the vortex core:$\lambda \geq 1$ and $\beta \geq 1$. To validate the model, the data adapted from the numerical studies of a vortex ring performed by Danaila and Helie (2008) and Danaila *et al.* (2009) are used. It is shown that the predicted temporal evolution of the translation velocity at high Reynolds numbers matches very well with the experiments and numerical simulations. Moreover, the model offers an accurate description of the vortex ring topology and gives a reasonable estimate of the volume of fluid carried inside the vortex bubble.

INTRODUCTION

An important progress in describing of vortex ring dynamics was made in the studies by Norbury and Fraenkel [1,2]. Predictions based on their ideal vortex ring model were found to agree well with measurements of the formation number [3,4]. However, the hypotheses used in the Norbury- Fraenkel model are far from being realistic: the vorticity distribution is linear and the dynamics is inviscid. In the paper [5] , a vortex ring model taking into account the viscosity was derived at low Reynolds numbers, in which the core has a Gausssian vorticity distribution. The parameters describing in this model a vortex of given impulse M are the vortex radius R_0 and the diffusivity scale of the ring core $L = \sqrt{2\nu t}$. A vortex from introduced family of vortices is identified by the ratio $\theta = R_0/L$. The vorticity distribution is

$$\zeta = \zeta_0 \exp\left(-\frac{\sigma^2+\eta^2+\theta^2}{2}\right) \mathrm{I}_1\left(\sigma\theta\right), \tag{1}$$

where $\sigma = r/L$, $\eta = (x - X_0)/L$, X_0 is the axial coordinate of the vortex center, I_1 is the modified Bessel function of the order one, and ζ_0 can be defined from the condition of the vorticity impulse conservation. The corresponding streamfunction Ψ is expressed through the integral and is given by the following eguation:

$$\Psi = \frac{M\sigma}{4\pi R_0}\int_0^\infty F(\mu,\eta)\mathrm{J}_1(\theta\mu)\mathrm{J}_1(\sigma\mu)d\mu, \tag{2}$$

where $F(\mu,\eta) = \exp(\eta\mu)\mathrm{erfc}(\frac{\mu+\eta}{\sqrt{2}}) + \exp(-\eta\mu)\mathrm{erfc}(\frac{\mu-\eta}{\sqrt{2}})$, $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)dt$ and J_1 is the Bessel function of the order one. Analytical expressions for the circulation, kinetic energy and translational velocity are derived in the form

$$\Gamma = \Gamma_0\left\{1-\exp\left(-\frac{\theta^2}{2}\right)\right\}, \Gamma_0 = \frac{M}{\pi {R_0}^2}, \tag{3}$$

$$E = \frac{M^2\theta}{2\pi^2 {R_0}^3}[\frac{1}{12}\theta^2_2\mathrm{F}_2\left(\{\frac{3}{2},\frac{3}{2}\},\{\frac{5}{2},3\},-\theta^2\right)], \tag{4}$$

$$U = \frac{M\theta\sqrt{\pi}}{4\pi^2 R_0^3}[3\exp(-\frac{\theta^2}{2})\mathrm{I}_1\left(\frac{\theta^2}{2}\right) + \frac{\theta^2}{12}{}_2\mathrm{F}_2\left(\{\frac{3}{2},\frac{3}{2}\},\{\frac{5}{2},3\},-\theta^2\right) - \frac{3\theta^2}{5}{}_2\mathrm{F}_2\left(\{\frac{3}{2},\frac{5}{2}\},\{2,\frac{7}{2}\},-\theta^2\right)\Bigg]. \tag{5}$$

The purpose of the present investigation is to derive the temporal variation of the streamfunction Ψ_e, circulation Γ_e, kinetic energy E_e and translation velocity U_e for the viscous vortex rings taking into account an elliptical core.

ELLIPTICAL VORTEX RING

The vortex ring induces a local straining field on itself which deforms the core into an elliptical structure. In order to model this structure Eq.(1) is altered to

$$\zeta = \zeta_{0e}\omega_e = \zeta_{0e}\exp\left(-\frac{\sigma_1^2+\eta_1^2/\beta^2+\theta_1^2}{2}\right)\mathrm{I}_1\left(\sigma_1\theta_1\right), \tag{6}$$

[a)] Corresponding author. E-mail: feliks.kaplanski@ttu.ee

$$\sigma_1 = r/L_1, \eta_1 = x/L_1, \theta_1 = R_0/L_1 = \lambda\theta, \quad (7)$$

where parameters $\beta \geq 1$ and $\lambda \geq 1$ define elongation and compression along axes x and r, respectively. Using the approach reported in the Ref. 5 we obtain

$$\Psi_e = \frac{M_e\sigma_1}{4\pi R_0}\int_0^\infty \exp((\beta^2-1)\mu^2/2)[\exp(-\eta_1\mu)\mathrm{erfc}(\frac{\mu\beta-\eta_1}{\sqrt{2}}) + \exp(\eta_1\mu)\mathrm{erfc}(\frac{\mu\beta+\eta_1}{\sqrt{2}})]\mathrm{J}_1(\mu\theta_1)\,\mathrm{J}_1(\sigma_1\mu)\,d\mu, \quad (8)$$

$$\Gamma_e = \frac{M_e}{\pi R_0^2}[1-\exp(-\frac{\theta_1^2}{2})] = \Gamma_{0e}[1-\exp(-\frac{\theta_1^2}{2})], \quad (9)$$

$$E_e = \frac{{M_e}^2\theta_1}{2\pi^2 R_0^3}\int_0^\infty \pi\exp((\beta^2-1)\mu^2)\mathrm{erfc}(\beta\mu)\mathrm{J}_1^2(\theta_1\mu)d\mu, \quad (10)$$

$$U_e = \frac{M_e\theta_1}{4\pi^2 R_0^3}\int_0^\infty \exp(-\mu^2)[6\sqrt{\pi}\beta\mu + \exp(\beta^2\mu^2)\pi(1-6\beta^2\mu^2)\mathrm{erfc}(\beta\mu)]\mathrm{J}_1^2(\theta_1\mu)d\mu, \quad (11)$$

where M_e is the impulse of the elliptical vortex. The results (6), (8)-(11) for $\beta = 1$, $\lambda = 1$ and $M = M_e$ reduce to the previous results (1) -(5) for the Stokes flow.

RESULTS

Equating the values of the impulse M, circulation Γ, kinetic energy E and translation velocity U, for the simulated vortex[6] with the values of these parameters for the elliptical vortex, we can obtain two nonlinear equations as follows:

$$E_d = E_e/(M_e^{1/2}\Gamma_e^{3/2}) = 0.276, \Gamma_d = \Gamma_e/(M_e^{1/3}U_e^{2/3}) = 2.128. \quad (12)$$

The solution of the system (12) is found to be ($\theta_1 = 4.14, \beta = 1.4$). This yields $\lambda = \theta_1/\theta = 4.14/3.56 = 1.16$. Also matching $\Gamma = \Gamma_e$ at $\theta_1 = 4.14$ in Eq.(9) allows to calculate Γ_{0e} and the vortex radius $R_0 = 0.783$. Substituting the solution with other reference data into (10) and (11) gives the energy $E = 0.73$ and translation velocity $U = 0.32$. These values are identical with numerical results [6]. Next we modeled the behavior of thinner and thicker rings by allowing the ellipticity parameters to depend on core thickness. For this purpose we allow the ellipticity parameters $\beta = 1+\epsilon$ and $\lambda = 1+\gamma$ to depend on the core thickness θ according to the following relations:

$$\epsilon = \epsilon_0\theta_0/\theta, \gamma = \gamma_0\theta_0/\theta \quad (\theta > \theta_0), \quad (13)$$

$$\epsilon = \epsilon_0\theta/\theta_0, \gamma = \gamma_0\theta/\theta_0 \quad (\theta \leq \theta_0). \quad (14)$$

The prediction for the speed versus time agreed very well with the large Reynolds number asymptotic solution of Fukumoto and Moffatt[7] as well as with the experiments of Weigand and Gharib [8]. The data based on the more accurate models for the velocity profile in the exit section of a piston/cylinder arrangement ((also called specified discharge velocity, SDV models) [9] supports this result as the upper bound on the translation velocity of a vortex ring in the entire time range ($t \geq 0$) . Analysis also shows that the shape of the vortex bubble predicted by the present study is different compared to the simulation [6]. The source of discrepancy in these results lies in the simulation of the velocity profile in the exit section [9]. Improved SDV model has a smaller vortex bubble volume than the conventional model and leads to a more accurate prediction of other geometric characteristics of the vortex ring, those are in reasonable agreement with our calculations. The estimation of such quantities is relavant to explanation of the phenomenon of formation number [3, 4] and the fluid entrainment in the vortex ring.

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