The dynamics of holomorphic germs near a curve of fixed points

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Abstract

One of the interesting areas in the study of the local dynamics in several complex variables is the dynamics near the origin $O$ of maps tangent to the identity, that is of germs of holomorphic self-maps $f : \mathbb{C}^n \to \mathbb{C}^n$ such that $f(O) = O$ and $d f_O = \text{id}$. When $n = 1$ the dynamics is described by the known Leau–Fatou flower theorem but when $n > 1$, we are still far from understanding the complete picture, even though very important results have been obtained in recent years (see, e.g., [2,7,10,19]). In this note we want to investigate conditions ensuring the existence of parabolic curves (the two-variable analogue of the petals in the Leau–Fatou flower theorem) for maps tangent to the identity in dimension 2. Using simple examples, we prove that these conditions are not, generally, sufficient.

Keywords: Complex dynamics; Parabolic curve; Characteristic direction; Residual index; Flower theorem

1. Introduction

Complex dynamics goes back to about 1871 with some work done by Schröder [16]. Suppose that $F$ is a holomorphic function near 0 in $\mathbb{C}$ with $F(0) = 0$. A main question is whether one can change coordinates so that $F$ becomes a linear function. This leads to the Schröder functional equation for the coordinate change $\phi$:

$$ \phi \circ F = \lambda \phi $$

where $\phi'(0) = 1$ and $\lambda = F'(0)$. For the case when $0 < |\lambda| < 1$, this was resolved by the end of the nineteenth century.

The case when $F'(0) = 1$ is more difficult. One cannot solve the Schröder equation in this case, but there is generically an open region which is attracted to 0. The precise statement is usually called the flower theorem.

In 1942 Siegel solved the Schröder equation for certain cases when $|\lambda| = 1$. Basically, $\lambda$ had to be far enough from roots of unity. His methods used power series and fall within the complex analytic methods.

In particular Siegel’s work shows that for sufficiently irrational $\lambda$ the derivative uniquely specifies the map up to holomorphic conjugation. The case when the derivative is 1 at the fixed point is more delicate. In 1983 Martinet and Ramis [13] showed that if $f$ and $g$ are two parabolic germs of holomorphic functions in $\mathbb{C}$ and they are formally conjugated then they are topologically conjugated. Moreover they showed that if they are $C^1$ conjugated they are actually holomorphically conjugated.
It is very natural to ask similar questions for higher dimension. So consider germs of holomorphic maps \( F \) defined near the origin in \( \mathbb{C}^2 \). Then one can also in this case study the Schröder equation and ask when it is solvable. Much as in one dimension the Theorem of Siegel carries over and shows that the derivative uniquely determines the map up to holomorphic conjugacy in the case when the eigenvalues are of modulus 1 and are sufficiently irrational and irrationally independent. However, the case when the maps are parabolic and the map is tangent to the identity is rather difficult. Works on such maps have been done by Weickert [19], Hakim [10,11], Bracci [4] and Abate [2,3] recently. A main open question is the following: If \( F \) is tangent to the identity in \( \mathbb{C}^2 \) and \( F \) has a curve of fixed points, and \( F \) is not the identity, must \( F \) always have a parabolic curve?

One of the interesting areas in the study of the local dynamics in several complex variable is the dynamics near the origin \( O \) of maps tangent to the identity, that is of germs of holomorphic self-maps \( f : \mathbb{C}^n \to \mathbb{C}^n \) such that \( f(O) = O \) and \( df_O = \text{id} \). When \( n = 1 \) the dynamics is described by the celebrated Leau–Fatou flower theorem but when \( n > 1 \), we are still far from understanding the complete picture, even though very important results have been obtained in recent years (see, e.g., [2,7,10,19]).

In this note we want to investigate conditions ensuring the existence of parabolic curves (the two-variable analogue of the petals in the Leau–Fatou flower theorem) for maps tangent to the identity in dimension 2.

2. Background and notation

Let us first recall some definitions and useful results. Let \( f \) be a germ of a holomorphic self-map of \( \mathbb{C}^n \) fixing the origin and tangent to the identity; we can write \( f = (f_1, \ldots, f_n) \), and let \( f_j = z_j + P_{j,v_1} + P_{j,v_1+1} + \cdots \) be the homogeneous expansion of \( f_j \) in a series of homogeneous polynomials, where \( \deg P_{j,k} = k \) (or \( P_{j,k} \equiv 0 \)) and \( P_{j,v_j} \neq 0 \), for \( j = 1, \ldots, n \). The order \( v(f) \) of \( f \) is defined by \( v(f) = \min\{v_1, \ldots, v_n\} \).

**Definition 1.** A parabolic curve for \( f \) at the origin is an injective holomorphic map \( \varphi : \Delta \to \mathbb{C}^n \) satisfying the following properties:

1. \( \Delta \) is simply connected domain in \( \mathbb{C} \), with \( 0 \in \partial \Delta \),
2. \( \varphi \) is continuous at the origin and \( \varphi(0) = O \),
3. \( \varphi(\Delta) \) is invariant under \( f \) (i.e. \( f(\varphi(\Delta)) \subset \varphi(\Delta) \)) and \( (f_{|\varphi(\Delta)})^n \to O \) as \( n \to \infty \).

Furthermore, if \( [\varphi(\xi)] \to [v] \in \mathbb{P}^{n-1} \) as \( \xi \to 0 \), where \([\cdot]\) denotes the canonical projection of \( \mathbb{C}^n \setminus \{O\} \) onto \( \mathbb{P}^{n-1} \) (projective space), we say that \( \varphi \) is tangent to \([v]\) at the origin.

**Definition 2.** A characteristic direction for \( f \) is a vector \( [v] = [v_1, \ldots, v_n] \in \mathbb{P}^{n-1} \) such that there is \( \lambda \in \mathbb{C} \) so that \( P_{j,v(f)}(v_1, \ldots, v_n) = \lambda v_j \) for \( j = 1, 2 \). If \( \lambda \neq 0 \), we say that \([v]\) is non-degenerate; otherwise, it is degenerate.

3. Dynamics

One of the most famous theorems in one-dimensional holomorphic dynamics is the following theorem:

**Theorem 1 (Leau–Fatou Flower Theorem [8,12]).** Let \( g(\xi) = \xi + a_k \xi^k + O(\xi^{k+1}) \), with \( k \geq 2 \) and \( a_k \neq 0 \), be a holomorphic function fixing the origin. Then, there are \( k-1 \) disjoint domains \( D_1, \ldots, D_{k-1} \) with the origin in their boundary, invariant under \( g \) (i.e. \( g(D_j) \subset D_j \)) and such that \( (g \setminus D_j) \to 0 \) as \( n \to \infty \), for \( j = 1, \ldots, k-1 \), where \( g^n \) denotes the composition of \( g \) with itself \( n \) times.

Any such domain is called a parabolic domain for the function \( g \) at the origin. Parabolic domains are (together with attracting basins, Siegel disks and Hermann rings) among the building blocks of Fatou sets of rational functions (see e.g. [6] for a modern exposition). A natural problem in higher dimensional holomorphic dynamics is that of finding a generalization of this result, where the function \( g \) is replaced by a germ \( f \) of a self-map of \( \mathbb{C}^n \) fixing the origin and tangent to the identity, that is, such that \( df_O = \text{id} \). Following preliminary results for in \( \mathbb{C}^2 \) obtained by Ueda [17] and Weickert [19], a very important step in this direction has been made by Hakim [10,11] (inspired by previous works by Ecalle [7]).

Characteristic directions arise naturally if we want to investigate the existence of parabolic curves tangent to some direction \([v]\). In fact, Hakim [10] observed that if there exist parabolic curves tangent to a direction \([v]\),
then this direction is necessary characteristic. However, Hakim was able to prove the converse for a non-degenerate characteristic direction only in the following theorem:

**Theorem 2** ([7,10,11]). Let \(f\) be a (germ of a) holomorphic self-map of \(\mathbb{C}^n\) fixing the origin and tangent to the identity. Then for every non-degenerate characteristic direction \([v]\) of \(f\) there are \(v(f) - 1\) parabolic curves tangent to \([v]\) at the origin.

When \(f\) has no non-degenerate characteristic direction, this theorem gives no information about the dynamics of \(f\). Furthermore, there are examples of parabolic curves tangent to degenerate characteristic directions such as the following one:

**Example 1.** Let us consider the germ \(f\) given by

\[
\begin{align*}
f_1(z, w) &= z + zw + w^2 - z^3 + O(z^2w, zw^2, w^3, z^4), \\
f_2(z, w) &= w\left(1 + z + w + O(z^2, zw, w^2)\right).
\end{align*}
\]

We observe that \([1, 0]\) is a degenerate characteristic direction. The line \([w = 0]\) is \(f\)-invariant and inside it \(f\) acts as the function \(z \mapsto z - z^3 + O(z^4)\). The classical Leau–Fatou theory then shows that there exist two parabolic curves for \(f\) tangent to \([1, 0]\) at the origin.

A similar situation occurred for continuous holomorphic dynamics. It has been known since the end of the nineteenth century, thanks to for example Poincaré [15], that a generic holomorphic vector field with an isolated singularity at the origin in \(\mathbb{C}^n\) admits an invariant submanifold passing through the singularity; but it remained unknown for more than one hundred years, even replacing “submanifold” by “complex analytic subvariety”, whether this was true for any holomorphic vector field with an isolated singularity. Finally, in 1982, C. Camacho and P. Sad proved the following theorem.

**Theorem 3** (Camacho and Sad [5]). Let \(F\) be a (germ of a) holomorphic vector field with an isolated singularity at \(O \in \mathbb{C}^2\). Then there exists a complex analytic subvariety invariant by \(F\) passing through the origin.

See [18] for a different proof of part of this result. It should also be mentioned that Camacho and Sad’s theorem is not true for in \(\mathbb{C}^3\); Gomez-Mont and Luengo [9] found a family of holomorphic vector fields with an isolated singularity at the origin in \(\mathbb{C}^3\) and no invariant complex analytic subvariety passing through the singularity.

A further step toward the understanding of the dynamics in a neighborhood of an isolated fixed point has been taken by Abate, who gave a complete generalization of the Leau–Fatou flower theorem for in \(\mathbb{C}^2\). Noting that the following theorem is an exact discrete analogue of the Camacho and Sad theorem

**Theorem 4** ([2]). Let \(f\) be a (germ of a) holomorphic self-map of \(\mathbb{C}^2\) tangent to the identity and such that the origin is an isolated fixed point. Then there exist at least \(v(f) - 1\) parabolic curves for \(f\) at the origin.

The proof of this theorem is based on the possibility of modifying the geometry of the ambient space via a finite number of blow-ups, and of defining a residual index \(\text{Ind}(\hat{f}, S, p) \in \mathbb{A}\) where \(\hat{f}\) is a holomorphic self-map of a complex 2-manifold \(M\) which is the identity on a one-dimensional submanifold \(S\), and \(p \in S\). It turns out that this index is either not defined anywhere on \(S\), in which case we say that \(\hat{f}\) is degenerate along \(S\) (or non-tangential to \(S\) in the terminology of [3], where there is described a far-reaching approach to indices for holomorphic self-maps), or it is everywhere defined, and then we say that \(\hat{f}\) is non-degenerate along \(S\) (respectively, tangential to \(S\).)

In particular, Abate gave a generalization of **Theorem 2** to those characteristic directions whose residual index is not a non-negative rational number.

**Corollary 1** ([2]). Let \(f\) be a (germ of a) holomorphic self-map of \(\mathbb{C}^2\) tangent to the identity and such that the origin is an isolated fixed point. Let \([v]\) be a characteristic direction of \(f\) such that \(\text{Ind}(\hat{f}, \mathbb{P}^1, [v]) \notin \mathbb{Q}^+\) (here \(\mathbb{P}^1\) is the exceptional divisor of the blow-up of the origin, and \(\hat{f}\) is the blow-up of \(f\)). Then there are, at least, \(v(f) - 1\) parabolic curves for \(f\) tangent to \([v]\) at the origin.

As another application, we can prove the existence of parabolic curves even when \(df_O\) is not diagonalizable.
Corollary 2. Let $f \in \text{End}(\mathbb{C}^2, O)$ be such that $df_O = J_2$, the canonical Jordan matrix associated with the eigenvalue 1, and assume that the origin is an isolated fixed point. Then, there is at least one parabolic curve tangent to $[1 : 0]$ for $f$ at the origin.

The theory concerning the existence of parabolic curves tangent to a direction $[v]$, for maps tangent to the identity, is thus almost complete, but there are still examples where the previous results cannot be applied.

Example 2. Let us consider the map $f$ given by

$$
\begin{align*}
&f_1(z, w) = z + zw + O(w^2, z^3, z^2w), \\
&f_2(z, w) = w + 2z^2 + bcz^3 + z^4 + O(z^5, z^2w, zw^2, w^3)
\end{align*}
$$

with $b \neq 0$.

We observe that $[v] = [1 : 0]$ is degenerate characteristic direction for $f$ with $\text{Ind}(\tilde{f}, \mathbb{P}^1, [v]) = 1$. Hence, we cannot say anything about the dynamics of $f$ in the direction $[1 : 0]$ using the Theorem 2 or the Theorem 1.

As a corollary of the work of Molino [14] (see Corollary 3), we shall able to prove the existence of parabolic curves tangent to $[1 : 0]$ also for this example (and many others). This will be a consequence of the following more general result proved by Molino [14].

Theorem 5. Let $S$ be a one-dimensional submanifold of a complex 2-manifold $M$ and let $f \in \text{End}(M, S)$ be such that $f|_S \equiv \text{Id}_S$. Assume that $df$ acts as the identity on the normal bundle of $S$ in $M$ and let $f$ be tangential to $S$. If $p \in S$ is a singular point of $f$, not a corner, with $v_0(f, p) = 1$ and $\text{Ind}(f, S, p) \neq 0$, then there exist parabolic curves for $f$ in $p$.

An important application of this result is the following one. Starting from a map $f \in \text{End}(\mathbb{C}^2, O)$ tangent to the identity and blowing up the origin, we obtain an $\tilde{f} \in \text{End}(M, S)$, where $S \cong \mathbb{P}^1$ is the exceptional divisor of the blow-up. It turns out that $\tilde{f}|_S = \text{Id}_S$, and that $d\tilde{f}$ acts as the identity on the normal bundle of $S$ in $M$. Furthermore, if the origin is an isolated fixed point of $f$, then no point $p \in S$ is a corner, and $\tilde{f}$ is tangential to $S$ if and only if the origin is non-dicritical for $f$. If the origin is dicritical for $f$, then all directions are characteristic, and there are parabolic curves tangent to all but a finite number of them; so we concentrate on the non-dicritical case. Every non-degenerate characteristic direction of $f$ is a singular point for $\tilde{f}$, and every singular point of $\tilde{f}$ is a characteristic direction of $f$. Moreover, if $p \in S$ is not singular then no infinite orbit can get arbitrarily close to $p$ (in particular, no infinite orbit can converge to $p$, and thus there can be no parabolic curves at $p$); therefore, from a dynamical point of view only singular points are interesting (for the proof of all these assertions see [1]).

Definition 3. We say that $f$ is regular along the characteristic direction $[v] \in \mathbb{P}^1$ if the pure order of $\tilde{f}$ at $[v]$ is 1, where $\tilde{f}$ is the blow-up of $f$ at the origin.

This is just a technical condition almost always satisfied (for instance, it is satisfied by the map in the Example 2).

Remark 1. If $[v]$ is a non-degenerate characteristic direction of $f$ then $f$ is regular along $[v]$. Generally, the converse is not true.

Example 3. Let us consider a map $f \in \text{End}(\mathbb{C}^2, O)$ of order 2:

$$
\begin{align*}
&f_1(z, w) = z + a_{2,0}z^2 + a_{1,1}zw + a_{0,2}w^2 + \cdots, \\
&f_2(z, w) = w + b_{2,0}z^2 + b_{1,1}zw + b_{0,2}w^2 + b_{3,0}z^3 + \cdots
\end{align*}
$$

We observe that $[1 : 0]$ is a characteristic direction for $f$ if $b_{2,0} = 0$. In this case, $f$ is regular along $[1 : 0]$ if the latter is non-degenerate or it is degenerate but $b_{3,0} \neq 0$.

Corollary 3. Let $f \in \text{End}(\mathbb{C}^2, O)$ be tangent to the identity with the origin as a non-dicritical isolated fixed point. Let $[v] \in \mathbb{P}^1$ be a characteristic direction of $f$ and assume that $f$ is regular along $[v]$ with $\text{Ind}(\tilde{f}, \mathbb{P}^1, [v]) \neq 0$ (here we identify $\mathbb{P}^1$ with the exceptional divisor of the blow-up of the origin, and $\tilde{f}$ is the blow-up of $f$). Then, there exist parabolic curves for $f$ tangent to $[v]$ at the origin.
References