THE APPLICATION OF COPULAS IN PRICING DEPENDENT CREDIT DERIVATIVES INSTRUMENTS

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Abstract:
The aim of this paper is to use copulas functions to capture the different structures of dependency when we deal with portfolios of dependent credit risks and a basket of credit derivatives. We first present the well-known result for the pricing of default risk, when there is only one defaultable firm. After that, we expose the structure of dependency with copulas in pricing dependent credit derivatives. Many studies suggest the inadequacy of multinormal distribution and then the failure of methods based on linear correlation for measuring the structure of dependency. Finally, we use Monte Carlo simulations for pricing Collateralized debt obligation (CDO) with Gaussian an Student copulas.

Key-Words: default risk, credit derivatives, CDO, copulas functions, Monte Carlo simulations.

JEL Classification: C15, G12

1. Introduction
Credit derivatives transactions are an increasingly important feature of modern financial markets. They provide an efficient means of hedging and separating credit risk from other market variables without jeopardizing relationships with borrowers. There are many different types of credit derivatives contracts. Most of them involve a fixed payment by the protection buyer to the protection seller. In return, the protection buyer receives a payment that is contingent upon the credit event. When dealing with credit derivatives and a portfolio of credit risky assets, default correlation is crucial. Also, for both the internal control and regulatory reporting, the financial industries are required to hold capital methodology, with respect to default correlation between different credits. However, most of existing credit risk models cannot be applied to analyze multiple defaults and default correlation. In a structural approach, one needs as given the dynamics of firm’s asset value.

A default occurs when the assets values are insufficient to cover liabilities. Kealhofer [20] shows that the firm’s default risk can be derived from the behavior of the firm’s asset value and the level of its obligations. The joint probability of default is the likelihood of both firms’ market asset values being below their respective default barrier in the same time. To determine this probability, it must know the market value of asset values of each firms, their asset volatilities and essentially, default correlation presented by correlation between the firm’s market asset values which are easily observable. Gersbach & Lipponer [12] explain that correlations of default risks depend on the correlations of asset returns which are assumed to be log normally distributed. The default correlations are affected by macro-economic risks. The default correlations are obtained with linear correlation coefficient between asset values. Zhou [29] provide a first passage time model for calculating default correlation and joint default probability which based on firm specific information. Default is triggered when the value of the firm hits a deterministic default boundary following Black & Cox [1]. Giesecke [13] provide a structural model of correlated multi-firms default with incomplete information concerning the default barrier (asset level) at which a firm is liquidated and therefore, the default is an unpredictable event. Stochastic dependence between default events is presented through correlated asset values (common macro-economic factors or macro- correlation) and correlated default barrier

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1 A list of credit events are presented by the international swaps and derivatives associations in the “2003 ISDA credit derivatives definitions”.

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(direct linkages between firms or micro-correlation). Davis & Lo [4] introduce a contagion model to capture the risk concentration in a portfolio of defaultable securities. They suppose that default of an issuer in a particular sector may trigger off defaults of other issuers in the same sector by an “infection” mechanism. The basic idea is that bonds i may default directly or may be “infected” by default of another bond j. Jeanblanc et al [18] discuss dependence mechanisms in a credit ratings-based framework. They proposed a multivariate Markov model for simulating the dynamics of correlated credit ratings of multiple firms because the Markov models do not account for default correlation when we dealing with a portfolio of credit risk. The rating changes across firms because the credit worthiness of issuers depends on a common set of economic factor. The model proposed is an extension of Jarrow, Lando and Turnbull [17]. Hull & White [15] develop an approach of modeling default correlation. They consider a variable describing the credit worthiness of company referred as “credit index”. They select correlated diffusion processes for the credit indices of the different firms and determine the default barrier of each company such that the company defaults of the credit indices reach default barrier. These indices start at zero and follow correlated wiener processes. Reduced form models can incorporate correlations between defaults by allowing hazard rates to be stochastic and correlated with macroeconomic variables.

However, more accurate and flexible approach is offered by Copula functions. The main potential advantage of copulas is the separation between marginal and the dependency because (in finance) the former are often understood and modelled in detail but dependency is almost interpreted as meaning Pearson’s correlation, which isn’t always appropriate. Embrechts et al [5] show the necessity to leave the Gaussian world since the normal joint distribution cannot catch some key futures of the dependence like the tail dependence and the classic correlation coefficient is only adapted for assessing linear dependence and can lead to a very strong underestimation of the real incurred risk. Then, the multivariate normal distribution is not a good model for the joint distribution of many economic variables. Many dependency measures have been proposed according to concepts such as concordance, quadrant dependency, etc. In the case of two random variables, structure of dependency can have a long variety of forms according to some specifications. Most popular examples are based on the concept of concordance and discordance which are scale invariant measures such as Kendall’s tau and the Spearman’s rho. More recently, the introduction of the theory of copulas in finance by Embrechts et al [6] has had a great impact in the study of dependence of random variables. In this paper, we use Archimedean copulas to model dependency between credit default swap prices and stock return volatility in view of the construction of bivariate distributions based on that dependency structure. Volatility is a statistical measure of the tendency of a market or security to rise or fall sharply within a short period of time. Volatile markets are characterized by wide price fluctuations and heavy trading. They are caused by things like company news, a recommendation from a well known analyst or unexpected earnings results. Mashal & Naldi [22] present a methodology for estimation, simulation and pricing of multiname credit derivatives. The dependence structure is modelled by a t-copula. Sircar and Zariphopoulou [26] study the impact of risk aversion on the valuation of basket credit derivatives. They use the technology of utility-indifference pricing in intensity based models of default risk.

In our paper, we use copulas functions to capture structures of dependency when we deal with portfolios of dependent credit risks and a CDO. The remainder of this paper is organised as follows: section two describes some mathematical background about the concept of copula and its properties. In section three, we illustrate different methods of parameters estimation of copula. Section four describes a methodology for pricing CDO instruments with Monte Carlo simulations and copulas functions. Section five concludes the paper.

2. Copulas functions

The theory of copula dates back to Sklar [27]. The copula function links the univariate margins with their full multivariate distribution. It presents a useful tool when modelling non Gaussian data since the Pearson’s correlation coefficient is adapted for linear dependence and normal distribution.
Using a copula approach, we can model the different relationships that can exist in different ranges of behavior. For \( n \) uniform random variables \( u_1, u_2, \ldots, u_n \), the joint distribution function \( C \) is defined as:

\[
C(u_1, u_2, \ldots, u_n, \theta) = Pr[U_1 \leq u_1, U_2 \leq u_2, \ldots, U_n \leq u_n]
\]

where \( \theta \) the dependence parameter.

As we only need the concept of copulas for two dimensions, we present the following definition:

A copula function is the restriction to \([0,1]^2\) of a continuous bivariate distribution function whose margins are uniform on \([0,1]\). A (bivariate) copula is a function \( C : [0,1]^2 \rightarrow [0,1] \) which satisfies the boundary conditions:

\[
C(t,0) = C(0,t) = 0 \text{ and } C(1,t) = t \text{ for } t \in [0,1] \tag{2}
\]

**Theorem 1:** Let \( F \) be a multivariate \( n \)-dimensional distribution function with marginals \( F_1, \ldots, F_n \).

Then it exists a copula such that \( F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \); \( (x_1, \ldots, x_n) \in \mathbb{R} \).

If the marginal distributions \( F_1, \ldots, F_n \) are continuous, then \( C \) is unique.

By definition, applying the cumulative distribution function (CDF) to a random variable (r.v.) results in a r.v. that is uniform on the interval \([0,1]\). Let \( X \) a random variable with continuous distribution function \( F_X \), \( F_X(X) \) is uniformly distributed on the interval \([0,1]\). This result is known as the probability integral transformation theorem and present many statistical procedures. With this result in hand, we may introduce the copula using basic statistical theory. In particular, the copula \( C \) for \((X,Y)\) is just the joint distribution function for the random couple \( F_X(X), F_Y(Y) \) provided \( F_X \) and \( F_Y \) are continuous.

The previous representation is called canonical representation of the distribution. Thus, copulas link joint distribution functions to their margins. Then, in continuous distribution, the problem of obtaining the joint distribution has reduced to selecting the appropriate copula. We can build multidimensional distributions with different marginals. Numerous copulas can be found in the literature (see Nelson [24] and Joe [19]).

**Clayton copula:** This family proposed by Clayton [3] is the following:

Let \( \Phi(t) = \frac{(t^{-\theta} - 1)}{\theta} \) with \( \theta \in (-1, \infty)/\{0\} \), then \( c_{\text{clayton}}(u, v) = \max \left[ \left( u^{-\theta} + v^{-\theta} - 1 \right)^{\frac{1}{\theta}}, 0 \right] \) \( \tag{3} \)

where \( \Phi(t) \) is the generator function. \( \theta \) expresses the degree of dependence among the marginal components. To illustrate the range of bivariate behaviour that can be represented by Clayton copula, consider the figure 1:

![Clayton copula density](https://via.placeholder.com/150)

**Figure 1.** Clayton copula density
**Gumbel Copula**: This family proposed by Gumbel [14] is the following:

Let \( \Phi(t) = (-\ln t)^\theta \), with \( \theta \geq 1 \).

\[
C^\text{Gumbel}_{\theta}(u, v) = \exp\left(-\left((-\ln u)^\theta + (-\ln v)^\theta\right)^{1/\theta}\right); \ 0 \leq u, v \leq 1.
\]  

(4)

Where \( \theta \in [1, \infty) \) controls the degree of dependence between \( u \) and \( v \).

The range of bivariate behaviour that can be represented by Gumbel copula is illustrated as follow:

![Figure 2. Gumbel copula density](image)

**Frank copula**: This family is proposed by Frank [7] as follows:

Let \( \Phi(t) = -\ln \left(\frac{e^{-\theta t} - 1}{e^{-\theta} - 1}\right) \) with

\[
C^\text{Frank}_{\theta}(u, v) = -\frac{1}{\theta} \ln\left(1 + \frac{e^{-\theta u} - 1}{e^{-\theta} - 1}\right). 
\]

(5)

The range of bivariate behaviour that can be represented by Frank copula is illustrated as follow:

![Figure 3. Frank copula density](image)
**Gaussian copula:** Let $X_1,\ldots,X_n$ be random variables which are standard normal distributed with means $\mu_1,\ldots,\mu_n$, standard deviations $\sigma_1,\ldots,\sigma_n$ and correlation matrix $\Sigma$. Then, the distribution function $C_\Sigma(u_1,\ldots,u_n)$ of the random variables $U_i = \Phi\left(X_i - \frac{\mu_i}{\sigma_i}\right), i \in \{1,\ldots,n\}$ is a Gaussian copula with correlation matrix $\Sigma$. $\Phi(.)$ denotes the cumulative univariate standard normal distribution function.

The Gaussian copula can be written as:

$$C_\Sigma(u_1,\ldots,u_n) = \frac{1}{(2\pi)^{n/2} \sqrt{\det \Sigma}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} (\mathbf{\theta} - \mathbf{\bar{\theta}})^T \Sigma^{-1} (\mathbf{\theta} - \mathbf{\bar{\theta}})\right) d\nu_1 \cdots d\nu_n \quad (6)$$

With $\Phi^{-1}$ is the inverse of the standard univariate Gaussian distribution function.

By differentiating the precedent equation with respect to $u_1,\ldots,u_n$, we obtain the density of the Gaussian copula:

$$C_\Sigma'(u_1,\ldots,u_n) = \frac{1}{\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2} (\mathbf{\nu}_1,\ldots,\mathbf{\nu}_n)ight) (\Sigma^{-1} - I) \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_n \end{pmatrix}; \quad (7)$$

With $\mathbf{\nu}_i = \Phi^{-1}(u_i), i \in \{1,\ldots,n\}$.

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**Student copula:** The t-student copula with the correlation matrix $\Sigma$ and $\nu$ degrees of freedom is presented as follow:

$$C_{\nu,\Sigma}(u_1,\ldots,u_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} k_{\nu,\Sigma} \left(1 + \frac{1}{\nu} (\mathbf{v} - \mathbf{\bar{\nu}})^T \Sigma^{-1} (\mathbf{v} - \mathbf{\bar{\nu}})\right)^{\frac{-\nu}{2}} \frac{1}{\nu} \frac{1}{\nu} \frac{1}{2} \cdots d\nu_1 \cdots d\nu_2 \quad (8)$$

With:
In terms of the appropriate choice for the number of degrees of freedom, it is often necessary to carry out some statistical tests with historical data to ascertain how fat we require the tails to be. Galiani [8] use an Exact Maximum Likelihood Method (EML). Other works explain how to calibrate t-student copula to real market data (Mashal and Zeevi [21], Romano [25], Meneguzzo and Vecchiato [23]...). The difference between Gaussian copulas and the t-Student copulas can be described with the concept of tail dependence. If a bivariate copula \( C(u, v) \) such as:

\[
\lim_{u \to 1} \frac{1 + C(u, u) - 2u}{1 - u} = \lambda_U > 0,
\]

Then \( C \) has upper tail dependence with parameter \( \lambda_U \). If:

\[
\lim_{u \to 0} \frac{C(u, u)}{u} = \lambda_L > 0
\]

Then \( C \) has lower tail dependence with parameter \( \lambda_L \).

### 3. Copulas parameters estimations

In this section, we will discuss nonparametric (Genest & Rivest [10]), parametric (Likelihood) and semi-parametric (Genest et al [11]) methods of estimating Archimedean copula parameters.

Genest & Rivest [10] suggested a nonparametric method for estimating the dependence function of a pair of random variables under the assumption that their uniform representation is Archimedean. Their method relies on the estimation of the univariate distribution function associated with the probability integral transformation and requires complete data. The best fitting Archimedean model is the one whose probability integral transformation distribution is the closest to its empirical estimate. The bivariate probability integral transformation of \((X, Y)\) with joint distribution function \(H\) is defined as

\[
V = H(X, Y).
\]

It is not generally true that the distribution function \( K \) of \( V \) is uniform on \([0,1]\) even if \( H \) is continuous. Similarly, \( K \) does not characterize \( H \) since \( K \) does not contain any information about the marginals \( F_X \) and \( F_Y \).

The problem of specifying a probability model for independent observations \((x_1, y_1), \ldots, (x_n, y_n)\) from a bivariate non normal distribution function \(H(X,Y)\) can be simplified by expressing \( H \) in terms of its marginals \( F_X \) and \( F_Y \) and its associated dependence function \( C \).

Then, Archimedean copulas are characterized by the stochastic behaviour of the random variate \( V = H(X, Y) \). The univariate distribution function is defined as:

\[
K(v) = \Pr[H(X, Y) \leq v] = \Pr[C\{F_X(X), F_Y(Y)\} \leq v] \quad \text{on the interval } (0,1).
\]

The estimation of \( K \) can be accomplished in two steps: they construct the empirical bivariate distribution \( H_n(X, Y)\) and they compute \( H_n(x_i, y_i)\) for \( i = 1, \ldots, n \) and use those pseudo observations to construct one-dimensional empirical distribution function for \( K \).

The Archimedean copula presents an appealing property: each copula has an analytical expression that links its parameters to its related Kendall tau. Here, we present the important theorem in the theory of Archimedean copula (Genest & MacKay, [9]):

**Theorem:** Let \((X, Y)\) be a pair of random variables whose distribution \(H\) is of the form

\[
[C \Phi(x, y) = \Phi^{-1}\{(\Phi(x) + \Phi(y))\} \text{ for some } \Phi, \text{ then: } \tau = 4 \int_0^1 \frac{\Phi(t)}{\Phi'(t)} dt + 1.
\]

Then, we can estimate the parameter from the parametric copula using a relationship between the Kendall’s \( \tau \) and the Archimedean copula.

Genest et al [11] proposed a semi-parametric procedure for estimating the dependence parameters in a family of multivariate distributions when one does not want to specify any parametric model to describe the marginal distribution. This procedure consists of transforming the marginal
observations into uniformly distributed vectors using the empirical distribution function. Then, the copula parameters are estimated by maximisation of a Pseudo log-likelihood function.

When nonparametric estimates are contemplated for the marginals, inference about the dependence parameter must be margin free. We have given a random sample \( \{X_{1k}, \ldots, X_{pk}\}; k = 1, \ldots, n \), from distribution \( F_{\alpha}(x_1, \ldots, x_p) = C_{\alpha}(F_1(x_1), \ldots, F_p(x_p)) \). In the construction of the likelihood function, we will be interested to the parametric representation of the copula, specifically, the copula density. The procedure consists of selecting the parameter value \( \hat{\alpha}_n \) that maximises the pseudo log-likelihood:

\[
L(\alpha) = \frac{n}{\log n} \log \left[ \prod_{k=1}^{n} c_{\alpha}(F_{in}(X_{1k}), \ldots, F_{pn}(X_{pk})) \right]
\]

in which \( c_{\alpha} \) is the copula density and \( F_{in} \) is the rescaled empirical distribution function given by: \( F_{in}(x) = \frac{1}{n+1} \sum_{j=1}^{n} \mathbb{I} \{X_{ij} \leq x\} \) for any \( 1 \leq i \leq p \).

Genest et al [11] examined the statistical properties of the proposed estimator. It is shown that it is consistent, asymptotically normal and fully efficient at independence.

4. Pricing Collateralized Debt Obligation

Following the classification of Tavakoli [28], a CDO is backed by portfolios of assets that may include a combination of bonds, loans, securitised receivables, asset-backed securities, tranches of other CDO’s, or credit derivatives referencing any of the former. Some market practitioners define a CDO as being backed by a portfolio including only bonds. A Collateralized loan obligation (CLO) is a type of CDO that is backed by a portfolio of loans. A Collateralized bond obligation (CBO) is a type of CDO that is backed by a portfolio of bonds issued by a variety of corporate or sovereign obligors.

The development of structured credit derivatives leads to the emergence of synthetic Collateralized Debt Obligations which transfer the risk of a pool of single-name Credit Default Swaps. This realizes an exposure to a variety of names.

Suppose that the total CDO notional is 100 millions and during the lifetime of CDO some debts in the collateral portfolio might default. At maturity, if the total default loss is less than 10 millions, only the equity tranche is affected. If the total loss is between 10 and 30 millions, the equity tranche does not get the principal back and the mezzanine gets only part of it. If the loss is more than 30 millions than the equity and mezzanine do not get anything back and senior tranche gets is left.

We present a methodology for pricing CDO with Monte Carlo simulations and Gaussian and student copulas. Consider an homogeneous CDO with \( n \) obligors with nominal amount \( A_i \) and recovery rate \( R_i \) with \( i = 1, 2, \ldots, n \), (assumed deterministic), maturity \( T \) years and we assume constant risk free interest rate. The total value of the portfolio is \( V_T = \sum_{i=1}^{n} A_i \) and \( L_i = (1 - R_i) A_i \) will denote the loss given default for the \( i^{th} \) credit. Let \( \tau_i \) be the default time of the \( i^{th} \) name and \( N_i(t) = \sum_{i=1}^{n} I_{[\tau_i, \infty)} \) be the counting process which jumps from 0 to 1 at default time of name \( i \).

\( L(t) \) will denote the cumulative loss on the collateral portfolio at time \( t \): \( L(t) = \sum_{i=1}^{n} L_i N_i(t) \). The tranche \([a, b]\) suffers a loss at time \( t \) if \( a\% V_T < L(t) \leq b\% V_T \), where \( a\% \) and \( b\% \) are respectively lower and upper bound. Suppose that \( a\% V_i = a' \) and \( b\% V_i = b' \), then, the tranche loss:

\[
L_{[a', b']} = \begin{cases} 
\mathbb{I}(L(t) - a') & \text{if } L(t) - a' \leq b' - a' \\
(b' - a') & \text{if } L(t) - a' > b' - a'
\end{cases}
\]

Using Monte Carlo simulation, the estimation of tranche loss becomes a straightforward task. According to Galiani [8], Pricing a CDO using Monte Carlo simulation involves creating sample paths of correlated default times. These default times are used to calculate the payments on two legs and value each leg. The first is the present value of tranche losses triggered by credit events during the CDO lifetime and is called default leg [DL] and the second is the present value of the premium payments weighted by the outstanding capital (original tranche amount minus accumulated losses) and
is called premium leg [PL]. The fair spread of CDO can be computed by dividing the present value of the default leg $E[DL]$ through the present value of the premium leg $E[PL]$: $S = \frac{E[DL]}{E[PL]}$ (11)

The $K^{th}$ default leg can be computed as: $DL^k = \sum_{i=1}^n e^{-r\tau^k_i} L_{x_i,b}(\tau^k_i)$ where $r$ are the free risk interest rate and $\{\tau^1_i, \tau^2_i, ..., \tau^n_i\}$ the sequence of default times with $K^{th}$ iteration of a Monte Carlo simulation. The accumulated loss is given by:

$$\mathcal{A}^k(t) = (1-R)\sum_{t=1}^n I_{[\tau^k_i < t]}.$$ (12)

The premium leg is paid over the outstanding capital in the tranche. If during the lifetime of the CDO the tranche is wiped out, there are no more premium payments:

$$PL^k = N \sum_{j=1}^{\delta_j} \delta_j e^{-\delta_j} \min \left\{ \max \left[ b - \mathcal{A}^k(t_j), 0 \right], b - a \right\}$$ (13)

Where $\{t_1, t_2, ..., t_\delta\}$ are the premium payment dates with frequency $\delta_j$.

Table 1 presents fair spread of a homogeneous CDO with Monte Carlo simulation. Standard errors of estimates are less than 1 basis point.

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Spread (basis point) (Gaussian copula)</th>
<th>Spread (basis point) (Student copulas)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0% à 10% (Equity)</td>
<td>2952.4</td>
<td>3172.895</td>
</tr>
<tr>
<td>10% à 30% (Mezzanine)</td>
<td>779.3024</td>
<td>762.065</td>
</tr>
<tr>
<td>30% à 100% (Senior)</td>
<td>43.4713</td>
<td>30.210</td>
</tr>
</tbody>
</table>

Hull & White [16] find that the double Student-t copula model with the same heavy tailed distributions for systematic and idiosyncratic risk performs very well in market price fitting. Burtschell et al. [2] report that the double Student-t copula model has very good calibration features to the CDO market in comparison to other models like t-Student copulas.

5. Conclusion

The aim of this paper is to use copulas functions to capture the different structures of dependency when we deal with portfolios of dependent credit risks and a basket of credit derivatives. The key idea of modelling correlated default is the usage of copulas functions. The valuation models are set up with Gaussian and Student copulas. We use Monte Carlo method for simulating the default times, with which multi-name credit derivatives can be priced. The advantage of Monte-Carlo is its simplicity and generality. Its main drawbacks, however, are the quality of the convergence, especially when one computes sensitivities. A good convergence is particularly hard to achieve for credit products since default events are usually rare, and probabilities in the tail of the distribution are difficult to estimate.

Furthermore, the Gaussian distribution has thin tails compared to other distributions. As we are concerned of default events that are by nature tail events, we use distributions with fat tails such as the Student distribution and we find that this change in assumption changes our results.

References


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