Turán Type inequalities for \((p, q)\)-Gamma function

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Abstract The aim of this paper is to establish new Turán-type inequalities involving the \((p, q)\)-polygamma functions. As an application, when \(p \to \infty, \ q \to 1\), we obtain some results from [14] and [15].

Keywords \((p, q)\)-Gamma function, \((p, q)\)-psi function.

2000 Mathematics Subject Classification: 33B15, 26A48.

§1. Introduction and preliminaries

The inequalities of the type

\[ f_n(x) f_{n+2}(x) - f_{n+1}^2(x) \leq 0 \]

have many applications in pure mathematics as in other branches of science. They are named by Karlin and Szegő in [8], Turán-type inequalities because the first of these type of inequalities was introduced by Turán in [18]. More precisely, he used some results of Szegő in [17] to prove the previous inequality for \(x \in (-1, 1)\), where \(f_n\) is the Legendre polynomial of degree \(n\). This classical result has been extended in many directions, as ultraspherical polynomials, Laguerre and Hermite polynomials, or Bessel functions, and so forth. Many results of Turán-type have been established on the zeros of special functions.

Recently, W. T. Sulaiman in [15] proved some Turán-type inequalities for some \(q\)-special functions as well as the polygamma functions, by using the following inequality:

Lemma 1.1. Let \(a \in R_+ \cup \{\infty\}\) and let \(f\) and \(g\) be two nonnegative functions. Then

\[ \left( \int_0^a g(x) f^{m+n} dx \right)^2 \leq \left( \int_0^a g(x) f^m dx \right) \left( \int_0^a g(x) f^n dx \right) \]  (1)

Let’s give some definitions for gamma and polygamma function.

The Euler gamma function \(\Gamma(x)\) is defined for \(x > 0\) by

\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \]
The digamma (or psi) function is defined for positive real numbers \( x \) as the logarithmic derivative of Euler’s gamma function, that is \( \psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \). The following integral and series representations are valid (see [2]):

\[
\psi(x) = -\gamma + \int_{0}^{\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n + x)},
\]

where \( \gamma = 0.57721 \cdots \) denotes Euler’s constant.

Euler gave another equivalent definition for the \( \Gamma(x) \) (see [12,13])

\[
\Gamma_p(x) = \frac{p!p^x}{x(x+1) \cdots (x+p)} = \frac{p^x}{x(1 + \frac{x}{1}) \cdots (1 + \frac{x}{p})}, \quad x > 0,
\]

where \( p \) is positive integer, and

\[
\Gamma(x) = \lim_{p \to \infty} \Gamma_p(x).
\]

The following representations are valid:

\[
\Gamma_p(x) = \int_{0}^{p} \left(1 - \frac{t}{p}\right)^p e^{-t} dt,
\]

\[
\psi_p(x) = \ln p - \int_{0}^{\infty} \frac{e^{-xt}(1 - e^{-(p+1)t})}{1 - e^{-t}} dt,
\]

\[
\psi_p^{(m)}(x) = (-1)^{m+1} \int_{0}^{\infty} \frac{t^m e^{-xt}(1 - e^{-pt})}{1 - e^{-t}} dt.
\]

Jackson defined the \( q \)-analogue of the gamma function as

\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(x; q)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1,
\]

\[
\Gamma_q(x) = \frac{(q^{-1}; q^{-1})_\infty}{(xq^{-1}; q^{-1})_\infty} (q - 1)^{1-x} q^{\binom{x}{2}}, \quad q > 1,
\]

where \( (a; q)_\infty = \prod_{j \geq 0} (1 - aq^j) \).

The \( q \)-gamma function has the following integral representation

\[
\Gamma_q(t) = \int_{0}^{\infty} x^{t-1} E_q^{-aq} dx,
\]

where \( E_q^x = \sum_{j=0}^{\infty} q^{\frac{j(j+1)}{2}} x^n = (1 + (1 - q)x)^{-\infty} \), which is the \( q \)-analogue of the classical exponential function.

It is well known that \( \Gamma_q(x) \to \Gamma(x) \) and \( \psi_q(x) \to \psi(x) \) as \( q \to 1 \).

**Definition 1.1.** For \( x > 0 \), \( p \in \mathbb{N} \) and for \( q \in (0, 1) \),

\[
\Gamma_{p,q}(x) = \frac{[p]_q! [p]_q!}{[x]_q [x+1]_q \cdots [x+p]_q},
\]

where \( [p]_q = \frac{1-q^p}{1-q} \).
The \((p, q)\)-analogue of the psi function is defined as the logarithmic derivative of the \((p, q)\)-gamma function, and has the following series representation and integral representation:

\[
\psi_{(p,q)}(x) = -\ln[p]_q - \log q \sum_{k=0}^{p} \frac{q^{x+k}}{1 - q^{x+k}},
\]

\[
\psi_{(p,q)}(x) = -\ln[p]_q - \int_{0}^{\infty} \frac{e^{-xt}}{1 - e^{-t}}(1 - e^{-(p+1)t})d\gamma_q(t),
\]

\[
\psi_{(p,q)}^{(n)}(x) = (-1)^{n+1} \int_{0}^{\infty} \frac{t^n e^{-xt}}{1 - e^{-t}}(1 - e^{-(p+1)t})d\gamma_q(t).
\]

where \(\gamma_q(t)\) is a discrete measure with positive masses-\(\log\) at the positive points-\(k\log q, k = 1, 2, \cdots\) i.e.

\[
\gamma_q(t) = -\log q \sum_{k=1}^{\infty} \delta(t + k \log q), \quad 0 < q < 1.
\]

In this paper, we give an extension of the main result of W. T. Sulaiman \cite{15}, V. Krasniqi etc. \cite{13} and C. Mortici \cite{14}.

\section*{2. Main results}

\textbf{Theorem 2.1.} For \(n = 1, 2, 3, \cdots\), let \(\psi_{(p,q),n} = \psi_{(p,q)}^{(n)}\) be the \(n\)-th derivative of the function \(\psi_{(p,q)}\). Then

\[
\psi_{(p,q),\frac{m}{s} + \frac{n}{t}}\left(\frac{x}{s} + \frac{y}{t}\right) \leq \psi_{(p,q),m}(x)\psi_{(p,q),n}(y),
\]

where \(\frac{m+n}{s} + \frac{1}{t} = 1\).

\textbf{Proof.} Let \(m\) and \(n\) be two integers of the same parity. From (10), it follows that:

\[
\psi_{(p,q),\frac{m}{s} + \frac{n}{t}}\left(\frac{x}{s} + \frac{y}{t}\right) = (-1)^{m+n} \int_{0}^{\infty} \frac{t^{\frac{m}{s} + \frac{n}{t}} e^{-\left(\frac{x}{s} + \frac{y}{t}\right)t}}{1 - e^{-t}}(1 - e^{-(p+1)t})d\gamma_q(t)
\]

\[
= (-1)^{m+n} (-1)^{n+1} \int_{0}^{\infty} \frac{t^{\frac{m+n}{s} + \frac{1}{t}} e^{-\left(\frac{x}{s} + \frac{y}{t}\right)t}}{1 - e^{-t} + \frac{1}{t}}(1 - e^{-(p+1)t})^\frac{1}{s} d\gamma_q(t)
\]

\[
\leq \left[(-1)^{m+1} \int_{0}^{\infty} \frac{t^{m} e^{-xt}}{1 - e^{-t}}(1 - e^{-(p+1)t})d\gamma_q(t)\right]^\frac{1}{s}
\]

\[
\cdot \left[(-1)^{n+1} \int_{0}^{\infty} \frac{t^{n} e^{-yt}}{1 - e^{-t}}(1 - e^{-(p+1)t})d\gamma_q(t)\right]^\frac{1}{t}
\]

\[
= \psi_{(p,q),m}(x)\psi_{(p,q),n}(y).
\]
Remark 2.1. Let \( p \) tends to \( \infty \), then we obtain Theorem 2.2 from [15]. On putting \( y = x \) then we obtain generalization of Theorem 2.1 from [15].

Another type via Minkowski’s inequality is the following:

**Theorem 2.2.** For \( n = 1, 2, 3, \ldots \), let \( \psi_{(p,q),n} = \psi^{(n)}(x) \) the \( n \)-th derivative of the function \( \psi_{(p,q),n} \). Then

\[
\left( \psi_{(p,q),m}(x) + \psi_{(p,q),n}(y) \right)^{\frac{1}{p}} \leq \psi_{(p,q),m}(x) + \psi_{(p,q),n}(y),
\]

where \( \frac{\alpha + \gamma}{\alpha} \) is an integer, \( p \geq 1 \).

**Proof.**

Since

\[
(a + b)^p \geq a^p + b^p, \quad a, b \geq 0, \quad p \geq 1,
\]

\[
\left( \psi_{(p,q),m}(x) + \psi_{(p,q),n}(y) \right)^{\frac{1}{p}} = \left[ (-1)^{n+1} \int_0^\infty \psi_{(p,q),m}(x) \frac{t^n e^{-xt}}{1 - e^{-t}} d\gamma_q(t) \right]^{\frac{1}{p}}
\]

\[
+ \left[ (-1)^{n+1} \int_0^\infty \psi_{(p,q),n}(y) \frac{t^n e^{-yt}}{1 - e^{-t}} d\gamma_q(t) \right]^{\frac{1}{p}}
\]

\[
\leq \left[ \int_0^\infty \left( \psi_{(p,q),m}(x) \frac{t^n e^{-xt}}{1 - e^{-t}} \right)^{\frac{1}{p}} d\gamma_q(t) \right]^{\frac{1}{p}}
\]

\[
+ \left[ \int_0^\infty \left( \psi_{(p,q),n}(y) \frac{t^n e^{-yt}}{1 - e^{-t}} \right)^{\frac{1}{p}} d\gamma_q(t) \right]^{\frac{1}{p}}
\]

Remark 2.2. Let \( p \) tends to \( \infty \), then we obtain generalization of Theorem 2.3 from [15].
Theorem 2.3. For every $x > 0$ and integers $n \geq 1$, we have:

1. If $n$ is odd, then \( \left( \exp \psi^{(n)}_{(p,q)}(x) \right)^2 \geq \exp \psi^{(n+1)}_{(p,q)}(x) \exp \psi^{(n-1)}_{(p,q)}(x) \);

2. If $n$ is even, then \( \left( \exp \psi^{(n)}_{(p,q)}(x) \right)^2 \leq \exp \psi^{(n+1)}_{(p,q)}(x) \exp \psi^{(n-1)}_{(p,q)}(x) \).

Proof. We use (10) to estimate the expression

\[
\psi^{(n)}_{(p,q)}(x) - \frac{\psi^{(n+1)}_{(p,q)}(x) + \psi^{(n-1)}_{(p,q)}(x)}{2} = (-1)^{n+1} \left( \int_{0}^{\infty} t^n e^{-xt} \frac{1-e^{-(p+1)t}}{1-e^{-t}} d\gamma_q(t) \right)
\]

\[+ \frac{1}{2} \int_{0}^{\infty} t^{n+1} e^{-xt} \left( 1 - e^{-(p+1)t} \right) d\gamma_q(t) \]

\[+ \frac{1}{2} \int_{0}^{\infty} t^{n-1} e^{-xt} \left( 1 - e^{-(p+1)t} \right) d\gamma_q(t) \]

\[= (-1)^{n+1} \left( \int_{0}^{\infty} t^{n-1} e^{-xt} \frac{(t+1)^2(1-e^{-(p+1)t})}{1-e^{-t}} d\gamma_q(t) \right). \]

Now, the conclusion follows by exponentiating the inequality

\[\psi^{(n)}_{(p,q)}(x) \geq \left( \frac{\psi^{(n+1)}_{(p,q)}(x) + \psi^{(n-1)}_{(p,q)}(x)}{2} \right) \]

as $n$ is odd, respectively even.

Remark 2.3. Let $p$ tends to $\infty$, $q$ tends to $1$, then we obtain generalization of Theorem 3.3 from [14].

References


